UNIVERSALITY FOR BOND PERCOLATION IN TWO DIMENSIONS

GEOFFREY R. GRIMMETT AND IOAN MANOLESCU

Abstract. All (in)homogeneous bond percolation models on the square, triangular, and hexagonal lattices belong to the same universality class, in the sense that they have identical critical exponents *at* the critical point (assuming the exponents exist). This is proved using the star–triangle transformation and the box-crossing property. The exponents in question are the one-arm exponent ρ, the $2j$-alternating-arms exponents ρ_{2j} for $j \geq 1$, the volume exponent δ, and the connectivity exponent η. By earlier results of Kesten, this implies universality also for the near-critical exponents β, γ, ν, Δ (assuming these exist) for any of these models that satisfy a certain additional hypothesis, such as the *homogeneous* bond percolation models on these three lattices.

1. Introduction and results

1.1. **Overview.** Two-dimensional percolation has enjoyed an extraordinary renaissance since Smirnov's proof in 2001 of Cardy's formula (see [15]). Remarkable progress has been made towards a full understanding of site percolation on the triangular lattice, at and near its critical point. Other critical two-dimensional models have, however, resisted solution. The purpose of the current work is to continue our study (beyond [6]) of the phase transition for inhomogeneous bond percolation on the square, triangular, and hexagonal lattices. Our specific target is to show that such models belong to the same universality class. We prove that critical exponents *at the critical point* are constant within this class of models (assuming that such exponents exist). We indicate a hypothesis under which exponents *near criticality* are constant also, and note that this is satisfied by the homogenous models.

We focus here on the one-arm exponent ρ, and the $2j$-alternating-arms exponents ρ_{2j} for $j \geq 1$. By transporting open primal paths

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and open dual paths, we shall show that these exponents are constant across (and beyond) the above class of bond percolation models. More precisely, if any one of these exponents, π say, exists for one of these models, then π exists and is equal for every such model. No progress is made here on the problem of existence of exponents.

Kesten [10] showed that the exponents δ and η are specified by knowledge of ρ, under the hypothesis that ρ exists. Therefore, δ and η are universal across this class of models. Results related to those of [10] were obtained in [11] for the 'near-critical' exponents β, γ, ν, Δ. This last work required a condition of rotation-invariance not possessed by the *strictly* inhomogeneous models. This is discussed further in Section 1.4.

It was shown in [6] that critical inhomogeneous models on the above three lattices have the box-crossing property; this was proved by transportations of open box-crossings from the homogeneous square-lattice model. This box-crossing property, and the star–triangle transformation employed to prove it, are the basic ingredients that permit the proof of universality presented here.

A different extension of the star–triangle method has been the subject of work described in [2, 18, 19]. That work is, in a sense, combinatorial in nature, and it provides connections between percolation on a graph embedded in $\mathbb{R}^2$ and on a type of dual graph obtained via a generalized star–triangle transformation. In contrast, the work reported here is closely connected to the property of isoradiality (see [3, 8]), and is thus more geometric in nature. It permits the proof of relations *between* a variety of two-dimensional graphs. The connection to isoradiality will be the subject of a later paper [5].

The paper is organized as follows. The relevant critical exponents are summarized in Section 1.3, and the main theorems stated in Section 1.4. Extensive reference will be made to [6], but the current work is fairly self-contained. Section 2 contains a short account of the star–triangle transformation, for more details of which the reader is referred to [6]. The proofs are to be found in Section 3.

1.2. **The models.** Let $G = (V, E)$ be a countable connected planar graph, embedded in $\mathbb{R}^2$. The bond percolation model on G is defined as follows. A *configuration* on G is an element $\omega = (\omega_e : e \in E)$ of the set $\Omega = \{0, 1\}^E$. An edge with endpoints u, v is denoted uv. The edge e is called *open*, or ω-*open*, in $\omega \in \Omega$ (respectively, *closed*) if $\omega_e = 1$ (respectively, $\omega_e = 0$).

For $\omega \in \Omega$ and $A, B \subseteq V$, we say A *is connected to* B (in ω), written $A \leftrightarrow B$ (or $A \xleftrightarrow{G,\omega} B$), if G contains a path of open edges from some

$a \in A$ to some $b \in B$. An *open cluster* of ω is a maximal set of pairwise-connected vertices, and the open cluster containing the vertex v is denoted C_v. We write $v \leftrightarrow \infty$ if v is the endpoint of an infinite open self-avoiding path.

The *homogeneous* bond percolation model on G is that associated with the product measure $\mathbb{P}_p$ on Ω with constant intensity $p \in [0,1]$. Let 0 denote a designated vertex of V called the *origin*. The *percolation probability* and *critical probability* are given by

$$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty),$$
$$p_{\mathrm{c}}(G) = \sup\{p : \theta(p) = 0\}.$$

We consider the square, triangular, and hexagonal (or honeycomb) lattices of Figure 1.1, denoted respectively as $\mathbb{Z}^2$, $\mathbb{T}$, and $\mathbb{H}$. It is standard that $p_{\mathrm{c}}(\mathbb{Z}^2) = \frac{1}{2}$, and $p_{\mathrm{c}}(\mathbb{T}) = 1 - p_{\mathrm{c}}(\mathbb{H})$ is the root in the interval $(0,1)$ of the cubic equation $3p - p^3 - 1 = 0$. See the references in [4, 6] for these and other known facts quoted in this paper.

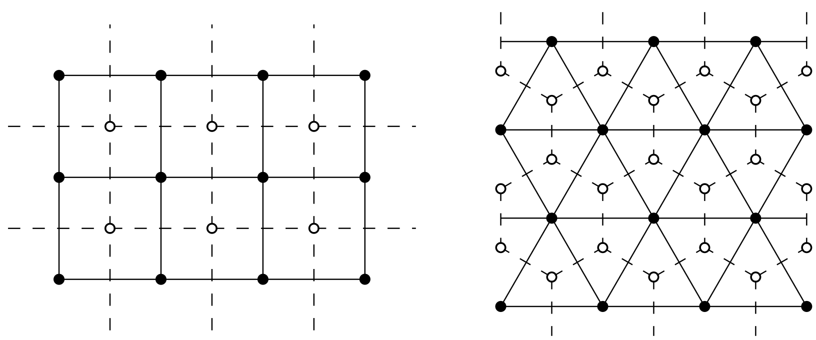

FIGURE 1.1. The square lattice and its dual square lattice. The triangular lattice and its dual hexagonal lattice.

We turn now to *inhomogeneous* percolation on the above three lattices. The edges of the square lattice are partitioned into two classes (horizontal and vertical) of parallel edges, while those of the triangular and hexagonal lattices may be split into three such classes. We allow the product measure on Ω to have different intensities on different edges, while requiring that any two parallel edges have the same intensity. Thus, inhomogeneous percolation on the square lattice has two parameters, p_0 for horizontal edges and p_1 for vertical edges, and we denote the corresponding measure $\mathbb{P}^{\square}_{\mathbf{p}}$ where $\mathbf{p} = (p_0, p_1)$. On the triangular and hexagonal lattices, the measure is defined by a triplet of parameters $\mathbf{p} = (p_0, p_1, p_2)$, and we denote these measures $\mathbb{P}^{\triangle}_{\mathbf{p}}$ and $\mathbb{P}^{\hexagon}_{\mathbf{p}}$, respectively. Let $\mathcal{M}$ denote the set of all such inhomogeneous bond percolation models on the square, triangular, and hexagonal lattices, with edge-parameters belonging to the half-open interval $[0, 1)$.

These models have percolation probabilities and critical surfaces, and the latter were given explicitly in [4, 6, 9]. Let

$$\begin{aligned}
\kappa_\square(\mathbf{p}) &= p_0 + p_1 - 1, & \mathbf{p} &= (p_0, p_1),\\
\kappa_\triangle(\mathbf{p}) &= p_0 + p_1 + p_2 - p_0p_1p_2 - 1, & \mathbf{p} &= (p_0, p_1, p_2),\\
\kappa_{\hexagon}(\mathbf{p}) &= -\kappa_\triangle(1 - p_0, 1 - p_1, 1 - p_2), & \mathbf{p} &= (p_0, p_1, p_2).
\end{aligned}$$

It is well known that the critical surface of the lattice $\mathbb{Z}^2$ (respectively, $\mathbb{T}$, $\mathbb{H}$) is given by $\kappa_\square = 0$ (respectively, $\kappa_\triangle(\mathbf{p}) = 0$, $\kappa_{\hexagon}(\mathbf{p}) = 0$). Bond percolation on $\mathbb{Z}^2$ may be obtained from that on $\mathbb{T}$ by setting one parameter to zero.

The triplet $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1)^3$ is called *self-dual* if $\kappa_\triangle(\mathbf{p}) = 0$. We write $\alpha \pm \mathbf{p}$ for the triplet $(\alpha \pm p_0, \alpha \pm p_1, \alpha \pm p_2)$, and also $\mathbb{N} = \{1, 2, \dots\}$ for the natural numbers, and $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ for the integers.

1.3. **Critical exponents.** The percolation singularity is of power-law type, and is described by a number of so-called 'critical exponents'. These may be divided into two groups of exponents: *at criticality*, and *near criticality*.

First, some notation: we write $f(t) \asymp g(t)$ as $t \to t_0 \in [0, \infty]$ if there exist strictly positive constants A, B such that

$$Ag(t) \le f(t) \le Bg(t)$$

in some neighbourhood of t_0 (or for all large t in the case $t_0 = \infty$). We write $f(t) \approx g(t)$ if $\log f(t)/\log g(t) \to 1$. Two vectors $\mathbf{p}_1 = (p_1(e))$, $\mathbf{p}_2 = (p_2(e))$ satisfy $\mathbf{p}_1 < \mathbf{p}_2$ if $p_1(e) \le p_2(e)$ for all e, and $\mathbf{p}_1 \ne \mathbf{p}_2$.

For simplicity we restrict ourselves to the percolation models of the last section. Let $\mathbb{L} = (V, E)$ be one of the square, triangular, and hexagonal lattices. Let $\mathbf{p} = (p(e) : e \in E) \in [0, 1)^E$ be invariant under translations of $\mathbb{L}$ as above, and let $\omega \in \Omega$. The lattice $\mathbb{L}$ has a dual lattice $\mathbb{L}^* = (V^*, E^*)$, each edge of which is called *open** if it crosses a closed edge of $\mathbb{L}$. Open paths of $\mathbb{L}$ are said to have colour 1, and open* paths of $\mathbb{L}^*$ colour 0. We shall make use of duality as described in [4, Sect. 11.2].

Let Λ_n be the set of all vertices within graph-theoretic distance n of the origin 0, with boundary $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$. Let $\mathcal{A}(N, n) = \Lambda_n \setminus \Lambda_{N-1}$ be the *annulus* centred at 0, with *interior radius* N and *exterior radius* n. We call $\partial\Lambda_n$ (respectively, $\partial\Lambda_N$) its *exterior* (respectively, *interior*) *boundary*. We shall soon consider embeddings of planar lattices in $\mathbb{R}^2$, and it will then be natural to use the L^∞ metric rather than graph-distance. The choice of metric is in fact of no fundamental important for what follows. For $v \in V$, we write

$$\mathrm{rad}(C_v) = \sup\{n : v \leftrightarrow v + \partial\Lambda_n\}.$$

Let $\mathbf{p}_c$ be a vector lying on the critical surface. Thus, $\mathbf{p}_c$ is critical in that

$$\theta(\mathbf{p}) := \mathbb{P}_{\mathbf{p}}(0 \leftrightarrow \infty) \begin{cases} = 0 & \text{if } \mathbf{p} < \mathbf{p}_c, \\ > 0 & \text{if } \mathbf{p} > \mathbf{p}_c. \end{cases}$$

Let $k \in \mathbb{N}$, and let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in \{0,1\}^k$; we call σ a *colour sequence*. The sequence σ is called *monochromatic* if either $\sigma = (0,0,\dots,0)$ or $\sigma = (1,1,\dots,1)$, and *bichromatic* otherwise. If k is even, σ is called *alternating* if either $\sigma = (0,1,0,1,\dots)$ or $\sigma = (1,0,1,0,\dots)$. For $0 < N < n$, the arm event $A_\sigma(N,n)$ is the event that the inner boundary of $\mathcal{A}(N,n)$ is connected to its outer boundary by k vertex-disjoint paths with colours $\sigma_1, \dots, \sigma_k$, taken in anticlockwise order. [Here and later, we require arms to be *vertex*-disjoint rather than *edge*-disjoint. This is an innocuous assumption since we work in this paper with alternating colour sequences only.]

The choice of N is in part immaterial to the study of the asymptotics of $\pi_\sigma(N,n)$ as $n \to \infty$, and we shall assume henceforth that $N = N(\sigma)$ is sufficiently large that, for $n \ge N$, there exists a configuration with the required j coloured paths. It is believed that there exist constants $\rho(\sigma)$ such that

$$\mathbb{P}_{\mathbf{p}_c}[A_\sigma(N,n)] \approx n^{-\rho(\sigma)},$$

and these are the *arm-exponents* of the model. [Such asymptotics are to be understood in the limit as $n \to \infty$.]

We concentrate here on the following exponents given in terms of $\mathbb{P}_{\mathbf{p}_c}$, with limits as $n \to \infty$:

(a) volume exponent: $\mathbb{P}_{\mathbf{p}_c}(|C_0| = n) \approx n^{-1-1/\delta}$,
(b) connectivity exponent: $\mathbb{P}_{\mathbf{p}_c}(0 \leftrightarrow x) \approx |x|^{-\eta}$,
(c) one-arm exponent: $\mathbb{P}_{\mathbf{p}_c}(\mathrm{rad}(C_0) = n) \approx n^{-1-1/\rho}$,
(d) $2j$-alternating-arms exponents: $\mathbb{P}_{\mathbf{p}_c}[A_\sigma(N,n)] \approx n^{-\rho_{2j}}$, for each alternating colour sequence σ of length $2j$.

It is believed that the above asymptotic relations hold for suitable exponent-values, and indeed with $\approx$ replaced by the stronger relation $\asymp$. Essentially the only two-dimensional percolation process for which these limits are proved (and, furthermore, the exponents calculated explicitly) is site percolation on the triangular lattice (see [15, 16]).

The arm events are defined above in terms of open primal and open* dual paths. When considering site percolation, one considers instead open paths in the primal and matching lattices. This is especially simple for the triangular lattice since $\mathbb{T}$ is self-matching. It is known for site percolation on the triangular lattice, [1], that for given $k \in \mathbb{N}$, the exponent for $\rho(\sigma)$ is constant for any bichromatic colour sequence σ of given length k. This is believed to hold for other two-dimensional models also, but no proof is known. In particular, it is believed for any model in $\mathcal{M}$ that

$$\mathbb{P}_{\mathbf{p}_c}[A_\sigma(N,n)] \approx n^{-\rho_{2j}},$$

for any bichromatic colour sequence σ of length $2j$, and any $j \geq 1$.

We turn now to the near-critical exponents which, for definiteness we define as follows. Let $\mathbf{p} = (p(e) : e \in E) \in [0,1)^E$ and $\epsilon \in \mathbb{R}$, and write $\mathbb{P}_{\mathbf{p}+\epsilon}$ for the product measure on Ω in which edge e is open with probability

$$(\mathbf{p}+\epsilon)_e := \max\bigl\{0, \min\{p(e)+\epsilon, 1\}\bigr\}.$$

By subcritical exponential-decay (see [4, Sect. 5.2]), for $\epsilon > 0$, there exists $\xi = \xi(\mathbf{p}_c - \epsilon) \in [0,\infty)$ such that

$$-\frac{1}{n}\log \mathbb{P}_{\mathbf{p}_c-\epsilon}(0 \leftrightarrow \partial\Lambda_n) \to 1/\xi \quad \text{as } n \to \infty.$$

The function ξ is termed the *correlation length*.

Here are the further exponents considered here:

(a) percolation probability: $\theta(\mathbf{p}_c + \epsilon) \approx \epsilon^\beta$ as $\epsilon \downarrow 0$,
(b) correlation length: $\xi(\mathbf{p}_c - \epsilon) \approx \epsilon^{-\nu}$ as $\epsilon \downarrow 0$,
(c) mean cluster-size: $\mathbb{P}_{\mathbf{p}_c+\epsilon}(|C_0|; |C_0| < \infty) \approx |\epsilon|^{-\gamma}$ as $\epsilon \to 0$,
(d) gap exponent: for $k \geq 1$, as $\epsilon \to 0$,

$$\frac{\mathbb{P}_{\mathbf{p}_c+\epsilon}(|C_0|^{k+1}; |C_0| < \infty)}{\mathbb{P}_{\mathbf{p}_c+\epsilon}(|C_0|^{k}; |C_0| < \infty)} \approx |\epsilon|^{-\Delta}.$$

We have written $\mathbb{P}(X)$ for the mean of X under the probability measure $\mathbb{P}$, and $\mathbb{P}(X; A) = \mathbb{P}(X 1_A)$ where 1_A is the indicator function of the event A.

As above, the near-critical exponents are known to exist essentially only for site percolation on the triangular lattice. See [4, Chap. 9] for a general account of critical exponents and scaling theory.

1.4. **Principal results.** A critical exponent π is said to *exist* for a model $M \in \mathcal{M}$ if the appropriate asymptotic relation (above) holds, and π is called $\mathcal{M}$*-invariant* if it exists for all $M \in \mathcal{M}$ and its value is independent of the choice of such M.

Theorem 1.1. *For every $\pi \in \{\rho\} \cup \{\rho_{2j} : j \geq 1\}$, if π exists for some model $M \in \mathcal{M}$, then it is $\mathcal{M}$-invariant.*

By the box-crossing property of [6, Thm 1.3], we may apply the theorem of Kesten [10] to deduce the following. If either ρ or η exists for some $M \in \mathcal{M}$, then:

(a) both ρ and η exist for M,
(b) δ exists for M,
(c) the scaling relations $\eta\rho = 2$ and $2\rho = \delta + 1$ are valid.

Taken in conjunction with Theorem 1.1, this implies in particular that δ and η are $\mathcal{M}$-invariant whenever either ρ or η exist for some $M \in \mathcal{M}$.

We note in passing that Theorem 1.1 may be extended to certain other graphs derived from the three main lattices of this paper by sequences of star–triangle transformations, as well as to their dual graphs. This includes a number of 2-uniform tessellations (see [7]) and, in particular, two further Archimedean lattices, namely those denoted $(3^3, 4^2)$ and $(3, 4, 6, 4)$ and illustrated in Figure 1.2. The measures on these two lattices are as follows. Let $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1)^3$ be self-dual. Edge e is open with probability $p(e)$ where:

(a) $p(e) = p_0$ if e is horizontal,
(b) $p(e) = p_1$ if e is parallel to the right edge of an upwards pointing triangle,
(c) $p(e) = p_2$ if e is parallel to the left edge of an upwards pointing triangle,
(d) the two parameters of any rectangle have sum 1.

Theorem 1.1 holds with $\mathcal{M}$ augmented by all such bond models on these two lattices. The proofs are essentially the same. The methods used here do not appear to extend to homogeneous percolation on these two lattices, and neither may they be applied to the other six Archimedean lattices. Drawings of the eleven Archimedean lattices and their duals may be found in [13].

There is a simple reason for the fact that Theorem 1.1 concerns the alternating-arm exponents rather than all arm exponents. We shall see in Section 2 that the star–triangle transformation conserves open primal and open* dual paths, but that, in certain circumstances, it allows distinct paths of the same colour to coalesce.

The box-crossing property of [6] implies a type of affine isotropy of these models at criticality, yielding in particular that certain directional exponents are independent of the choice of direction, For example, if one insists on one-arm connections in a specific direction, the ensuing exponent equals the undirected exponent ρ. A similar statement holds for arm-directions in the alternating-arm exponents.

Kesten has shown in [11] (see also [12]) that the above near-critical exponents may be given explicitly in terms of exponents at criticality,

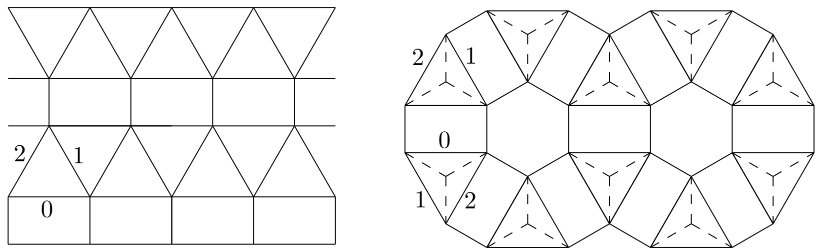

FIGURE 1.2. Isoradial embeddings of the Archimedean lattices $(3^3, 4^2)$ and $(3, 4, 6, 4)$. The second may be transformed into the hexagonal lattice by one sequence of star–triangle transformations as marked. An edge parallel to one labelled i has edge-parameter p_i, and the two parameters on any square have sum 1.

for two-dimensional models satisfying certain hypotheses. Homogeneous percolation on our three lattices satisfy these hypotheses, but it is not known whether the strictly inhomogeneous models have sufficient regularity for the conclusions to hold for them. The basic problem is that, while the box-crossing property of [6] implies an isotropy for these models at criticality, the corresponding isotropy away from criticality is unknown. For this reason we restrict the statement of the next theorem to homogeneous models.

Theorem 1.2. *Assume that ρ and ρ_4 exist for some $M \in \mathcal{M}$. Then β, γ, ν, and Δ exist for homogeneous percolation on the square, triangular and hexagonal lattices, and they are invariant across these three models. Furthermore, they satisfy the scaling relations*

$$\rho\beta = \nu, \quad \rho\gamma = \nu(\delta - 1), \quad \rho\Delta = \nu\delta.$$

The proof is an adaptation of the arguments and conclusions of [11, 12], and is omitted here.

Other authors have observed hints of universality, and we mention for example [14], where it is proved that certain dual pairs of lattices have equal exponents (whenever these exist).

2. Star–triangle transformation

Consider the triangle $G = (V, E)$ and the star $G' = (V', E')$, as drawn in Figure 2.1. Let $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1)^3$. Write $\Omega = \{0, 1\}^E$ with associated product probability measure $\mathbb{P}^{\triangle}_{\mathbf{p}}$, and $\Omega' = \{0, 1\}^{E'}$ with associated measure $\mathbb{P}^{\text{Y}}_{1-\mathbf{p}}$, as illustrated in the figure. Let $\omega \in \Omega$ and $\omega' \in \Omega'$. For each graph we may consider open connections between its vertices, and we abuse notation by writing, for example, $x \xleftrightarrow{G,\omega} y$ for the *indicator function* of the event that x and y are connected by an open path of ω. Thus connections in G are described by the family $(x \xleftrightarrow{G,\omega} y : x, y \in V)$ of random variables, and similarly for G'.

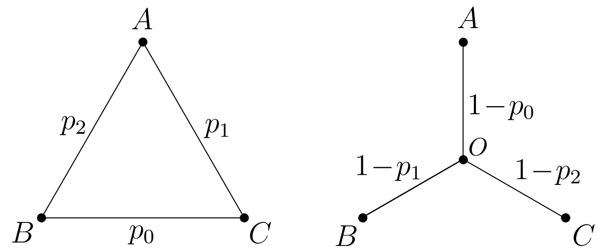

FIGURE 2.1. The star–triangle transformation when $\kappa_\triangle(\mathbf{p}) = 0$.

It may be shown that the two families

$$\left(x \xleftrightarrow{G,\omega} y : x, y = A, B, C\right), \quad \left(x \xleftrightarrow{G',\omega'} y : x, y = A, B, C\right),$$

of random variables have the same joint law whenever $\kappa_\triangle(\mathbf{p}) = 0$. That is to say, if $\mathbf{p}$ is self-dual, the existence (or not) of open connections is preserved (in law) under the star–triangle transformation. See [4, Sect. 11.9].

The two measures $\mathbb{P}^\triangle_{\mathbf{p}}$ and $\mathbb{P}^{\hexagon}_{1-\mathbf{p}}$ may be coupled in a natural way. Let $\mathbf{p} \in [0,1)^3$ be self-dual, and let Ω (respectively, Ω') have associated measure $\mathbb{P}^\triangle_{\mathbf{p}}$ (respectively, $\mathbb{P}^{\hexagon}_{1-\mathbf{p}}$) as above. The random mappings $T : \Omega \to \Omega'$ and $S : \Omega' \to \Omega$ of Figure 2.2 are such that: $T(\omega)$ has law $\mathbb{P}^{\hexagon}_{1-\mathbf{p}}$, and $S(\omega')$ has law $\mathbb{P}^\triangle_{\mathbf{p}}$. Under this coupling, the presence or absence of connections between the corners A, B, C is preserved.

The maps S and T act on configurations on stars and triangles. They act simultaneously on the duals of these graph elements, illustrated in Figure 2.3. Let $\omega \in \Omega$, and define $\omega^*(e^*) = 1 - \omega(e)$ for each primal/dual pair e/e^* of the left side of the figure. The action of T on Ω induces an action on the dual space Ω^*, and it is easily checked that this action preserves ω^*-connections. The map S behaves similarly. This property of the star–triangle transformation has been generalized and studied in [2] and the references therein.

So-called mixed lattices were introduced in [6]. These are hybrid embeddings of the square lattice with either the triangular or hexagonal lattice, the two parts being separated by a horizontal interface. By means of appropriate star–triangle transformations, the interface

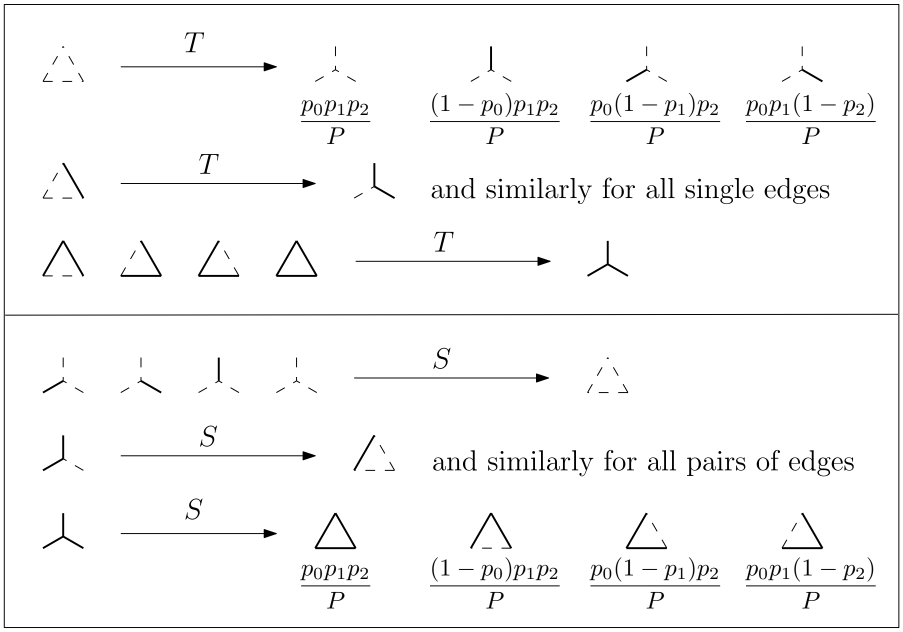

FIGURE 2.2. The 'kernels' T and S and their transition probabilities, with $P := (1-p_0)(1-p_1)(1-p_2)$.

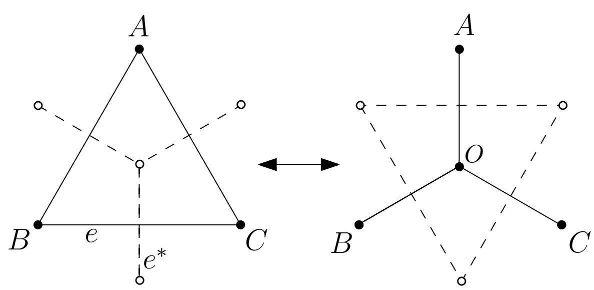

FIGURE 2.3. The star–triangle transformation acts simultaneously on primal and dual graph elements.

may be moved up or down, and this operation permits the transportation of open box-crossings between the square lattice and the other lattice. Whereas this was suited for proving the box-crossing property, a slightly altered hybrid is useful for studying arm exponents.

Let $m \geq 0$, and consider the mixed lattice $\mathbb{L}^m = (V^m, E^m)$ drawn on the left of Figure 2.4, formed of a horizontal strip of the square lattice centred on the x axis of height $2m$, with the triangular lattice above and beneath it. The embedding of each lattice is otherwise as in [6]: the triangular lattice comprises equilateral triangles of side length $\sqrt{3}$, and the square lattice comprises rectangles with horizontal (respectively, vertical) dimension $\sqrt{3}$ (respectively, 1). We require also that the origin of $\mathbb{R}^2$ be a vertex of the mixed lattice.

Let $\mathbf{p} \in [0,1)^3$, and let $\mathbb{P}^m_{\mathbf{p}}$ be the product measure on $\Omega^m = \{0,1\}^{E^m}$ for which edge e is open with probability $p(e)$ given by:

(a) $p(e) = p_0$ if e is horizontal,
(b) $p(e) = 1 - p_0$ if e is vertical,
(c) $p(e) = p_1$ if e is the right edge of an upwards pointing triangle,
(d) $p(e) = p_2$ if e is the left edge of an upwards pointing triangle.

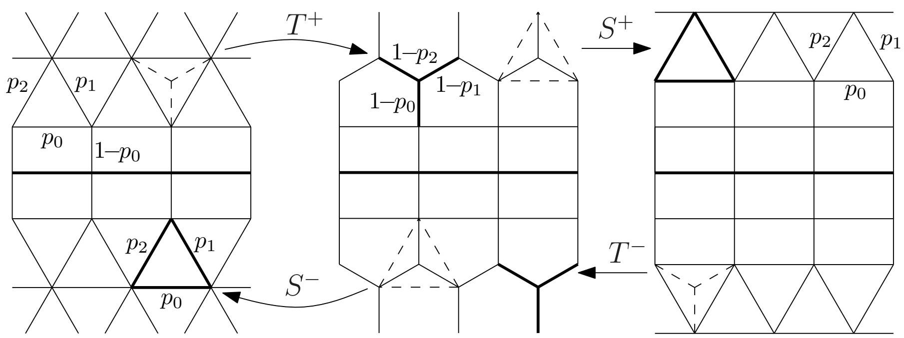

FIGURE 2.4. The transformation $S^+ \circ T^+$ (respectively, $S^- \circ T^-$) transforms $\mathbb{L}^1$ into $\mathbb{L}^2$ (respectively, $\mathbb{L}^2$ into $\mathbb{L}^1$). They map the dashed graphs to the bold graphs.

Suppose further that $\mathbf{p}$ is self-dual, in that $\kappa_{\triangle}(\mathbf{p}) = 0$, and let $\omega^m \in \Omega^m$. We denote by $T^{\vartriangle}$ (respectively, $T^{\triangledown}$) the transformation T of Figure 2.2 applied to an upwards (respectively, downwards) pointing triangle. Write T^+ for the transformation of ω obtained by applying $T^{\vartriangle}$ to every upwards pointing triangle in the upper half plane, and $T^{\triangledown}$ similarly in the lower half plane; sequential applications of star–triangle transformations are required to be independent of one another.

Similarly, we denote by $S^{\curlywedge}$ (respectively, $S^{\curlyvee}$) the transformation S of Figure 2.2 applied to an upwards (respectively, downwards) pointing star. Write S^+ for the transformation of $(T^+\mathbb{L}^m, T^+(\omega^m))$ obtained by applying $S^{\curlywedge}$ to all upwards pointing stars in the upper half-plane and similarly $S^{\curlyvee}$ in the lower half-plane. It may be checked that $\omega^{m+1} = S^+ \circ T^+(\omega^m)$ lies in Ω^{m+1} and has law $\mathbb{P}^{m+1}_{\mathbf{p}}$. That is, viewed as a transformation acting on measures, we have $(S^+ \circ T^+)\mathbb{P}^m_{\mathbf{p}} = \mathbb{P}^{m+1}_{\mathbf{p}}$.

The transformations T^- and S^- are defined similarly, and illustrated in Figure 2.4. As in that figure, for $m \geq 0$,

$$(S^+ \circ T^+)\mathbb{L}^m = \mathbb{L}^{m+1}, \qquad (S^+ \circ T^+)\mathbb{P}^m_{\mathbf{p}} = \mathbb{P}^{m+1}_{\mathbf{p}},$$
$$(S^- \circ T^-)\mathbb{L}^{m+1} = \mathbb{L}^m, \qquad (S^- \circ T^-)\mathbb{P}^{m+1}_{\mathbf{p}} = \mathbb{P}^m_{\mathbf{p}}.$$

We turn to the operation of these two transformations on open paths, and will concentrate on $S^+ \circ T^+$; similar statements are valid for $S^- \circ T^-$. Let $\omega^m \in \Omega^m$, and let π be an ω^m-open path of $\mathbb{L}^m$. It is not difficult to see (and is explained fully in [6]) that the image of π under $S^+ \circ T^+$ contains some ω^{m+1}-open path π'. Furthermore, π' lies within the 1-neighbourhood of π viewed as a subset of $\mathbb{R}^2$, and has endpoints within unit Euclidean distance of those of π. Any vertex of π in the square part of $\mathbb{L}^m$ is unchanged by the transformation. The corresponding statements hold also for open* paths of the dual of $\mathbb{L}^m$. These facts will be useful in observing the effect of $S^+ \circ T^+$ on the arm events.

Arm exponents are defined in Section 1.3 in terms of boxes that are adapted to the lattice viewed as a graph. It will be convenient to work also with boxes of $\mathbb{R}^2$. Let $\mathbb{L} = (V, E)$ be a mixed lattice duly embedded in $\mathbb{R}^2$, and write V_0 for the subset of V lying on the x-axis. Let $\omega \in \Omega = \{0,1\}^E$. For $R \subseteq \mathbb{R}^2$ and $A, B \subseteq R \cap V_0$, we write $A \xleftrightarrow{R,\omega} B$ (with negation written $A \overset{R,\omega}{\not\longleftrightarrow} B$) if there exists an ω-open path joining some $a \in A$ and some $b \in B$ using only edges that intersect R. Let D be the unit (Euclidean) disk of $\mathbb{R}^2$ and write $R + D$ for the direct sum $\{r + d : r \in R,\ d \in D\}$.

Proposition 2.1. *Let $m \geq 0$, $\omega \in \Omega^m$, $R \subseteq \mathbb{R}^2$, and $u, v \in R \cap V_0$. For $\tau \in \{S^+ \circ T^+, S^- \circ T^-\}$,*

(a) *if $u \xleftrightarrow{R,\omega} v$ then $u \xleftrightarrow{R+D,\tau(\omega)} v$,*
(b) *if $u \overset{R+D,\omega}{\not\longleftrightarrow} v$ then $u \overset{R,\tau(\omega)}{\not\longleftrightarrow} v$.*

Proof. (a) Let $\tau = S^+ \circ T^+$; the case $\tau = S^- \circ T^-$ is similar (we assume $m \geq 1$ where necessary). If $u \xleftrightarrow{R,\omega} v$, there exists an ω-open path π of $\mathbb{L}$ from u to v using edges that intersect R. Since u, v are not moved by τ, the image $\tau(\pi)$ contains a $\tau(\omega)$-open path of $\tau\mathbb{L}$ from u to v. It is elementary that τ transports paths through a distance not exceeding 1 (see [6, Prop. 2.4]). Therefore, every edge of $\tau(\pi)$ intersects $R + D$.
(b) Suppose $u \xleftrightarrow{R,\tau(\omega)} v$. By considering the star–triangle transformations that constitute the mapping τ (as in part (a)), we have that $u \xleftrightarrow{R+D,\omega} v$. □

3. Universality of arm exponents

This section contains the proof of Theorem 1.1. The reader is reminded that we work with translation-invariant measures associated with the square, triangular, and hexagonal lattices.

3.1. The arm exponents.

Let $k \in \mathbb{N}$ and $\sigma \in \{0,1\}^k$. The arm event $A_\sigma(N,n)$ is empty if N is too small to support the existence of the required j disjoint paths to the exterior boundary of the annulus $\mathcal{A}(N,n)$. As explained in [12] for example, for each σ, there exists $N = N_0(\sigma)$ such that the arm exponent (assuming existence) is independent of the choice of $N \geq N_0(\sigma)$. We assume henceforth that N is chosen sufficiently large for this to be the case.

It is a significant open problem of probability theory to prove the existence and invariance of arm exponents for general lattices. This amounts to the following in the present situation.

Conjecture 3.1. *Let $\mathbf{p} \in [0,1)^3$ be self-dual. For $k \in \mathbb{N}$ and a colour sequence $\sigma \in \{0,1\}^k$, there exists $\rho = \rho(\sigma, \mathbf{p}) > 0$ such that*

$$\mathbb{P}_{\mathbf{p}}^{\triangle}[A_\sigma(N,n)] \approx n^{-\rho}.$$

Furthermore, $\rho(\sigma,\mathbf{p})$ is constant for all self-dual $\mathbf{p}$.

This is phrased for the triangular lattice, but it embraces also the square and hexagonal lattices, the first by setting a component of $\mathbf{p}$ to 0, and the second by a single application of the star–triangle transformation. (See also [14].) We make no contribution towards a proof of the first part of this conjecture. Theorem 1.1 may be viewed as the resolution of the second part when $k \in \{1,2,4,\dots\}$.

Hereafter, we consider only the one-arm event with $\sigma = \{1\}$, and the $2j$-alternating-arms events with $\sigma = (1,0,1,0,\dots)$, with associated exponents denoted respectively as ρ_1 and ρ_{2j}. Thus $\rho_1 = 1/\rho$ with ρ as in Section 1.3.

3.2. Main proposition.

Let $\mathbb{L}$ be one of the square, triangular, and hexagonal lattices, or a hybrid thereof as in Section 2. We embed $\mathbb{L}$ in $\mathbb{R}^2$ in the manner described in that section. Let $x_i = (i\sqrt{3}, 0)$, $i \geq 0$, denote the vertices common to these lattices to the right of the origin, and $y_i = ((i+\frac{1}{2})\sqrt{3}, \frac{1}{2})$, $i \geq 0$, the vertices of the dual lattice $\mathbb{L}^*$ corresponding to the faces of $\mathbb{L}$ lying immediately above the edge $x_i x_{i+1}$. For $r \in (0,\infty)$, let $B_r = [-r,r]^2 \subseteq \mathbb{R}^2$, with boundary ∂B_r. We recall that C_x (respectively, C_y^*) denotes the open cluster of $\mathbb{L}$ containing x (respectively, the open* cluster of $\mathbb{L}^*$ containing y). For $n \geq 1$ and any connected subgraph C of either $\mathbb{L}$ or $\mathbb{L}^*$, we write

$C \cap \partial B_r \neq \varnothing$ if C contains vertices in both B_r and $\mathbb{R}^2 \setminus (-r, r)^2$. Note that we may have $C \cap \partial B_r \neq \varnothing$ even when there are no vertices of C belonging to ∂B_r.

For $j, n \in \mathbb{N}$ with $j \geq 2$, let

$$\begin{aligned} A_1(n) &= \{C_{x_0} \cap \partial B_n \neq \varnothing\}, \\ A_2(n) &= \{C_{x_0} \cap \partial B_n \neq \varnothing,\, C^*_{y_0} \cap \partial B_n \neq \varnothing\}, \\ A_{2j}(n) &= \bigcap_{0 \leq i < j} \Big\{C_{x_i} \cap \partial B_n \neq \varnothing, \text{ and } x_i \overset{B_n,\omega}{\not\longleftrightarrow} \{x_0, x_1, \dots, x_{i-1}\}\Big\}. \end{aligned}$$

We write $A_k^{\mathbb{L}}(n)$ when the role of $\mathbb{L}$ is to be stressed. Note the condition of disconnection in the definition of $A_{2j}(n)$: it is required that the x_i are not connected by open paths of edges all of which intersect B_n.

A proof of the following elementary lemma is sketched at the end of this subsection. An alternative proof of the second inequality of the lemma may be obtained with the help of the forthcoming separation theorem, Theorem 3.5, as in the final part of the proof of Proposition 3.7. The latter route is more general since it assumes less about the underlying lattice, but it is also more complex since it relies on a version of the separation theorem of [11] whose somewhat complicated proof is omitted from the current work.

Lemma 3.2. *Let $\mathbf{p} \in [0,1)^E$ be self-dual. Let $k \in \{1, 2, 4, 6, \dots\}$, and let σ be an alternating colour sequence of length k (when $k = 1$ we set $\sigma = \{1\}$). There exists $N_0 = N_0(k) \in \mathbb{N}$ and $c = c(\mathbf{p}, N, k) > 0$ such that*

$$\mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n\sqrt{3})] \leq \mathbb{P}^{\triangle}_{\mathbf{p}}[A_\sigma(N, n)] \leq c\mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)]$$

for $n \geq N \geq N_0$.

Let $\mathbf{p} \in [0,1)^3$ be self-dual with $p_0 > 0$, and consider the two measures $\mathbb{P}^{\square}_{(p_0, 1-p_0)}$ (respectively, $\mathbb{P}^{\triangle}_{\mathbf{p}}$) on the square (respectively, triangular) lattice. The proof of the universality of the box-crossing property was based on a technique that transforms one of these lattices into the other while preserving primal and dual connections. The same technique will be used here to prove the following result, the proof of which is deferred to Section 3.3.

Proposition 3.3. *For any $k \in \{1, 2, 4, 6, \dots\}$ and any self-dual triplet $\mathbf{p} \in [0,1)^3$ with $p_0 > 0$, there exist $c_0, c_1, n_0 > 0$ such that, for all $n \geq n_0$,*

$$c_0 \mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)] \leq \mathbb{P}^{\square}_{(p_0, 1-p_0)}[A_k(n)] \leq c_1 \mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)].$$

Proof of Theorem 1.1. Suppose there exist $k \in \{1, 2, 4, 6, \dots\}$, a self-dual $\mathbf{p} \in [0, 1)^3$, and $\alpha > 0$, such that

$$\mathbb{P}^{\triangle}_{\mathbf{p}}[A_\sigma(N, n)] \approx n^{-\alpha}, \tag{3.1}$$

with σ the alternating colour sequence of length k (when $k = 1$, we take $\sigma = \{1\}$). By Lemma 3.2, (3.1) is equivalent to

$$\mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)] \approx n^{-\alpha}. \tag{3.2}$$

We say that '$\mathbb{P}$ satisfies (3.2)' if (3.2) holds with $\mathbb{P}^{\triangle}_{\mathbf{p}}$ replaced by $\mathbb{P}$. By self-duality, there exists i such that $p_i > 0$, and we assume without loss of generality that $p_0 > 0$. By Proposition 3.3, $\mathbb{P}^{\square}_{(p_0,1-p_0)}$ satisfies (3.2). Similarly, $\mathbb{P}^{\triangle}_{\mathbf{p}'}$ satisfies (3.2) for any self-dual $\mathbf{p}' \in [0, 1)^3$ of the form $\mathbf{p}' = (p_0, p'_1, p'_2)$. The claim is proved after one further application of the proposition. $\square$

Outline proof of Lemma 3.2. First, a note concerning the event $A_{2j}(n)$ with $j \geq 2$. If $\omega \in A_{2j}(n)$, vertices x_i, $0 \leq i < j$, are connected to ∂B_n by open paths. We claim that j such open paths may be found that are vertex-disjoint and interspersed by j open* paths joining the y_i to ∂B_n. This will imply the existence of $2j$ arms of alternating types joining $\{x_0, y_0, x_1, y_1, \dots, x_{j-1}\}$ to ∂B_n, such that the open primal paths are vertex-disjoint, and the open* dual paths are vertex-disjoint except at the y_i. The claim may be seen as follows (see also Figure 3.1). The dual edge e with endpoints $\pm y_0$ is necessarily open*. By exploring the boundary of C_{x_0} at e, one may find two open* paths denoted π_0, π'_0, joining y_0 to ∂B_n, and vertex-disjoint except at y_0. Let $0 \leq r \leq j - 2$. Since $x_r, x_{r+1} \xleftrightarrow{B_n,\omega} \partial B_n$ and $x_r \not\xleftrightarrow{B_n,\omega} x_{r+1}$, we may similarly explore the boundary of C_{x_r} to find an open* path π_r of B_n that joins y_r to ∂B_n, and is vertex-disjoint from either π_0 or π'_0, and in addition from π_s, $s \neq r$. The dual paths $\pi'_0, \pi_0, \pi_1, \dots, \pi_{j-2}$ are the required open* arms.

The set Λ_n induces a subgraph of $\mathbb{T}$ whose boundary is denoted $\partial\Lambda_n$. We denote the inside of $\partial\Lambda_n$ (that is, the closure of the bounded component of $\mathbb{R}^2 \setminus \partial\Lambda_n$) by Λ_n also. It is easily seen that $\Lambda_n \subseteq B_{n\sqrt{3}}$, and the first inequality follows immediately.

For the second inequality, we shall use the fact that $B_n \subseteq \Lambda_n$, together with a suitable construction of open and open* paths within Λ_N. Let $k = 2j \in \{2, 4, 6, \dots\}$ and suppose $A_\sigma(N, n)$ occurs. On an anticlockwise traverse of $\partial\Lambda_N$, we find points $a_0, b_0, a_1, b_1, \dots, a_{j-1}, b_{j-1}$ such that the a_i (respectively, b_i) are endpoints of open (respectively, open*) paths crossing the annulus $\mathcal{A}(N,n)$. Note that the b_i are not vertices of $\mathbb{L}^*$, but instead lie in open* edges. Write $\mathbf{a} = (a_0, a_1, \dots, a_{j-1})$, $\mathbf{b} = (b_0, b_1, \dots, b_{j-1})$.

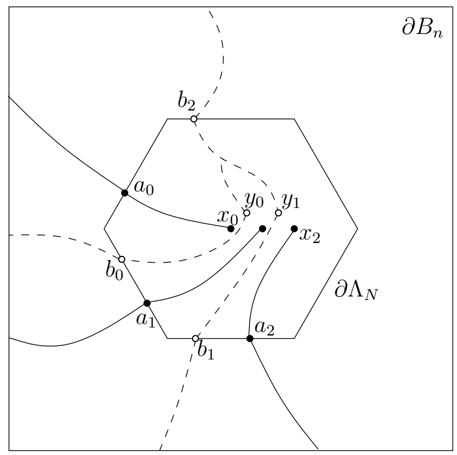

FIGURE 3.1. The vertices x_r (respectively, y_r) may be connected by open (respectively, open*) paths to the a_r (respectively, b_r) in such a way that the event $A_6(n)$ results.

As illustrated in Figure 3.1, for sufficiently large N and all vectors $\mathbf{a}$, $\mathbf{b}$ of length j, there exists a configuration $\omega_{\mathbf{a},\mathbf{b}}$ of primal edges of Λ_N such that $x_r \overset{\Lambda_N}{\longleftrightarrow} a_r$ for $0 \le r \le j-1$, and $y_r \overset{\Lambda_N}{\longleftrightarrow}{}^* b_r$ and $y_r \overset{\Lambda_N}{\longleftrightarrow}{}^* b_{j-1}$ for $0 \le r \le j-2$. That is, conditional on $A_\sigma(N,n)$, if $\omega_{\mathbf{a},\mathbf{b}}$ occurs then so does $A_k(n)$. Assume for the moment that $p_i > 0$ for all i. The configurations $\omega_{\mathbf{a},\mathbf{b}}$ may be chosen in such a way that

$$c' = c'(\mathbf{p}, N, k) := \min_{\mathbf{a},\mathbf{b}} \mathbb{P}_{\mathbf{p}}^{\triangle}(\omega_{\mathbf{a},\mathbf{b}} \mid A_\sigma(N,n))$$

satisfies $c' > 0$. The details of the construction of the $\omega_{\mathbf{a},\mathbf{b}}$ are slightly complex but follow standard lines and are omitted (similar arguments are used in [4, Sect. 8.2] and [17, Chap. 2]). It follows as required that

$$c'\mathbb{P}_{\mathbf{p}}^{\triangle}[A_\sigma(N,n)] \le \mathbb{P}_{\mathbf{p}}^{\triangle}[A_k(n)].$$

Whereas a naive construction of the $\omega_{\mathbf{a},\mathbf{b}}$ succeeds when $p_i > 0$ for all i, a minor variant of the argument is needed if $p_i = 0$ for some i. The details are elementary and are omitted.

The case $k = 1$ is similar but simpler. $\square$

3.3. **Proof of Proposition 3.3.** A significant step in the arguments of [11] is called the 'separation theorem' (see also [12, Thm 11]). This states roughly that, conditional on the arm event $A_\sigma(N,n)$, there is probability bounded away from 0 that arms with the required colours can be found whose endpoints on the exterior boundary of the annulus are separated from one another by a given distance or more. A formal statement of this appears at Theorem 3.5; the proof is rather technical and very similar to those of [11, 12] and is therefore omitted. It is followed in Section 3.4 by an application (Proposition 3.7) to the lattices $\mathbb{L}^m$ of which we make use here.

The proof of Proposition 3.3 relies on the following lemma, in which the measure $\mathbb{P}_{\mathbf{p}}$ is utilized within the star–triangle transformations comprising the map τ. Let $k \in \{1, 2, 4, 6, \dots\}$.

Lemma 3.4. *Let $\mathbb{L} = (V, E)$ be a mixed lattice, and let $\mathbb{P}_{\mathbf{p}}$ be a self-dual measure on $\Omega = \{0,1\}^E$. For $n/\sqrt{3} > k+2$ and $\tau \in \{S^+ \circ T^+, S^- \circ T^-\}$,*

$$\tau A_k^{\mathbb{L}}(n) \subseteq A_k^{\tau\mathbb{L}}(n-1).$$

The proof of the lemma is deferred to the end of this section. Let c and N_1 be as in Proposition 3.7. By making n applications of $\tau = S^+ \circ T^+$ to $\mathbb{L}^0$, we deduce that $\tau^n A_k^{\mathbb{L}^0}(2n) \subseteq A_k^{\mathbb{L}^n}(n)$. Therefore, for $n \geq N_1$,

$$\begin{aligned}
\mathbb{P}^{\square}_{(p_0,1-p_0)}[A_k(n)] &= \mathbb{P}^n_{\mathbf{p}}[A_k(n)] \\
&\geq \mathbb{P}^0_{\mathbf{p}}[A_k(2n)] \quad \text{by Lemma 3.4} \\
&= \mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(2n)] \\
&\geq c\mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)] \quad \text{by Proposition 3.7.}
\end{aligned}$$

This proves the first inequality of Proposition 3.3.

Fix $n \geq \max\{k\sqrt{3}, N_1\}$, and consider the event $A_k(n)$ on the lattice $\mathbb{L}^n$. If we apply n times the transformation $S^- \circ T^-$ to $\mathbb{L}^n$, we obtain via Lemma 3.4 applied to the event $A_k(2n)$ that:

$$\begin{aligned}
\mathbb{P}^{\square}_{(p_0,1-p_0)}[A_k(n)] &= \mathbb{P}^n_{\mathbf{p}}[A_k(n)] \\
&\leq c^{-1}\mathbb{P}^n_{\mathbf{p}}[A_k(2n)] \quad \text{by Proposition 3.7} \\
&\leq c^{-1}\mathbb{P}^0_{\mathbf{p}}[A_k(n)] \quad \text{by Lemma 3.4} \\
&= c^{-1}\mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)].
\end{aligned}$$

Proposition 3.3 is proved.

Proof of Lemma 3.4. Let $k \in \{1, 4, 6, \dots\}$, we shall consider the case $k = 2$ separately. Let $\tau \in \{S^+ \circ T^+, S^- \circ T^-\}$ and $\omega \in A_k^{\mathbb{L}}(n)$. Note that the points x_r, $r = 0, 1, \dots$, are invariant under τ.

It is explained in Section 2 (see also [6, Sect. 2]) that the image $\tau(\pi)$ of an ω-open path π contains a $\tau(\omega)$-open path of $\tau\mathbb{L}$ lying within distance 1 of π. Therefore, for $n/\sqrt{3} > 2r + 2$, if $C_{x_r}(\omega) \cap \partial B_n \neq \varnothing$, then $C_{x_r}(\tau(\omega)) \cap \partial B_{n-1} \neq \varnothing$. The proof when $k = 1$ is complete, and we assume now that $k \geq 4$. Let $j = k/2$ and $n/\sqrt{3} > k + 2$. By Proposition 2.1, $x_r \stackrel{B_{n-1},\tau(\omega)}{\not\longleftrightarrow} x_s$ for $0 \leq r < s \leq j - 1$, whence $\tau(\omega) \in A_k^{\tau\mathbb{L}}(n-1)$.

Finally, let $k = 2$. Let $\tau \in \{S^+ \circ T^+, S^- \circ T^-\}$ and $\omega \in A_2^{\mathbb{L}}(n)$. Let Γ (respectively, Γ^*) be an open primal (respectively open* dual) path starting at x_0 (respectively y_0), that intersects ∂B_n. Since x_0 and y_0 are unchanged under τ, they are contained, respectively, in $\tau(\Gamma)$ and $\tau(\Gamma^*)$. By the remarks in Section 2 concerning the operation of τ on open* dual paths, we conclude that $C_{x_0} \cap \partial B_{n-1} \neq \varnothing$ in $\tau\mathbb{L}$, and similarly $C^*_{y_0} \cap \partial B_{n-1} \neq \varnothing$ in $\tau\mathbb{L}^*$. The proof is complete. $\square$

3.4. **Separation theorem.** The so-called 'separation theorem' is a basic element in Kesten's work on scaling relations in two dimensions. It asserts roughly that, conditional on the occurrence of a given arm event, there is probability bounded from 0 that such arms may be found whose endpoints on the interior and exterior boundaries of the annulus are distant from one another. The separation theorem is useful since it permits the extensions of the arms using box-crossings.

Kesten proved his lemma in [11] for homogeneous site percolation models, while noting that it is valid more generally. The proof has been reworked in [12], also in the context of site percolation. The principal tool is the box-crossing property of the critical model. In this section, we state a general form of the separation theorem, for use in both the current paper and the forthcoming [5]. The proof follows closely that found in [11, 12], and is omitted.

Let $G = (V, E)$ be a connected planar graph, embedded in the plane in such a way that each edge is a straight line segment, and let $\mathbb{P}$ be a product measure on $\Omega = \{0, 1\}^E$. As usual we denote by G^* the dual graph of G, and more generally the superscript $*$ indicates quantities defined on the dual. We shall use the usual notation from percolation theory, [4], and we assume there exists a uniform upper bound $L < \infty$ on the lengths of edges of G and G^*, viewed as straight line segments of $\mathbb{R}^2$.

The hypothesis required for the separation theorem concerns a lower bound on the probabilities of open and open* box-crossings. Let $\omega \in \Omega$ and let R be a (non-square) rectangle of $\mathbb{R}^2$. A lattice-path π is said to *cross* R if π contains an arc (termed a *box-crossing*) that lies in the interior of R except for its two endpoints, which are required to lie, respectively, on the two shorter sides of R. Note that box-crossings lie in the longer direction. The rectangle R is said to *possess an open crossing* (respectively, *open* dual crossing*) if there exists an open path of G (respectively, open* path of G^*) crossing R, and we write $C(R)$ (respectively, $C^*(R)$) for the event that this occurs. Let $\mathcal{T}$ be the set of translations of $\mathbb{R}^2$, and $\tau \in \mathcal{T}$. Let $H_n = [0, 2n] \times [0, n]$ and $V_n = [0, n] \times [0, 2n]$, and let $n_0 = n_0(G) < \infty$ be minimal with the property that, for all τ and all $n \geq n_0$, τH_n and τV_n possess crossings in both G and G^*. Let

$$b(G, \mathbb{P}) = \inf\big\{\mathbb{P}(C(\tau H_n)), \mathbb{P}(C(\tau V_n)) : n \geq n_0,\ \tau \in \mathcal{T}\big\}, \tag{3.3}$$

$$b^*(G, \mathbb{P}) = \inf\big\{\mathbb{P}(C^*(\tau H_n)), \mathbb{P}(C^*(\tau V_n)) : n \geq n_0,\ \tau \in \mathcal{T}\big\}, \tag{3.4}$$

and

$$\beta = \beta(G, \mathbb{P}) = \min\{b, b^*\}. \tag{3.5}$$

The pair $(G, \mathbb{P})$ (respectively, $(G^*, \mathbb{P}^*)$) is said to have the *box-crossing property* if and only if $b(G, \mathbb{P}) > 0$ (respectively, $b^*(G, \mathbb{P}) > 0$). In [6], the box-crossing property is given in terms of boxes of arbitrary orientation and aspect-ratio. It is shown there that it suffices to consider only horizontal and vertical boxes. It is a consequence of the FKG inequality that the box-crossing property does not depend on the chosen aspect-ratio (so long as it is strictly greater than 1). By [6, Thm 1.3], $\mathbb{P}^{\triangle}_{\mathbf{p}}$ has the box-crossing property whenever $\mathbf{p} \in [0, 1)^3$ is self-dual.

Let $k \in \mathbb{N}$ and $\sigma \in \{0, 1\}^k$. Rather than working with the arm events $A_\sigma(N, n)$ of Section 1.3, we use instead the events $\bar{A}_\sigma(N, n)$ defined in the same way except that Λ_n is replaced throughout the definition by $B_n = [-n, n]^2$, and arms are required to comprise edges that intersect B_n. All constants in the following statements are permitted to depend on the colour sequence σ.

Let $\bar{\mathcal{A}}(N, n) = B_n \setminus (-N, N)^2$ be the annulus with interior boundary ∂B_N and exterior boundary ∂B_n. We shall consider open and open* crossings between the interior and exterior boundaries. We emphasize that the endpoints of these crossings are not required to be lattice-points.

For clarity, we concentrate first on the behaviour of crossings at their exterior endpoints. Let $\eta \in (0, 1)$. A primal (respectively, dual) *η-exterior-fence* is a set Γ of connected open (respectively, open*) paths comprising the union of:

(i) a crossing of $\bar{\mathcal{A}}(N, n)$ from its interior to its exterior boundary, with exterior endpoint denoted $\operatorname{ext}(\Gamma)$,

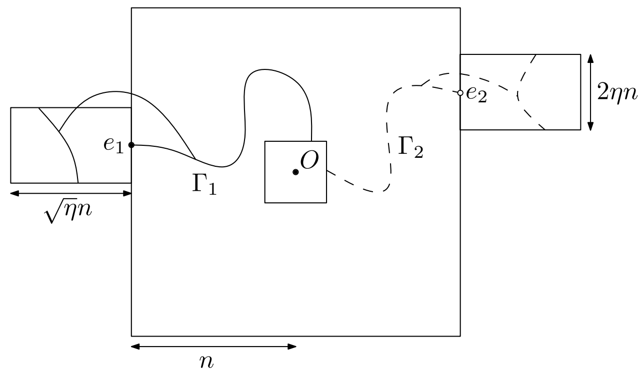

FIGURE 3.2. A primal η-exterior-fence Γ_1 with exterior endpoint e_1, and a dual η-exterior-fence Γ_2.

together with certain further paths which we describe thus under the assumption that $\mathrm{ext}(\Gamma) = (n, y)$ is on the right side of ∂B_n:

(ii) a vertical crossing of the box $[n, (1 + \sqrt{\eta})n] \times [y - \eta n, y + \eta n]$,
(iii) a connection between the above two crossings, contained in $\mathrm{ext}(\Gamma) + B_{\sqrt{\eta}n}$.

If $\mathrm{ext}(\Gamma)$ is on a different side of ∂B_n, the event of condition (ii) is replaced by an appropriately rotated and translated event. This definition is illustrated in Figure 3.2.

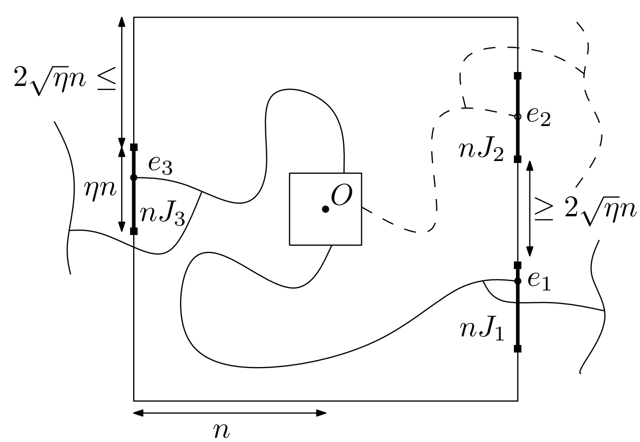

FIGURE 3.3. The event $A_\sigma^{\varnothing,J}(N, n)$ with $\sigma = (1, 0, 1)$ and η-landing-sequence J. Each crossing Γ_i is an η-exterior-fence with exterior endpoint $e_i \in nJ_i$.

One may similarly define an η-*interior*-fence by considering the behaviour of the crossing near its interior endpoint. We introduce also the concept of a primal (respectively, dual) η-*fence*; this is a union of an open (respectively, open*) crossing of $\bar{A}(N,n)$ together with further paths in the vicinities of both interior and exterior endpoints along the lines of the above definitions.

An η-*landing-sequence* is a sequence of closed sub-intervals $I = (I_i : i = 1,2,\dots,k)$ of ∂B_1, taken in anticlockwise order, such that each I_i has length η, and the minimal distance between any two intervals, and between any interval and a corner of B_1, is greater than $2\sqrt{\eta}$. We shall assume that

$$0 < k(\eta + 2\sqrt{\eta}) < 8, \tag{3.6}$$

so that η-landing-sequences exist.

Let η, η' satisfy (3.6), and let I (respectively, J) be an η-landing-sequence (respectively, η'-landing-sequence). Write $\bar{A}^{I,J}_\sigma(N,n)$ for the event that there exists a sequence of η-fences $(\Gamma_i : i = 1,2,\dots,k)$ in the annulus $B_n \setminus (-N,N)^2$, with colours prescribed by σ, such that, for all i, the interior (respectively, exterior) endpoint of Γ_i lies in NI_i (respectively, nJ_i). Let $\bar{A}^{I,\varnothing}_\sigma(N,n)$ (respectively, $\bar{A}^{\varnothing,J}_\sigma(N,n)$) be given similarly in terms of η-interior-fences (respectively, η'-exterior-fences). Note that

$$\bar{A}^{I,J}_\sigma(N,n) \subseteq \bar{A}^{\varnothing,J}_\sigma(N,n), \bar{A}^{I,\varnothing}_\sigma(N,n) \subseteq \bar{A}_\sigma(N,n). \tag{3.7}$$

These definitions are illustrated in Figure 3.3.

In the proof of the forthcoming Proposition 3.7 (and nowhere else), we shall make use of a piece of related notation introduced here. Let $k = 2j \geq 2$, and let η and I be as above. As explained in the proof of Lemma 3.2, the event $A_k(n)$ of Section 3.2 requires the existence of an alternating sequence of open and open* paths joining the set $\{x_0, y_0, x_1, y_1, \dots, x_{j-1}\}$ to the boundary ∂B_n. Let $A^I_k(n)$ be the sub-event in which the exterior endpoints of the open (respectively, open*) paths lie in $I_1, I_3, \dots, I_{k-1}$ (respectively, $I_2, I_4, \dots, I_k$), and in addition these exterior endpoints have associated paths as given in (ii)–(iii) of the above definition of an η-exterior-fence.

We now state the separation theorem. The proof is omitted, and may be constructed via careful readings of the appropriate sections of [11, 12]. There is a small complication arising from the fact that the endpoints of box-crossings are not necessarily vertices of the lattice, and this is controlled using the uniform upper bound L on the lengths of embeddings of edges.

Theorem 3.5 (Separation theorem)**.** *Let $k \in \mathbb{N}$, and $\sigma \in \{0,1\}^k$. For $\beta_0 > 0$, $M \in \mathbb{N}$, and $\eta_0 > 0$, there exist constants $c > 0$ and $n_1 \in \mathbb{N}$ such that: for any pair $(G, \mathbb{P})$ with $\beta(G,\mathbb{P}) > \beta_0$ and $n_0(G) \leq M$, for all $\eta, \eta' > \eta_0$ satisfying* (3.6)*, all η-landing-sequences I and η'-landing-sequences J, and all $N \geq n_1$ and $n \geq 2N$, we have*

$$\mathbb{P}\left[\bar{A}_\sigma^{I,J}(N,n)\right] \geq c\mathbb{P}\left[\bar{A}_\sigma(N,n)\right].$$

Amongst the consequences of Theorem 3.5 is the following. The proof (also omitted) is essentially that of [12, Prop. 12], and it uses the extension of paths by judiciously positioned box-crossings.

Corollary 3.6. *Suppose that $\beta = \beta(G,\mathbb{P}) > 0$. For $k \in \mathbb{N}$ and $\sigma \in \{0,1\}^k$, there exists $c = c(\beta,\sigma) > 0$ and $N_0 \in \mathbb{N}$ such that, for all $N \geq N_0$ and $n \geq 2N$,*

$$\mathbb{P}[\bar{A}_\sigma(N,2n)] \geq c\mathbb{P}[\bar{A}_\sigma(N,n)].$$

The proof of Proposition 3.3 makes use of the following application of Corollary 3.6 to the pairs $(\mathbb{L}^m, \mathbb{P}^m_{\mathbf{p}})$.

Proposition 3.7. *For $k \in \{1,2,4,6,\dots\}$ and a self-dual triplet $\mathbf{p} \in [0,1)^3$ with $p_0 > 0$, there exist $c > 0$ and $N_1 \in \mathbb{N}$ such that, for $m \geq 0$ and $n \geq N_1$,*

$$\mathbb{P}^m_{\mathbf{p}}[A_k(2n)] \geq c\mathbb{P}^m_{\mathbf{p}}[A_k(n)].$$

Proof. The box-crossing property has been studied in [6] in the context of hybrids of $\mathbb{Z}^2/\mathbb{T}$ or $\mathbb{Z}^2/\mathbb{H}$ type. The arguments of [6] may be adapted as follows to obtain that, for given self-dual $\mathbf{p}$ with $p_0 > 0$, the pairs $(\mathbb{L}^m, \mathbb{P}^m_{\mathbf{p}})$ satisfy a *uniform* box-crossing property in the sense that there $\beta_0 > 0$ such that

$$\beta(\mathbb{L}^m, \mathbb{P}^m_{\mathbf{p}}) > \beta_0, \quad m \geq 0. \tag{3.8}$$

Write $B_{M,N} = [0,M] \times [0,N]$, and denote by $C_{\mathrm{h}}(B_{M,N})$ (respectively, $C_{\mathrm{v}}(B_{M,N})$) the event that there exists a horizontal (respectively, vertical) open crossing of $B_{M,N}$ (with a similar notation C^*_{h}, C^*_{v} for dual crossings). Since every translate of $B_{M,3N}$ contains a rectangle with dimensions $M \times N$ lying in either the square or triangular part of $\mathbb{L}^m$,

$$\mathbb{P}^m_{\mathbf{p}}[C_{\mathrm{h}}(\tau B_{M,3N})] \geq \min\bigl\{\mathbb{P}^{\triangle}_{\mathbf{p}}[C_{\mathrm{h}}(B_{M,N})], \mathbb{P}^{\square}_{(p_0,1-p_0)}[C_{\mathrm{h}}(B_{M,N})]\bigr\},$$

for all $\tau \in \mathcal{T}$. The dual model lives on a mixed square/hexagonal lattice with parameter $1 - \mathbf{p}$, and the same inequality holds with C_{h} replaced by C^*_{h}. By [6, Thm 1.3], there exists $b_1 = b_1(\mathbf{p}) > 0$ such that

$$\mathbb{P}^m_{\mathbf{p}}[C_{\mathrm{h}}(\tau B_{M,3N})], \mathbb{P}^m_{\mathbf{p}}[C^*_{\mathrm{h}}(\tau B_{M,3N})] \geq b_1, \quad m \geq 0,\ M,N \geq 1,\ \tau \in \mathcal{T}. \tag{3.9}$$

Adapting the proof of [6, Prop. 3.8], we obtain that

$$\mathbb{P}_{\mathbf{p}}^m[C_{\mathrm{v}}(\tau B_{3N,N})] \ge \mathbb{P}_{\mathbf{p}}^{\triangle}[C_{\mathrm{v}}(B_{N,2N})], \quad m \ge 0,\ N \ge 1,\ \tau \in \mathcal{T}.$$

The same inequality holds with C_{v} replaced by C_{v}^*, as above, and therefore there exists $b_2 = b_2(\mathbf{p}) > 0$ such that

$$\mathbb{P}_{\mathbf{p}}^m[C_{\mathrm{v}}(\tau B_{3N,N})], \mathbb{P}_{\mathbf{p}}^m[C_{\mathrm{v}}^*(\tau B_{3N,N})] \ge b_2, \quad m \ge 0,\ N \ge 1,\ \tau \in \mathcal{T}. \tag{3.10}$$

Inequalities (3.9)–(3.10) imply as in the proof of [6, Prop. 3.1] that (3.8) holds for some $n_0 < \infty$ and $\beta_0 > 0$, and we choose them accordingly.

Let σ be an alternating colour sequence of length k (we set $\sigma = \{1\}$ when $k = 1$), and denote $\bar{A}_\sigma(N, n)$ by $\bar{A}_k(N, n)$.

Let η satisfy (3.6) and let I be an η-landing sequence. Let $c = c_2$ and n_1 be as in Theorem 3.5. By Corollary 3.6, there exists $c_0 = c_0(\beta_0, k) > 0$ and $N_0 \ge n_1$ such that

$$\mathbb{P}_{\mathbf{p}}^m[\bar{A}_k(N, 2n)] \ge c_0 \mathbb{P}_{\mathbf{p}}^m[\bar{A}_k(N, n)], \quad m \ge 0,\ n \ge 2N \ge 2N_0. \tag{3.11}$$

Therefore,

$$\mathbb{P}_{\mathbf{p}}^m[A_k(n)] \le \mathbb{P}_{\mathbf{p}}^m[\bar{A}_k(N, n)] \le c_0^{-1} \mathbb{P}_{\mathbf{p}}^m[\bar{A}_k(N, 2n)]. \tag{3.12}$$

By an elementary consideration of paths of $\mathbb{L}^m$, there exist $N_1 \ge 2N_0$ and $c_1 = c_1(\mathbf{p}, N_1)$ such that

$$\mathbb{P}_{\mathbf{p}}^m[A_k^I(\tfrac{1}{2}N_1)] \ge c_1, \quad m \ge 0. \tag{3.13}$$

By Theorem 3.5 and (3.7),

$$\mathbb{P}_{\mathbf{p}}^m[\bar{A}_k^{I,\varnothing}(N_1, 2n)] \ge c_2 \mathbb{P}_{\mathbf{p}}^m[\bar{A}_k(N_1, 2n)], \quad m \ge 0,\ n \ge N_1. \tag{3.14}$$

Furthermore, by the uniform box-crossing property as in [12, Prop. 12], there exists $c_3 = c_3(\mathbf{p}, \eta) > 0$ such that

$$\begin{aligned} \mathbb{P}_{\mathbf{p}}^m[A_k(2n)] &\ge c_3 \mathbb{P}_{\mathbf{p}}^m[A_k^I(\tfrac{1}{2}N_1)] \mathbb{P}_{\mathbf{p}}^m[\bar{A}_k^{I,\varnothing}(N_1, 2n)] \\ &\ge c_1 c_2 c_3 \mathbb{P}_{\mathbf{p}}^m[\bar{A}_k(N_1, 2n)] \quad \text{by (3.13)–(3.14)}. \end{aligned}$$

The claim follows by (3.12) with $c = c_0 c_1 c_2 c_3$. □

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Statistical Laboratory, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WB, UK

E-mail address: {g.r.grimmett, i.manolescu}@statslab.cam.ac.uk

URL: http://www.statslab.cam.ac.uk/∼grg/