RANDOM WALKS IN RANDOM LABYRINTHS

G. R. Grimmett, M. V. Menshikov, S. E. Volkov

Abstract. A random labyrinth is a disordered environment of scatterers on a lattice. A beam of light travels through the medium, and is reflected off the scatterers. The set of illuminated vertices is studied, under the assumption that there is a positive density of points, called 'normal points', at which the light behaves in the manner of a simple symmetric random walk. The ensuing 'random walk in a labyrinth' is found to be recurrent in two dimensions, and also non-localised under certain extra assumptions on the underlying probability distribution. The walk is shown to be transient (with strictly positive probability) in three and higher dimensions, subject to the assumption that the density of 'non-trivial' scatterers is sufficiently small. The principal arguments used in deriving such results originate in percolation theory. In addition, we utilise the relationship between random walks and electrical networks, namely that a random walk is recurrent if and only if a certain electrical network has infinite resistance.

1. Introduction

Suppose that we distribute obstacles within a Euclidean space $\mathbb{R}^d$, and then we shine light through the space. If the light is reflected by the obstacles, then its trajectory can be tortuous. When the placements of the obstacles are disordered (or 'random') then it seems difficult to derive rigorous results concerning the path followed by the light. The problem is difficult even in restricted versions, such as when the light is constrained to a lattice, and when the obstacles have a limited number of possible shapes. The combination of the disordered medium and the (conditionally) deterministic flow leads apparently to mathematical complications of substantial difficulty (but great appeal).

Let us recall one version of such a question which has gained a certain notoriety. Let $0 \le p \le 1$. At each vertex of the square lattice is placed, with probability p, a plane two-sided mirror. Each mirror is placed in one of two possible orientations, termed 'NW' and 'NE', each such possibility having probability $\frac{1}{2}$. The effect of a NW or NE mirror is illustrated in Figure 1. Now we place a candle at the origin, and light rays emanate outwards from the origin in the four axial directions. This light is deflected by the mirrors, and we ask whether or not the light remains 'localised', in the sense that it illuminates only finitely many vertices.

Writing $\eta(p)$ for the probability that the light is *not* localised (i.e., that it illuminates infinitely many vertices), we may wish to determine for which values of p it is the case the $\eta(p)$ is *strictly* positive. It is elementary that $\eta(0) = 1$, and it is

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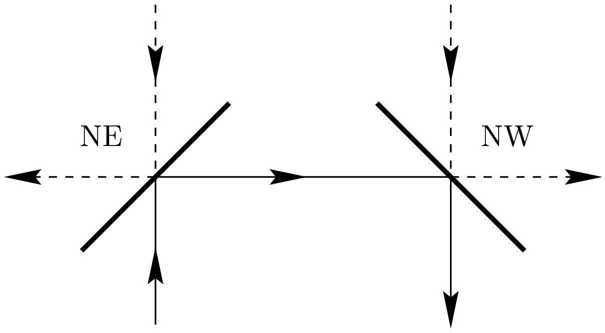

Fig. 1. NE and NW mirrors reflect rays of light thus.

known that $\eta(1) = 0$ (the latter statement is in the folklore of the problem). We believe that it is unknown whether or not η is monotone in p, and whether $\eta(p) = 0$ for p ($\neq 1$) sufficiently close to 1. We remark that there is short proof that $\eta(1) = 0$ using a percolation argument. (This method is referred to by Grimmett (1989, p. 240), and is closely related to the proof of the forthcoming Theorem 4(b); see also Bunimovitch and Troubetzkoy (1992).)

In this paper we consider a variant of the above problem of mirrors. In this variant, there is a positive density of vertices at which the light behaves as a random walk. This extra randomness makes possible an analysis of certain properties of the model, including partial results on non-localisation and recurrence/transience. The ensuing process is a type of 'random walk in a random environment', but the environment is sufficiently rigid that we prefer to use the term 'labyrinth' rather than 'environment'.

In advance of giving a formal definition of the labyrinths of this paper, we present a brief discussion of an important two-dimensional example. Let $\mathbf{p} = (p_\varnothing, p_\tau, p_{\mathrm{NW}}, p_{\mathrm{NE}})$ be a vector of non-negative numbers with sum 1, and write S for the set $\{\varnothing, \tau, \mathrm{NW}, \mathrm{NE}\}$ of possible vertex-states. The vertices of the square lattice $\mathbb{L}^2$ are (independently) allocated states, so that the vertex x is in state s with probability p_s (for each $s \in S$). We write Z_x for the (random) state of x, and we call the family $Z = (Z_x : x \in \mathbb{Z}^2)$ a *random labyrinth*. A vertex x is called:

(a) a *normal point* (or *normal vertex*), if $Z_x = \varnothing$,
(b) a *tunnel*, if $Z_x = \tau$,
(c) a *north-west (NW) mirror*, if $Z_x = \mathrm{NW}$,
(d) a *north-east (NE) mirror*, if $Z_x = \mathrm{NE}$.

The physical meanings of these designations are as follows. At a normal point, a ray of light behaves as a symmetric random walk (that is, on arriving at the point, it leaves in a random direction, each possibility having probability $\frac{1}{4}$, and being independent of the labyrinth and all previous steps). At a tunnel, a ray of light is undeflected (that is, it leaves the vertex in the same direction as it was travelling just before it arrived). At a NW or NE mirror, the ray is deflected according to the type of mirror (see Figure 1).

We now construct a (random) sequence $X = (X_0, X_1, \dots)$ of vertices as in the following informal description. Let x be a normal point. We set $X_0 = x$, the starting point of the sequence. Next we choose X_1 uniformly at random from the set of neighbours of x. Subsequent values of the X_j are those visited by a light ray obeying the above rules in the labyrinth Z.

Let the labyrinth Z be sampled according to the above rules. We call the point

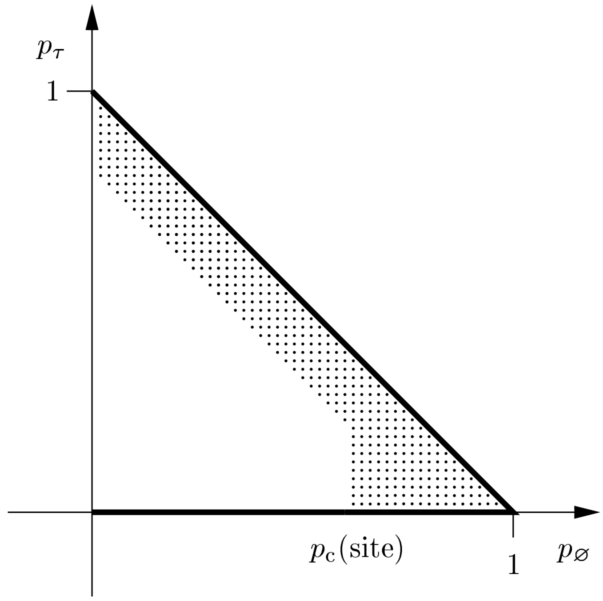

Fig. 2. Suppose that the only scatterers in $\mathbb{L}^2$ are tunnels and NW/NE mirrors. The dotted region and the heavy lines of the figure indicate the part of $(p_\varnothing, p_\tau)$ space for which non-localisation is proved. The labyrinth is a.s. localised when $p_\varnothing = p_\tau = 0$.

x *Z-recurrent* if x is normal and in addition there exists a.s. N (> 0) such that $X_N = x$. We call the labyrinth Z *recurrent* if all normal points are Z-recurrent, otherwise we call the labyrinth *transient*. We shall prove that, if $p_\varnothing > 0$, then the above two-dimensional random labyrinth Z is a.s. recurrent.

Remembering that irreducible Markov chains on *finite* state spaces are necessarily recurrent, we turn our attention to a question of 'localisation'. We call the random labyrinth Z *non-localised* if there exists a starting point x such that the sequence X visits (almost surely) infinitely many vertices, and *localised* otherwise.

We shall show that the above two-dimensional labyrinth Z is a.s. non-localised if one of the following holds:

(a) $p_\varnothing$ exceeds the critical probability of site percolation on $\mathbb{L}^2$,

(b) $p_\varnothing > 0$ and $p_\tau = 0$,

(c) $p_\varnothing > 0$ and $p_\varnothing + p_\tau > 1 - A$, for some prescribed $A > 0$.

See Figure 2 for an illustration of these conditions.

We turn next to higher-dimensional labyrinths. Our main conclusion will be that such labyrinths are a.s. transient, so long as the density $p_\varnothing$ of normal points satisfies $p_\varnothing > 0$, and additionally the density of scatterers (other than tunnels) is sufficiently small. We present formal statements of our results in two and more dimensions in Section 2, after giving a proper definition of a labyrinth. This definition will allow scatterers of more general geometry than the simple mirrors referred to above.

In Section 3, we present a proof of recurrence in two-dimensional labyrinths. A comparison with percolation appears in Section 4, and this is exploited in Sections 5 and 6 to obtain partial results concerning non-localisation in two dimensions, and transience in higher dimensions.

2. Random labyrinths

Consider the d-dimensional cubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ with vertex-set $\mathbb{Z}^d$ and with edges joining all pairs of vertices which are (Euclidean) distance 1 apart. An edge joining two vertices u, v is written as $\langle u, v\rangle$. The origin is denoted as 0. For any set A of vertices, we define the *surface* ∂A to be the set of x ($\in A$) at least one of whose neighbours does not lie in A. Throughout, we assume that $d \geq 2$.

Let $I = \{u_1, u_2, \dots, u_d\}$ be the set of unit vectors, and let $I^{\pm} = \{-1, +1\} \times I$. Members of $I^{\pm}$ are written as $\pm u_j$.

We define a *scatterer* to be a mapping $\sigma : I^{\pm} \to I^{\pm}$ satisfying $\sigma(-\sigma(u)) = -u$ for all $u \in I^{\pm}$. The set of all such scatterers is denoted as Σ. The identity scatterer (i.e., the identity map on $I^{\pm}$) is called a *tunnel* and denoted by τ.

The physical interpretation of a scatterer is as follows. If light impinges on a scatterer σ, heading in the direction u ($\in I^{\pm}$) then it departs the scatterer in direction $\sigma(u)$. The requirement that $\sigma(-\sigma(u)) = -u$ is in response to the reversibility of reflections. For any given $\sigma \in \Sigma$ and $u \in I^{\pm}$ exactly one of three possibilities occurs for a light ray incident at the scatterer in direction u, namely:

(a) $\sigma(u) = -u$, the light is reflected back on itself,
(b) $\sigma(u) = u$, the light passes through undeflected,
(c) $\sigma(u) \neq \pm u$, the light is deflected through $90°$.

For each σ, there will be an even number of u ($\in I^{\pm}$) belonging to each category.

Next we introduce randomness. Let $p_\varnothing$, p_τ be non-negative reals satisfying $p_\varnothing + p_\tau \leq 1$, and let π be a probability mass function on $\Sigma \backslash \{\tau\}$. let $Z = (Z_x : x \in \mathbb{Z}^d)$ be independent random variables taking values in $\Sigma \cup \{\varnothing\}$ having common distribution given by

$$\mathbb{P}(Z_0 = \alpha) = \begin{cases} p_\varnothing & \text{if } \alpha = \varnothing, \\ p_\tau & \text{if } \alpha = \tau, \\ (1 - p_\varnothing - p_\tau)\pi(\sigma) & \text{if } \alpha = \sigma \in \Sigma\backslash\{\tau\}. \end{cases} \tag{2.1}$$

The family Z is termed a *labyrinth*. We call a vertex x a *tunnel* if $Z_x = \tau$, and a *normal point* if $Z_x = \varnothing$. (Throughout, we use $\mathbb{P}$ to denote the probability measure associated with the labyrinth.)

We now construct a random walk $X = (X_0, X_1, \dots)$ within the labyrinth Z. Let x be a normal point, and let $X_0 = x$. We choose X_1 at random from the set of neighbours of x in $\mathbb{L}^d$, each of its $2d$ neighbours having equal probability. Having defined vertices $X_0, X_1, \dots, X_n$, we define X_{n+1} by:

(a) if X_n is a normal point, then X_{n+1} is chosen uniformly from the neighbour-set of X_n, independently of all earlier choices and of the scatterers at other points of the labyrinth,
(b) if X_n is occupied by the scatterer σ ($\in \Sigma$) then $X_{n+1} - X_n = \sigma(X_n - X_{n-1})$.

Such a sequence $X = (X_0, X_1, \dots)$ conforms to the scatterers which it encounters, and behaves as a symmetric random walk at normal points. It is called a 'random walk in the labyrinth Z'.

As described in the last section, we call a normal point x ($\in \mathbb{Z}^d$)

$$\left.\begin{matrix} Z\text{-}recurrent \\ Z\text{-}transient \end{matrix}\right\} \text{ if } P_x^Z\big(X_N = x \text{ for some } N \geq 1\big) \begin{cases} = 1 \\ < 1, \end{cases} \tag{2.2}$$

where P_x^Z denotes the law of X conditional on the labyrinth Z (and $X_0 = x$ as before). We call Z a *recurrent* labyrinth if all normal points are Z-recurrent; otherwise we call Z a *transient* labyrinth. The event $\{Z \text{ is recurrent}\}$ is invariant with respect to translations of the underlying lattice $\mathbb{L}^d$, and in addition $\mathbb{P}$ is a product measure. Therefore, Z is either $\mathbb{P}$-a.s. recurrent or $\mathbb{P}$-a.s. transient.

Lemma 1. *Assume $p_\varnothing > 0$. Then Z is a.s. recurrent if and only if*

$$\mathbb{P}\big(0 \text{ is } Z\text{-recurrent} \mid 0 \text{ is normal}\big) = 1.$$

Proof. If $\mathbb{P}\big(0 \text{ is } Z\text{-recurrent} \mid 0 \text{ is normal}\big) = 1$, then

$$\mathbb{P}(Z \text{ is transient}) \le \sum_{x\in\mathbb{Z}^d} \mathbb{P}(x \text{ is normal and } Z\text{-transient}) = 0.$$

Conversely, if $\mathbb{P}\big(0 \text{ is } Z\text{-recurrent} \mid 0 \text{ is normal}\big) < 1$, then

$$\mathbb{P}(Z \text{ is transient}) \ge \mathbb{P}(0 \text{ is normal and } Z\text{-transient}) > 0,$$

and the claim follows by the above zero–one observation. □

In advance of a discussion of localisation, we introduce some further notation. A $\mathbb{Z}^d$*-path* is a sequence $x_0, e_0, x_1, e_1, \dots$ of alternating vertices x_i and distinct edges e_j such that $e_j = \langle x_j, x_{j+1}\rangle$ for all j. If the path has a final vertex x_n, then it is said to have length n and to join x_0 to x_n. If the path is infinite, then it is said to join x_0 to ∞. We allow a $\mathbb{Z}^d$-path to visit vertices more than once. A Z*-path* is a $\mathbb{Z}^d$-path $x_0, e_0, x_1, e_1, \dots$ with the property that, for all j,

$$x_{j+1} - x_j = Z_{x_j}(x_j - x_{j-1}) \quad \text{whenever } Z_{x_j} \ne \varnothing,$$

which is to say that the path conforms to the scatterers at all non-normal points.

Let N be the set of normal points. We define an equivalence relation $\leftrightarrow$ on N by $x \leftrightarrow y$ if and only if there exists a Z-path with endpoints x and y. We denote by C_x the equivalence class of $(N, \leftrightarrow)$ containing the normal point x, and by $\mathcal{C}$ the set of equivalence classes of $(N, \leftrightarrow)$.

For any Z-path with vertex sequence $V = (v_0, v_1, \dots)$, let $N(V) = (v_{i_1}, v_{i_2}, \dots)$ be the (ordered) sequence of normal points lying in V. Now consider a random walk $X = (X_0, X_1, \dots)$ in Z, beginning at the normal point x. It is not difficult to see that the normal subsequence $N(X)$ constitutes an irreducible Markov chain on the equivalence class C_x. Also, X is recurrent if and only if $N(X)$ is a recurrent Markov chain. Certainly $N(X)$ is recurrent if $|C_x| < \infty$, but it may generally be the case that $|C_x| = \infty$ for some x.

Let x be a normal point of the labyrinth Z. A random walk in the labyrinth, starting at x, gives rise to an irreducible Markov chain on the equivalence class C_x. We call x Z*-localised* if $|C_x| < \infty$, and Z*-non-localised* otherwise. We have, by elementary Markov chain theory, that

$$P_x^Z\big(X \text{ visits infinitely many vertices}\big) = \begin{cases} 0 & \text{if } |C_x| < \infty, \\ 1 & \text{if } |C_x| = \infty. \end{cases} \tag{2.3}$$

We call the labyrinth *localised* if all its normal points are Z-localised, and otherwise we call it *non-localised*. The proof of the following lemma is very similar to that of Lemma 1, and is omitted.

Lemma 2. *Assume $p_\varnothing > 0$. Then Z is a.s. non-localised if and only if*

$$\mathbb{P}\big(0 \text{ is } Z\text{-localised} \mid 0 \text{ is normal}\big) = 1.$$

Next we state our main results, beginning with the case $d = 2$.

Theorem 3. *Suppose that $d = 2$ and $p_\varnothing > 0$. Then the two-dimensional random labyrinth Z is a.s. recurrent.*

Theorem 4. *Suppose that $d = 2$. There exists a constant A (> 0) such that Z is a.s. non-localised whenever one of the following conditions holds:*
(a) $p_\varnothing > p_c(\text{site})$,
(b) $p_\varnothing > 0$ *and* $p_\sigma = 0$ *for* $\sigma \in \Sigma \setminus \{\text{NW}, \text{NE}\}$,
(c) $p_\varnothing > 0$ *and* $p_\varnothing + p_\tau > 1 - A$.

Here, $p_c(\text{site})$ denotes the critical probability of site percolation on $\mathbb{L}^2$. We do not know whether or not the inequality $p_\varnothing > 0$ is sufficient for Z to be a.s. non-localised. The scatterers NW and NE in (b) are those introduced in Section 1 and sketched in Figure 1. More precisely, NW is the scatterer σ with $\sigma(u_1) = -u_2$, $\sigma(-u_2) = u_1$, where u_1 and u_2 are unit vectors in the increasing axial directions. (A similar definition holds for NE.)

In Figure 2 is sketched a picture of the part of $(p_\varnothing, p_\tau)$ space for which non-localisation is proved, in the case when the only scatterers are tunnels, NW mirrors, and NE mirrors (cf. condition (b) above). For a fixed probability measure π on $\{\text{NW}, \text{NE}\}$, let

$$\text{NL} = \Big\{(p_\varnothing, p_\tau) : Z \text{ is a.s. non-localised}\Big\}.$$

It is easy to see that $(p'_\varnothing, p'_\tau) \in \text{NL}$ if $(p_\varnothing, p_\tau) \in \text{NL}$ and $p'_\varnothing \geq p_\varnothing$, $p'_\tau \leq p_\tau$, $p'_\varnothing + p'_\tau \geq p_\varnothing + p_\tau$. This is so since the labyrinth with such parameters $(p'_\varnothing, p'_\tau)$ may be obtained from that with parameters $(p_\varnothing, p_\tau)$ by replacing each NW/NE mirror by a normal point with probability α, and each tunnel by a normal point with probability β, where α and β satisfy

$$p'_\varnothing = p_\varnothing + (1 - p_\varnothing - p_\tau)\alpha + p_\tau\beta, \quad p'_\tau = p_\tau(1 - \beta),$$

which is to say that

$$\alpha = \frac{p'_\varnothing + p'_\tau - p_\varnothing - p_\tau}{1 - p_\varnothing - p_\tau}, \quad \beta = 1 - \frac{p'_\tau}{p_\tau}.$$

Such conversions to normality can only make the labyrinth 'more non-localised'.

We now turn to the question of transience in three and more dimensions.

Theorem 5. *Assume that $d \geq 3$. There exists a constant $A = A(d) > 0$ such that Z is a.s. transient when $p_\varnothing > 0$ and $p_\varnothing + p_\tau > 1 - A$.*

It is reasonable to ask whether transience is valid under weaker assumptions on the pair $p_\varnothing, p_\tau$.

Our basic strategy in proving recurrence and transience is to relate the labyrinth to a certain electrical network, and to estimate the effective resistance of this network. When $d = 2$, this effective resistance will be infinite, which will in turn imply recurrence. When $d \geq 3$, we shall show that the effective resistance is at most a bounded multiple of that of the infinite open cluster in a certain percolation model; this is known to be a.s. finite (see Grimmett, Kesten, Zhang (1993)), and therefore the labyrinth is a.s. transient. Such comparisons constitute a fairly standard method for understanding certain properties of random walks and, more generally, time-reversible Markov chains. For further details, see Nash-Williams (1959), Lyons (1983), and Doyle and Snell (1984).

3. Recurrence in two dimensions

Assume that $d = 2$ and $p_\varnothing > 0$. We shall concentrate on a random walk beginning at the origin, and therefore we assume also that the origin is a normal point.

Let $e = \langle u, v\rangle$ be an edge of $\mathbb{L}^2$. We call e a *normal* edge if there exists a Z-path $\eta(e) = (x_0, e_0, x_1, \dots, x_n)$, with the properties that:

(3.1) x_0 (resp. x_n) is either a normal point or is such that $Z_{x_0}(x_0 - x_1) = x_1 - x_0$ (resp. $Z_{x_n}(x_n - x_{n-1}) = x_{n-1} - x_n$), and

(3.2) no x_j with $0 < j < n$ is a normal point.

If e is normal, we write $L(e)$ for the number of edges in $\eta(e)$, and define

$$\rho(e) = \begin{cases} L(e)^{-1} & \text{if } x_0 \text{ and } x_n \text{ are distinct normal points,} \\ \infty & \text{otherwise.} \end{cases} \tag{3.3}$$

If e is not normal, we set $\rho(e) = \infty$.

The $\rho(e)$ are used to define an electrical network $E(\mathbb{L}^2, \rho)$ in the following way. Consider the square lattice $\mathbb{L}^2$, and think of each edge e as an electrical resistor having resistance $\rho(e)$. Let $R(Z)$ be the effective resistance of this network between the origin and infinity. (More rigorously, let $R_n(Z)$ be the resistance between 0 and a composite vertex obtained by identifying all the vertices of $\mathbb{L}^2$ at distance n from the origin. Then $R(Z) = \lim_{n\to\infty} R_n(Z)$.)

Lemma 6. *It is the case that*

$$\mathbb{P}\big(R(Z) = \infty \mid 0 \text{ is normal}\big) = 1. \tag{3.4}$$

Once this lemma is proved, Theorem 3 follows easily, as follows. Let X be a random walk in the labyrinth with $X_0 = 0$, and consider the Markov chain $N(X)$ on the equivalence class C_0. There is a corresponding electrical network $E(C_0)$ with vertex set C_0 and with resistors arranged as follows. Between any two normal points n_1, n_2 in C_0 there is placed a unit resistor if and only if there is a Z-path η joining n_1 to n_2 of which no other vertex is normal. Such a Z-path η has some length, L say, and the required unit resistance may be obtained by replacing each edge of η by a resistor of resistance L^{-1}. The effective resistance of $E(C_0)$, between 0 and infinity, is at least that of $E(\mathbb{L}^2, \rho)$, since $E(C_0)$ may be obtained from $E(\mathbb{L}^2, \rho)$ by the device of separating Z-paths whenever they intersect at a non-normal vertex. However, by Lemma 6, $E(\mathbb{L}^2, \rho)$ a.s. has infinite resistance, and therefore so has $E(C_0)$. This in turn implies (see Doyle and Snell (1984)) that the Markov chain $N(X)$ is recurrent. Theorem 3 now follows by Lemma 1.

Next we prove Lemma 6.

Proof of Lemma 6. For $x = (x_1, x_2) \in \mathbb{Z}^2$ we define $\|x\| = |x_1| + |x_2|$. Let $B(n) = \{x \in \mathbb{Z}^2 : \|x\| \le n\}$ and $\partial B(n) = B(n) \backslash B(n-1)$. We define the 'edge-boundary' $\Delta_{\mathrm{e}} B(n)$ to be the set of edges $e = \langle x, y\rangle$ for which $x \in \partial B(n)$ and $y \in \partial B(n+1)$. We claim that there exists a positive constant c and a random $M = M(Z)$ (≥ 1) such that

$$\rho(e) \ge \frac{c}{\log n} \quad \text{for all} \quad e \in \Delta_{\mathrm{e}} B(n) \quad \text{and} \quad n \ge M. \tag{3.5}$$

To show this, we argue as follows. Let $e = \langle x, y\rangle$ be a normal edge, and let $\eta = (x_0, e_0, x_1, \dots, e_{L-1}, x_L)$ be the unique Z-path containing e and satisfying (3.1) and (3.2); we may assume that x_0 and x_L are normal, since otherwise $\rho(e) = \infty$. Assume that $e = e_K$ for $0 \le K < L$, so that e 'splits' η into the two sub-paths $\eta_1 = (x_0, e_0, \dots, x_K)$ and $\eta_2 = (x_{K+1}, e_{K+1}, \dots, x_L)$. These sub-paths may be constructed in the following sequential manner. Imagine a ray of light proceeding along e from x to y. If y is normal, we cease the construction. If y is not normal, we allow the light to proceed according to the scatterer at y. We iterate this procedure until the light meets a normal point, obtaining thereby the subpath of η lying on 'one side' of the edge e.

Write λ_1 for the number of edges (excepting e) which are traversed by the path thus constructed. We now repeat the construction but in the other direction, beginning with a light ray which proceeds along e in the direction from y to x. In this way we obtain a path of length λ_2.

Each of the two paths obtained above may contain self-intersections, but it is easily seen that no vertex may appear more than twice in the union of the two paths.

Now, if $\lambda_1 \ge k$, then the first k vertices in the corresponding path η_1 must be non-normal, implying that the first $\lfloor \frac{1}{2}k \rfloor$ distinct vertices encountered in the construction are non-normal. Therefore,

$$\begin{aligned} \mathbb{P}\big(L(e) > 2k,\ e \text{ is normal}\big) &= \mathbb{P}\big(\lambda_1 + \lambda_2 \ge 2k,\ e \text{ is normal}\big) \\ &\le 2\mathbb{P}\big(\lambda_1 \ge k,\ e \text{ is normal}\big) \\ &\le 2(1 - p_\varnothing)^{\frac{1}{2}(k-1)} \end{aligned} \tag{3.6}$$

for $k \ge 0$. It follows by (3.3) that, for $c > 0$ and all $n \ge 2$,

$$\begin{aligned} \mathbb{P}\left(\rho(e) < \frac{c}{\log n} \text{ for some } e \in \Delta_e B(n)\right) &\le |\Delta_e B(n)| \mathbb{P}\left(L(e) > \frac{\log n}{c},\ e \text{ is normal}\right) \\ &\le \beta |\Delta_e B(n)| n^{-\alpha(c)} \end{aligned}$$

where $\alpha(c) = -(4c)^{-1} \log(1 - p_\varnothing)$ and $\beta = \beta(c, p_\varnothing)$ is a finite constant. We choose c such that $\alpha(c) > \frac{5}{2}$, whence

$$\sum_{n \ge 2} \mathbb{P}\left(\rho(e) < \frac{c}{\log n} \text{ for some } e \in \Delta_e B(n)\right) < \infty.$$

Statement (3.5) now follows by the Borel–Cantelli lemma.

The fact that $R(Z) = \infty$ a.s. is a fairly immediate consequence of (3.5), using the usual argument which follows. From the electrical network $E(\mathbb{L}^2, \rho)$ we construct another network with no larger resistance. This we do by, for each $n \ge 1$, identifying all vertices contained in $\partial B(n)$. In this new network there are $|\Delta_e B(n)|$ parallel connections between $\partial B(n)$ and $\partial B(n+1)$, each of which has (for $n \ge M = M(Z)$) a resistance exceeding $c/\log n$ (by (3.5)). The effective resistance from the origin to infinity is therefore at least

$$\sum_{n=M}^{\infty} \frac{c}{|\Delta_e B(n)| \log n} \ge \sum_{n=M}^{\infty} \frac{c}{12n \log n} = \infty,$$

and the proof is complete. □

4. A comparison with a percolation model

In this section we compare the random labyrinth Z with a certain percolation process. It will follow that there exists an infinite equivalence class of normal points whenever the percolation process has an infinite cluster.

The first application of this comparison is to the case of two dimensions. We shall deduce that the labyrinth is a.s. non-localised in two dimensions, for certain parameter values. See Section 5 for more details.

The second application is to the case of three or more dimensions. It will emerge that a random walk on the above infinite equivalence class is transient whenever a random walk on the infinite percolation cluster is transient. The latter fact is known to hold a.s. when $d \geq 3$ (see Grimmett, Kesten, Zhang (1993)), and therefore certain higher-dimensional labyrinths are a.s. transient; see Section 6 for a full proof.

There is more than one way of performing the required comparison, and we choose here to proceed by a block argument. Let $d \geq 2$, and let V be a positive integer to be chosen later. Consider the box $S = [0, V-1]^d$. For $x, y \in \partial S$, we define $M(x, y)$ to be the event that there exists a Z-path joining x to y, using vertices in S only, and having length not exceeding $2(d+1)V$.

We declare the box S to be *occupied* if the two following conditions hold:
(a) S contains only normal points and tunnels, and
(b) $M(x, y)$ occurs for all $x, y \in \partial S$.
More generally, for $k = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d$, we declare k to be *occupied* if the box $B_k = kV + S$ satisfies conditions (a) and (b). Since these conditions involve the states of vertices lying inside the box only, the set of occupied vertices of $\mathbb{Z}^d$ constitutes a site percolation process. This percolation process is supercritical if $\mathbb{P}(S \text{ is occupied})$ is sufficiently large, and we therefore seek a lower bound for this probability.

Theorem 7. *Let $d \geq 2$ and $p_\varnothing > 0$. There exists an integer V and a strictly positive constant $A = A(d)$ such that*

$$\mathbb{P}(S \text{ is occupied}) > p_c(\text{site}) \quad \textit{if } p_\varnothing + p_\tau > 1 - A \tag{4.1}$$

where $p_c(\text{site})$ is the critical probability of site percolation on $\mathbb{L}^d$.

We shall see applications of this block comparison in the next sections, to the cases $d = 2$ and $d \geq 3$ respectively.

Proof. First, the event $\text{NT} = \{S \text{ contains only normal points and tunnels}\}$ has probability

$$\mathbb{P}(\text{NT}) = (p_\varnothing + p_\tau)^{V^d}. \tag{4.2}$$

Suppose now that NT occurs, and turn to condition (b) above. We claim that

$$\mathbb{P}\big(M(x, y) \text{ for all } x, y \in \partial S \mid \text{NT}\big) \geq 1 - 2dV^{d-1}(1 - \widetilde{p}_\varnothing^{\,d-1})^{V-1} \tag{4.3}$$

where

$$\widetilde{p}_\varnothing = \frac{p_\varnothing}{p_\varnothing + p_\tau}, \tag{4.4}$$

and we prove this as follows.

Consider a plane face of ∂S containing the origin, say the face

$$F = \big\{(m_1, m_2, \dots, m_{d-1}, 0) : 0 \le m_j < V \text{ for } 1 \le j < d\big\}.$$

For $m = (m_1, m_2, \dots, m_{d-1}, 0) \ne (0, 0, \dots, 0)$ and an integer k satisfying $1 \le k < V$, we let $T_k(m)$ be the set of points

$$\begin{aligned} T_k(m) = \big\{&(0, \dots, 0, k), (m_1, 0, \dots, 0, k), \\ &(m_1, m_2, 0, \dots, 0, k), \dots, (m_1, m_2, \dots, m_{d-1}, k)\big\}. \end{aligned}$$

Note that $|T_k(m)| = d - z$ $(\le d)$ where $z = |\{i : 1 \le i < d \text{ and } m_i = 0\}|$. Let $U_k(m)$ be the event that all vertices in $T_k(m)$ are normal. Since $T_k(m) \cap T_\ell(m) = \varnothing$ when $k \ne \ell$, the events $U_k(m)$, $1 \le k < V$, are independent. Furthermore $U_k(m) \subseteq M'(0, m)$, where $M'(x, y)$ is the event that vertices $x, y \in \partial S$ are joined by a Z-path of S having length not exceeding $(d+1)V$. Therefore

$$\begin{aligned} \mathbb{P}\big(M'(0,m) \mid \mathrm{NT}\big) &\ge \mathbb{P}\Bigg(\bigcup_{k=1}^{V-1} U_k(m) \,\Bigg|\, \mathrm{NT}\Bigg) \\ &= 1 - \prod_{k=1}^{V-1} \Big\{1 - \mathbb{P}\big(U_k(m) \mid \mathrm{NT}\big)\Big\} \\ &\ge 1 - (1 - \widetilde{p}_\varnothing^{\,d-1})^{V-1} \end{aligned}$$

where

$$\widetilde{p}_\varnothing = \mathbb{P}\big(Z_0 = \varnothing \mid Z_0 \in \{\varnothing, \tau\}\big) = \frac{p_\varnothing}{p_\varnothing + p_\tau}. \tag{4.5}$$

It follows that

$$\mathbb{P}\big(M'(0,m) \text{ for all } m \in F \mid \mathrm{NT}\big) \ge 1 - V^{d-1}(1 - \widetilde{p}_\varnothing^{\,d-1})^{V-1}. \tag{4.6}$$

Inequality (4.6) is valid for any face F of ∂S containing the origin, of which there are exactly d. It is similarly valid with $M'(0,m)$ replaced by $M'(u,m)$ where $u = (V-1, V-1, \dots, V-1)$, and with F replaced by any face of ∂S containing u (of which there are exactly d). If all the corresponding events $M'(0,m)$, $M'(u,m')$ occur for all appropriate m, m', then so does $M(x,y)$ for all $x, y \in \partial S$. Inequality (4.3) follows, as promised earlier.

Combining (4.2) and (4.3), we obtain that

$$\mathbb{P}(S \text{ is occupied}) \ge \big\{1 - 2dV^{d-1}(1 - \widetilde{p}_\varnothing^{\,d-1})^{V-1}\big\}(p_\varnothing + p_\tau)^{V^d}. \tag{4.7}$$

Let $p_{\mathrm{c}} = p_{\mathrm{c}}(\text{site})$ be the critical probability of site percolation on $\mathbb{L}^d$, and assume that $p_\varnothing > 0$ (so that $\widetilde{p}_\varnothing > 0$). Pick V large enough that

$$2dV^{d-1}(1 - \widetilde{p}_\varnothing^{\,d-1})^{V-1} < \tfrac{1}{2}(1 - p_{\mathrm{c}}),$$

and then pick A small enough that

$$(1 - A)^{V^d} > \tfrac{1}{2}(1 + p_{\mathrm{c}}).$$

It follows from (4.7) that

$$\mathbb{P}(S \text{ is occupied}) > \left(\frac{1 + p_{\mathrm{c}}}{2}\right)^2 > p_{\mathrm{c}} \quad \text{if} \quad p_\varnothing + p_\tau > 1 - A. \tag{4.8}$$

The proof is complete. $\square$

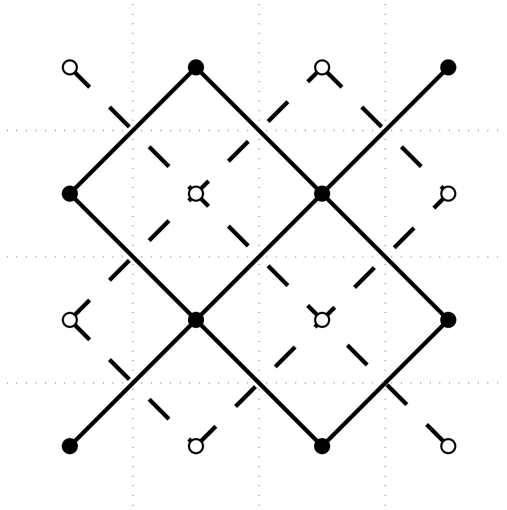

Fig. 3. The heavy lines form the lattice $\mathbb{L}_A^2$, and the dashed lines form the lattice $\mathbb{L}_B^2$. The central point is the origin of $\mathbb{L}^2$.

5. Non-localisation in two dimensions

Next we prove Theorem 4, and we begin with part (a). If $p_\varnothing > p_c(\text{site})$, then there exists a.s. a unique infinite cluster I of normal points having density at least $\theta(p_\varnothing)$ (> 0), where θ is the percolation probability function; see Grimmett (1989). If a vertex x lies in I, then $I \subseteq C_x$, implying that the labyrinth is lon-localised.

Moving to part (b), we assume next that the only types of point are normal points, NW mirrors, and NE mirrors, occurring with respective probabilities $p_\varnothing$, p_{NW}, p_{NE}, where $p_\varnothing > 0$. From $\mathbb{L}^2$ we construct two interlaced copies of $\mathbb{L}^2$, as follows. Let

$$A = \Big\{(m+\tfrac{1}{2}, n+\tfrac{1}{2}) : m+n \text{ is even}\Big\},\ B = \Big\{(m+\tfrac{1}{2}, n+\tfrac{1}{2}) : m+n \text{ is odd}\Big\}.$$

On the respective sets A and B we define the relation $(m+\frac{1}{2}, n+\frac{1}{2}) \sim (r+\frac{1}{2}, s+\frac{1}{2})$ if and only if $|m-r| = 1$ and $|n-s| = 1$, obtaining thereby two copies of $\mathbb{L}^2$ denoted respectively as $\mathbb{L}_A^2$ and $\mathbb{L}_B^2$. See Figure 3.

We now use the labyrinth Z to define bond percolation processes on $\mathbb{L}_A^2$ and $\mathbb{L}_B^2$. Here are the rules for $\mathbb{L}_A^2$, exactly similar rules are valid for $\mathbb{L}_B^2$. We declare the edge of $\mathbb{L}_A^2$ joining $(m-\frac{1}{2}, n-\frac{1}{2})$ to $(m+\frac{1}{2}, n+\frac{1}{2})$ to be *open* if there is a NE mirror at (m,n); similarly we declare the edge joining $(m-\frac{1}{2}, n+\frac{1}{2})$ to $(m+\frac{1}{2}, n-\frac{1}{2})$ to be *open* if there is a NW mirror at (m,n). Edges which are not open are designated *closed*. This defines percolation models on $\mathbb{L}_A^2$ and $\mathbb{L}_B^2$ in which north-easterly edges (resp. north-westerly edges) are open with probability p_{NE} (resp. p_{NW}). These processes are subcritical since $p_{\text{NE}} + p_{\text{NW}} = 1 - p_\varnothing < 1$. Therefore, there exists a.s. no infinite open path in either $\mathbb{L}_A^2$ or $\mathbb{L}_B^2$, and we assume henceforth that no such infinite open path exists.

Let $N(A)$ (resp. $N(B)$) be the number of open circuits in $\mathbb{L}_A^2$ (resp. $\mathbb{L}_B^2$) which contain the origin in their interiors. Since the above percolation processes are subcritical, there exists $\alpha = \alpha(p_{\text{NW}}, p_{\text{NE}}) > 0$ such that

$$\mathbb{P}\Big(x \text{ lies in an open cluster of } \mathbb{L}_A^2 \text{ of size at least } n\Big) \le e^{-\alpha n} \quad \text{for all } n, \tag{5.1}$$

where x is any given vertex of $\mathbb{L}_A^2$. The same fact is valid for $\mathbb{L}_B^2$. (This follows from standard percolation arguments; see [3, 7].) We claim that

$$\mathbb{P}\big(0 \text{ is normal, and } N(A) = N(B) = 0\big) > 0, \tag{5.2}$$

and we prove this as follows. Let $\Lambda(k) = [-k, k]^2$, and let $N_k(A)$ (resp. $N_k(B)$) be the number of circuits contributing to $N(A)$ (resp. $N(B)$) which contain only points lying strictly outside $\Lambda(k)$. If $N_k(A) \geq 1$ then there exists some vertex $(m + \frac{1}{2}, \frac{1}{2})$ of $\mathbb{L}_A^2$, with $m \geq k$, which belongs to an open circuit of length exceeding m. Using (5.1),

$$\mathbb{P}\big(N_k(A) \geq 1\big) \leq \sum_{m=k}^{\infty} e^{-\alpha m} < \tfrac{1}{3}$$

for sufficiently large k. We pick k accordingly, whence

$$\mathbb{P}\big(N_k(A) + N_k(B) \geq 1\big) \leq \tfrac{2}{3}.$$

Now, if $N_k(A) = N_k(B) = 0$, and in addition all points of $\mathbb{L}^2$ inside $\Lambda(k)$ are normal, then $N(A) = N(B) = 0$. These last events have strictly positive probabilities, and (5.2) follows.

Let J be the event that there exists a normal point $x = x(Z)$ which lies in the interior of no open circuit of either $\mathbb{L}_A^2$ or $\mathbb{L}_B^2$. Since J is invariant with respect to translations of $\mathbb{L}^2$, and since $\mathbb{P}$ is product measure, we have that $\mathbb{P}(J)$ equals either 0 or 1. Using (5.2), we deduce that $\mathbb{P}(J) = 1$. Therefore we may find a.s. some such vertex $x = x(Z)$. We claim that x is Z-non-localised, which will imply that the labyrinth if a.s. non-localised as claimed.

We may generate the equivalence class C_x in the following way. We allow light to leave x along the four axial directions. When a light ray hits a mirror, it is reflected; when a ray hits a normal point, it causes light to depart the point along each of the other three axial directions. Following this physical picture, let F be the set of 'frontier mirrors', i.e., the set of mirrors only one side of which are illuminated. Assume that F is non-empty, say F contains a mirror at some point (m, n). Now this mirror must correspond to an open edge e in either $\mathbb{L}_A^2$ and $\mathbb{L}_B^2$ (see Figure 3 again), and we may assume without loss of generality that this open edge e is in $\mathbb{L}_A^2$. We write $e = \langle u, v\rangle$ where $u, v \in A$, and we assume that $v = u + (1, 1)$; an exactly similar argument holds otherwise. There are exactly three other edges of $\mathbb{L}_A^2$ which are incident to u (resp. v), and we claim that one of these is open. To see this, argue as follows. If none is open, then

$u + (-\frac{1}{2}, \frac{1}{2})$ either is normal or has a NE mirror,
$u + (-\frac{1}{2}, -\frac{1}{2})$ either is normal or has a NW mirror,
$u + (\frac{1}{2}, -\frac{1}{2})$ either is normal or has a NE mirror.

See Figure 4 for a diagram of the eight possible combinations. By inspection, each such combination contradicts the fact that $e = \langle u, v\rangle$ corresponds to a frontier mirror.

Therefore, u is incident to some other open edge f of $\mathbb{L}_A^2$, other than e. By a further consideration of each of $2^3 - 1$ possibilities, we may deduce that there exists such an edge f lying in F. Iterating the argument, we find that e lies in either an open circuit or an infinite open path of F lying in $\mathbb{L}_A^2$. Since there exists (by assumption) no infinite open path, this proves that f lies in an open circuit of F

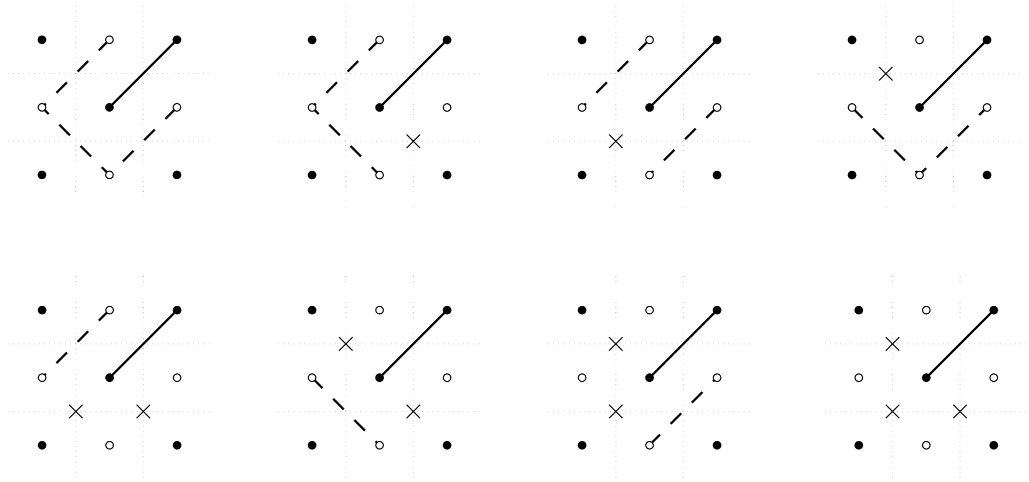

Fig. 4. The solid line in each picture is the edge $e = \langle u, v\rangle$, and the central vertex is u. If all three of the other edges of $\mathbb{L}^2_A$ incident with the vertex u are closed in $\mathbb{L}^2_A$, then there are eight possibilities for the corresponding edges of $\mathbb{L}^2_B$. The dashed lines indicate open edges of $\mathbb{L}^2_B$, and the crosses mark normal points of $\mathbb{L}^2$. In every picture, light incident with one side of the mirror at e will illuminate the other side also.

in $\mathbb{L}^2_A$. By taking the union over all $e \in F$, we obtain that F is a union of open circuits of $\mathbb{L}^2_A$ and $\mathbb{L}^2_B$. Each such circuit has an interior and an exterior, and x lies (by assumption, above) in every exterior. There are various ways of deducing that x is Z-non-localised, and here is such a way.

Assume that x is Z-localised. Amongst the set of vertices $\{x + (n, 0) : n \geq 1\}$, let y be the rightmost vertex at which there lies a frontier mirror. By the above argument, y lies in some open circuit G of F (belonging to either $\mathbb{L}^2_A$ or $\mathbb{L}^2_B$), whose exterior contains x. Since y is rightmost, we have that $y' = y + (-1, 0)$ is illuminated by light originating at x, and that light traverses the edge $\langle y', y\rangle$. Similarly, light does not traverse the edge $\langle y, y''\rangle$, where $y'' = y + (1, 0)$. Therefore, the point $y + (\frac{1}{2}, 0)$ of $\mathbb{R}^2$ lies in the interior of G, which contradicts the fact that y is rightmost. This completes the proof for part (b).

Finally we turn to part (c). Assume that $p_\varnothing > 0$ and $p_\varnothing + p_\mathcal{T} > 1 - A$ where A is as in Theorem 7. Using that theorem, and the vocabulary of Section 4, there exists a.s. an infinite cluster I of occupied sites in the renormalised lattice obtained by replacing each box $B_k = kV + S$ by a 'block site' at $k = (k_1, k_2) \in \mathbb{Z}^2$. For each $k \in I$, the box B_k must contain some normal point $n(k)$, and we choose such a $n(k)$ according to some rule. The equivalence class $C_{n(k)}$ contains all normal points lying in boxes B_ℓ with $\ell \in I$. Therefore $|C_{n(k)}| = \infty$, implying that the labyrinth is $\mathbb{P}$-a.s. non-localised.

6. Transience in three and more dimensions

Finally we prove Theorem 5, using the comparison with percolation which was established in Section 4. Let $d \geq 3$, $p_\varnothing > 0$, and $p_\varnothing + p_\mathcal{T} > 1 - A$ where A is given in Theorem 7. For each occupied block $B_k = kV + S$ (where V is given as in Theorem 7), we may find a normal point $n(k)$ within the block; in general there will be many of these, and we pick one according to some arbitrary rule. If k and ℓ are neighbouring vertices of $\mathbb{Z}^d$ such that B_k and B_ℓ are occupied, then, by definition of the 'occupied' state, there exists a Z-path $\eta(k,\ell)$ joining $n(k)$ and $n(\ell)$ and having length at most $4(d+1)V$. We may assume that each $\eta(k,\ell)$ visits any vertex at most once. We now use such paths to construct an electrical network on $\mathbb{L}^d$ as follows. The nodes of the network are the set of all k for which B_k is occupied, and there is a resistor between k and ℓ if and only if k and ℓ are adjacent in $\mathbb{L}^d$, and in addition B_k, B_ℓ are both occupied. Such a resistor is allocated resistance $4(d+1)V$. On the event that S is occupied, we write $R(\text{block})$ for the effective resistance of the network between the vertex $n(0)$ and the points at infinity (this resistance is defined as a limit in the usual way).

Writing $\{S \overset{\text{occ}}{\leftrightarrow} \infty\}$ for the event that S belongs to an infinite path of occupied blocks of $\mathbb{L}^d$, we have, by the above comparison and Theorem 7, that

$$\mathbb{P}\big(R(\text{block}) < \infty \mid S \overset{\text{occ}}{\leftrightarrow} \infty\big) \geq P\big(R(\text{perc}) < \infty \mid 0 \leftrightarrow \infty\big), \tag{6.1}$$

where $R(\text{perc})$ is the effective resistance between 0 and ∞ in the infinite open cluster of a supercritical site percolation process on $\mathbb{L}^d$ (and P is the corresponding probability measure). Using results of Grimmett, Kesten, and Zhang (1993), the latter conditional probability equals 1. (Actually, the a.s. finiteness of $R(\text{perc})$ on the event $\{0 \leftrightarrow \infty\}$ was proved for bond percolation only, but the proof is equally valid for site percolation. Indeed it relies on two key inequalities which were derived for site percolation rather than bond percolation by Grimmett and Marstrand (1990).) Therefore

$$\mathbb{P}\big(R(\text{block}) < \infty \mid S \overset{\text{occ}}{\leftrightarrow} \infty\big) = 1. \tag{6.2}$$

Assume that S belongs to an infinite path of occupied blocks. Let $E(C_{n(0)})$ be the electrical network on the equivalence class $C_{n(0)}$, defined as was $E(C_0)$ after the statement of Lemma 6, but with 0 replaced by $n(0)$ and with $d \geq 3$. We write $R(\mathcal{E})$ for the effective resistance from 0 to ∞ in any appropriate electrical network $\mathcal{E}$. We shall alter $E(C_{n(0)})$ in certain ways, and at each stage the resistance $R(\cdot)$ will not decrease. First we remove all vertices, and incident resistors, not lying in the union of the $\eta(k,\ell)$ as k and ℓ vary over the index set of the cluster of occupied blocks containing S. The ensuing electrical network is infinite and contains the normal point $n(0)$. We may construct it from the lattice $\mathbb{L}^d$ by deleting edges and vertices not in the $\eta(k,\ell)$ specified above, and by allocating to each edge e the resistance $\rho(e) = L(e)^{-1}$, as in (3.3). The paths $\eta(k,\ell)$ may have vertices and edges in common. If two such paths intersect at a normal vertex other than an endvertex, then we 'separate' the paths at this point, thereby not decreasing the effective resistance. Finally, each edge e has some resistance $L(e)^{-1} \leq 1$, and we replace this edge by a unit resistor. These actions result in a network whose resistance from $n(0)$ to infinity is no greater than $R(\text{block})$. Using (6.2), we obtain that

$$\mathbb{P}\Big(R\big(E(C_{n(0)})\big) < \infty \mid S \overset{\text{occ}}{\leftrightarrow} \infty\Big) \geq \mathbb{P}\big(R(\text{block}) < \infty \mid S \overset{\text{occ}}{\leftrightarrow} \infty\big) = 1.$$

Since the block lattice a.s. contains an infinite occupied cluster of blocks, we find that there exists a.s. some normal vertex x such that

$$P_x^Z(X_N = x \text{ for some } N \geq 1) < 1,$$

where P_x^Z is the law of the random walk X (starting at x) conditional on the labyrinth Z. That is to say, Z is a.s. transient.

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G. R. G., Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, UK

M. V. M. and S. E. V., Laboratory of Large Random Systems, Department of Probability, Faculty of Mechanics and Mathematics, Moscow State University, 119899 Moscow, Russia