INEQUALITIES AND ENTANGLEMENTS FOR PERCOLATION AND RANDOM-CLUSTER MODELS

GEOFFREY R. GRIMMETT

ABSTRACT. We discuss inequalities and applications for percolation and random-cluster models. The relevant areas of methodology concern the following two types of inequality: inequalities involving the probability of a general increasing event, and certain differential inequalities involving the percolation probability. We summarise three areas of application of such inequalities, namely strict inequality between the bond and site critical percolation probabilities of a general graph, the general study of entanglements in percolation, and strict inequalities for critical points of disordered random-cluster models.

1. Introduction

Harry Kesten's achievements across probability theory continue to be enormously influential and stimulating, and nowhere more so than in the study of spatial random processes. The results reported in this paper have been inspired in part by Harry's beautiful work on percolation.

Inequalities are central to the mathematics of disordered physical systems such as percolation and random-cluster models. They occur in several different ways, some of which are discussed here.

The methodological uses of inequalities include applications of the FKG and BK inequalities; these inequalities are now well understood and appreciated (see [6]). Less well known is an inequality used in [9, 11] in order to study exponential decay in random-cluster models. We present this inequality in Section 3.1, together with an application to percolation entanglements in Section 4.

Our second 'methodological' inequality is more a frame of mind than a theorem, and concerns the problem of proving that enhancements of certain processes cause *strict* changes in the values of the critical point. We present a very brief account of the relevant methods of [2] in Section 3.2. This method will be applied in Section 5 to obtain a theorem concerning strict inequalities between critical points of 'disordered' (or 'quenched') random-cluster models.

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Some of the results of this paper have appeared or will appear elsewhere, and therefore no proofs are included here, although references are listed. The results of Section 5 concerning disordered random-cluster models are however new, and proofs are included in that section.

2. Percolation and Random-Cluster Models

Let $\mathbb{L}^d$ denote the d-dimensional cubic lattice having vertex set $\mathbb{Z}^d$ and edge set $\mathbb{E}^d$, where $d \geq 2$. We consider bond percolation on $\mathbb{L}^d$. The appropriate sample space is $\Omega = \{0,1\}^{\mathbb{E}^d}$, and the probability measure is the product measure P_p with density p, where $0 \leq p \leq 1$. As usual, we call an edge e *open* in the configuration ω ($\in \Omega$) if $\omega(e) = 1$, and we call e *closed* otherwise.

A path in $\mathbb{L}^d$ is called *open* if and only if all its edges are open. For $A, B \subseteq \mathbb{Z}^d$, we write $A \leftrightarrow B$ if there exists an open path having one endpoint in A and the other in B. We write $A \leftrightarrow \infty$ if there exists some vertex in A which is the endpoint of an infinite open path. For $x \in \mathbb{Z}^d$, we write $C_x = \{y \in \mathbb{Z}^d : x \leftrightarrow y\}$ for the *open cluster* at x. The origin of $\mathbb{L}^d$ is denoted as 0, and we abbreviate C_0 to C.

We shall be particularly interested in the existence (or not) of infinite open clusters. The principal objects of study in percolation theory are the *percolation probability*

$$\theta(p) = P_p(|C| = \infty) = P_p(0 \leftrightarrow \infty), \tag{2.1}$$

together with the associated *critical probability*

$$p_c = \sup\{p : \theta(p) = 0\}. \tag{2.2}$$

See [6, 15] for detailed accounts of the percolation model.

Site percolation is a variant of the above model in which the vertices rather than the edges of $\mathbb{L}^d$ are designated either open or closed. The 'site' percolation probability is defined as in (2.1), with the difference that a path is called *open* if and only if all its *vertices* are open.

The random-cluster model of this paper will be defined in a slightly more general way than was percolation. Let $\mathbf{p} = (p_e : e \in \mathbb{E}^d)$ be a vector of numbers from the interval $[0,1]$, and let $q \geq 1$. For a finite box Λ in $\mathbb{L}^d$, we write $\mathbb{E}_\Lambda$ for the set of edges induced by Λ. For $\xi \in \Omega = \{0,1\}^{\mathbb{E}^d}$, we write Ω_Λ^ξ for the set of all configurations $\omega \in \Omega$ satisfying $\omega(e) = \xi(e)$ for all $e \notin \mathbb{E}_\Lambda$. The random-cluster measure $\phi_{\Lambda,\mathbf{p},q}^\xi$ on Ω_Λ^ξ is given by

$$\phi_{\Lambda,\mathbf{p},q}^\xi(\omega) = \frac{1}{Z_{\Lambda,\mathbf{p},q}^\xi} \left\{ \prod_{e \in \mathbb{E}_\Lambda} p_e^{\omega(e)} (1-p_e)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)}, \quad \omega \in \Omega_\Lambda^\xi, \tag{2.3}$$

where $Z^{\xi}_{\Lambda,\mathbf{p},q}$ is the appropriate normalising constant, and $k(\omega,\Lambda)$ is the number of open clusters of ω which intersect Λ. When $\xi = 0$ (respectively $\xi = 1$), this is called the 'free' (respectively 'wired') measure. For the purposes of this paper, it suffices to consider the wired measure, and we abbreviate henceforth $\phi^1_{\Lambda,\mathbf{p},q}$ to $\phi_{\Lambda,\mathbf{p},q}$. For a general guide to such random-cluster measures, see [8] and the references therein.

For $A \subseteq \mathbb{Z}^d$, the *surface* ∂A is the subset of A containing all vertices which have a neighbour in $\mathbb{L}^d$ not lying in A. We write $\Lambda_k = [-k,k]^d$ for the box of $\mathbb{L}^d$ having side-length $2k$.

The following facts are standard (see [8]):

(a) the limit measure $\phi_{\mathbf{p},q} = \lim_{\Lambda\to\mathbb{Z}^d} \phi_{\Lambda,\mathbf{p},q}$ exists in the sense of weak convergence,

(b) for any finite subset A of $\mathbb{Z}^d$,

$$\phi_{\Lambda,\mathbf{p},q}(A \leftrightarrow \partial\Lambda) \to \phi_{\mathbf{p},q}(A \leftrightarrow \infty) \quad \text{as } \Lambda \uparrow \mathbb{Z}^d,$$

(c) $\phi_{\Lambda,\mathbf{p},q}$ and $\phi_{\mathbf{p},q}$ satisfy the FKG inequality.

The random-cluster percolation probability is given by

$$\theta(\mathbf{p},q) = \phi_{\mathbf{p},q}(0 \leftrightarrow \infty). \tag{2.4}$$

For reasons which will become clear in Section 5, we shall work not with $\theta(\mathbf{p},q)$ but with the function $\psi(\mathbf{p},q)$ defined by

$$\psi(\mathbf{p},q) = \phi_{\mathbf{p},q}(I) \tag{2.5}$$

where I is the event that there exists at least one infinite open cluster.

We note that the random-cluster model with parameters $\mathbf{p}$, q reduces to the above bond percolation model when $q = 1$ and $\mathbf{p} = p$, the vector all of whose entries equal p. It is a standard fact concerning percolation that

$$\theta(p,1) = 0 \text{ if and only if } \psi(p,1) = 0.$$

Therefore, the percolation critical probability satisfies

$$p_c = \sup\{p : \psi(p,1) = 0\}.$$

3. Inequalities

3.1. An inequality for increasing events

There is a partial order on Ω given by $\omega \le \omega'$ if and only if $\omega(e) \le \omega'(e)$ for all $e \in \mathbb{E}^d$. A random variable X is called *increasing* if $X(\omega) \le X(\omega')$ whenever $\omega \le \omega'$. An event A is called *increasing* if its indicator function 1_A is increasing.

For any $\omega \in \Omega$ and any increasing event A, we define the 'distance' $F_A(\omega)$ from ω to A by

$$F_A(\omega) = \inf\left\{\sum_e \bigl(\omega'(e) - \omega(e)\bigr) : \omega' \ge \omega,\ \omega' \in A\right\}. \tag{3.1}$$

That is to say, $F_A(\omega)$ is the minimal number of extra edges which must be designated 'open' in order for A to occur.

Let E be a finite set of edges of $\mathbb{L}^d$, and let A be an event defined in terms of the states of edges belonging to E. Let

$$N(\omega) = \sum_{e \in E} \omega(e),$$

the total number of open edges of E in the configuration ω. The following proposition follows by Russo's formula and the FKG inequality, on noting that $F_A 1_A = 0$ and that $N + F_A$ is an increasing random variable.

Proposition 3.1. *Let $0 < p < 1$. For any non-empty increasing cylinder event A,*

$$\frac{d}{dp}\{\log P_p(A)\} \ge \frac{P_p(F_A)}{p(1-p)}. \tag{3.2}$$

[We write $\mu(X)$ for the mean of the random variable X under the probability measure μ.]

Proposition 3.1 relates the gradient of $\log P_p(A)$ to the mean of F_A. A different type of inequality is needed in order to bound this mean value below. In a way similar to the 'sprinkling' argument of [1], one may obtain the following.

Proposition 3.2. *For all p_1, p_2 satisfying $0 < p_1 < p_2 < 1$, there exist strictly positive numbers $a = a(p_1, p_2)$ and $b = b(p_1, p_2)$ such that, for any increasing cylinder event A,*

$$P_{p_1}(F_A) \ge -b - a \log P_{p_2}(A). \tag{3.3}$$

These two propositions do not appear to be sufficient by themselves in applications, and it is useful in practice to have recourse to the following additional general proposition.

Proposition 3.3. *Let $A, B_1, B_2, \dots, B_m$ be increasing cylinder events such that $A \subseteq B_1 \cap B_2 \cap \cdots \cap B_m$ and such that the B_i are defined on disjoint sets of edges. Then*

$$F_A \geq \sum_{i=1}^{m} F_{B_i}. \tag{3.4}$$

Similar propositions are valid in the more general context of random-cluster models, and their proofs may be found in [9]. They may be applied to study the decay rates of the connectivity functions of subcritical random-cluster models. They also have applications to percolation, and such an application to entanglements in percolation is described in Section 4.

3.2. Multiparameter processes

The following general situation occurs frequently. One encounters some random process having two (or more) real-valued parameters p, s, say. This random process has a phase transition, in the sense that some 'macroscopic function' $\theta = \theta(p, s)$ satisfies

$$\theta(p,s) \begin{cases} = 0 & \text{if } \phi(p,s) < 0 \\ > 0 & \text{if } \phi(p,s) > 0, \end{cases}$$

for some smooth function ϕ. The set of pairs (p, s) satisfying $\phi(p, s) = 0$ is sometimes called the 'critical surface' of the process. In many situations, one may be able to prove the existence of such a function ϕ, but its detailed properties, such as continuity or strict monotonicity, can be difficult to ascertain.

Here are two examples of questions which may be formulated in this way. First, one may ask whether or not there exists a critical probability $p_{\mathrm{c}}^{\mathrm{ent}}$ for the existence of an infinite entanglement in bond percolation, and in addition whether or not $p_{\mathrm{c}}^{\mathrm{ent}}$ differs from p_{c}. (See Section 4.) Secondly, if some of the strengths of interactions of a disordered ferromagnetic Ising or Potts model are increased, does the critical temperature necessarily change? We discuss this latter question further in Section 5.

A useful method for approaching and sometimes answering such questions has been described in [2]. In a broad class of situations including the two examples above, one may reformulate the question in the manner of the first paragraph of this section. One may then find a sequence $\theta_n(p, s)$, $n \geq 1$, of non-decreasing real-analytic functions satisfying $\theta_n(p, s) \to \theta(p, s)$, and in addition such that

$$\alpha \frac{\partial \theta_n}{\partial p} \leq \frac{\partial \theta_n}{\partial s} \leq \alpha^{-1} \frac{\partial \theta_n}{\partial p} \tag{3.5}$$

for some continuous function $\alpha = \alpha(p, s)$ which is strictly positive and finite on the interior of the parameter space. Such differential inequalities may be used to gain information about the gradient vector of θ_n, and this in turn implies certain properties of continuity and strict monotonicity for a natural parametrization of the critical surface of the process.

This approach has recently yielded, amongst other results, a solution to the problem of proving strict inequality between the critical probabilities of bond and site percolation on a given graph, and we state the relevant theorem here.

Let G be an arbitrary connected graph, and write $p_{\mathrm{c}}^{\mathrm{bond}}(G)$ (respectively $p_{\mathrm{c}}^{\mathrm{site}}(G)$) for the critical point of bond percolation (respectively site percolation) on G. The automorphism group of G acts on the vertices of G in a natural way, and we call G *finitely transitive* if this group action has only finitely many orbits. An edge e of G is called a *bridge* if its removal disconnects G; G is said to be *bridgeless* if it contains no bridges. We write $\Delta = \Delta(G)$ for the maximum vertex degree of a graph G.

Theorem 3.4. *Let G be an infinite connected graph with $\Delta = \Delta(G) < \infty$.*
(a) *If G is finitely transitive and bridgeless, then either*
  (i) $p_{\mathrm{c}}^{\mathrm{bond}}(G) = p_{\mathrm{c}}^{\mathrm{site}}(G) = 1$, *or*
  (ii) $0 < p_{\mathrm{c}}^{\mathrm{bond}}(G) < p_{\mathrm{c}}^{\mathrm{site}}(G) < 1$.
(b) *We have that*

$$p_{\mathrm{c}}^{\mathrm{site}}(G) \leq 1 - \big(1 - p_{\mathrm{c}}^{\mathrm{bond}}(G)\big)^{\Delta-1}.$$

The proof may be found in [12]. This result generalises related inequalities valid for certain two-dimensional lattices; see [15] and elsewhere. For more details and applications of the general argument around (3.5), see [2, 9].

4. Entanglements in Percolation

The question was posed in [14] whether or not a percolation model on $\mathbb{Z}^3$ can contain large entangled clusters but no large connected clusters. Numerical work reported in [14] suggested the existence of an 'entanglement critical point' $p_{\mathrm{c}}^{\mathrm{ent}}$ satisfying $p_{\mathrm{c}}^{\mathrm{ent}} \approx p_{\mathrm{c}} - 1.8 \times 10^{-7}$. No formal definition of this critical point was presented, and indeed the discussion of this initial paper concerned the contents of finite boxes only, rather than the configuration on the entire infinite lattice. We summarise in this section recent progress towards a rigorous formulation of the problem of entanglements in percolation, and we present an application of Propositions 3.1–3.3, together with some open problems.

Here is some terminology. With each edge e of $\mathbb{E}^3$, we associate the closed straight line segment of $\mathbb{R}^3$ joining its endpoints. For $E \subseteq \mathbb{E}^3$, we write $[E]$ for the union of the corresponding line segments. A 'sphere' shall be taken to mean any subset of $\mathbb{R}^3$ which is homeomorphic to the unit sphere. The complement of a sphere S has two connected components, an unbounded component called the *outside* of S and denoted $\mathrm{out}(S)$, and a bounded component called the *inside* and denoted $\mathrm{ins}(S)$.

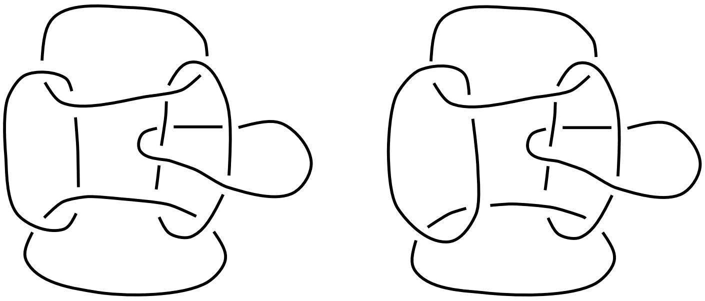

Figure 1. Sketches of two graphs. The first is entangled, the second is not.

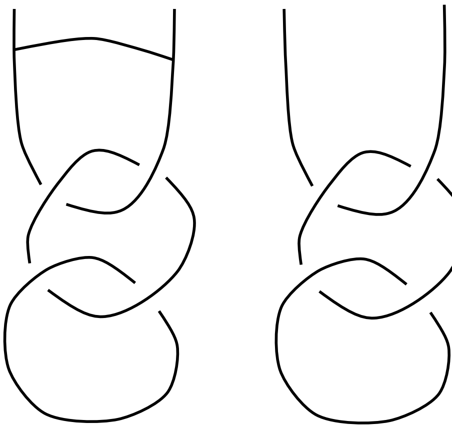

Figure 2. The four uppermost points lie in disjoint infinite paths not shown in this figure. The first graph is strongly entangled; the second graph is weakly entangled but not strongly entangled.

Let E be a finite subset of $\mathbb{E}^3$. We call E *entangled* if, for any sphere S not intersecting $[E]$, either $[E] \subseteq \mathrm{ins}(S)$ or $[E] \subseteq \mathrm{out}(S)$. This definition is illustrated in Figure 1.

There is more than one way of extending the notion of entanglement to an infinite set E of edges. Here are two such ways.

(a) We call E *strongly entangled* if, for every finite subset F of E, there exists a finite entangled subset F' of E satisfying $F \subseteq F'$.

(b) We call E *weakly entangled* if, for any sphere S not intersecting $[E]$, either $[E] \subseteq \mathrm{ins}(S)$ or $[E] \subseteq \mathrm{out}(S)$.

Such definitions are explored in [10, 13], where it is shown that E is weakly entangled whenever it is strongly entangled; the converse statement is false. See Figure 2.

Let J^{w} (respectively J^{s}) be the event that the origin is an endvertex of some edge lying in an infinite weakly (respectively strongly) entangled set E of open edges. It may be shown that J^{w} and J^{s} are indeed events, and it is clear that they are increasing. One may therefore define the *weak* and *strong entanglement* *probabilities*

$$\theta^{\mathrm{w}}(p) = P_p(J^{\mathrm{w}}), \quad \theta^{\mathrm{s}}(p) = P_p(J^{\mathrm{s}}), \tag{4.1}$$

and associated *entanglement critical points*

$$\begin{aligned} {}^{\mathrm{w}}p_{\mathrm{c}}^{\mathrm{ent}} &= \sup\{p : \theta^{\mathrm{w}}(p) = 0\}, \\ {}^{\mathrm{s}}p_{\mathrm{c}}^{\mathrm{ent}} &= \sup\{p : \theta^{\mathrm{s}}(p) = 0\}. \end{aligned} \tag{4.2}$$

It may be conjectured that

$${}^{\mathrm{w}}p_{\mathrm{c}}^{\mathrm{ent}} = {}^{\mathrm{s}}p_{\mathrm{c}}^{\mathrm{ent}}. \tag{4.3}$$

Article [10] contains a further discussion of types of entanglement, but no proof of this conjecture. In order to be more concrete in the remainder of this section, we concentrate henceforth on 'strong entanglement', and shall suppress further reference to the word 'strong'. Thus, for example, we write $p_{\mathrm{c}}^{\mathrm{ent}} = {}^{\mathrm{s}}p_{\mathrm{c}}^{\mathrm{ent}}$.

Since all connected graphs are entangled, it is immediate that $p_{\mathrm{c}}^{\mathrm{ent}} \le p_{\mathrm{c}}$. The technique of Section 3.2 was used in [2] in such a way as to imply the strict inequality $p_{\mathrm{c}}^{\mathrm{ent}} < p_{\mathrm{c}}$. Only recently was it proved in [13] that $p_{\mathrm{c}}^{\mathrm{ent}} > 0$. We summarise these two facts in a theorem.

Theorem 4.1. *It is the case that* $0 < p_{\mathrm{c}}^{\mathrm{ent}} < p_{\mathrm{c}}$.

We consider next a further problem, namely to ascertain the manner of decay of the sizes of large finite entanglements. Let E_x be the maximal entanglement touching the vertex x, and write $E = E_0$; it is not hard to see that E_x is well defined for any x. For $n \ge 1$, let $B(n)$ be the box $[-n, n]^3$, and $\partial B(n) = B(n) \backslash B(n-1)$. It seems reasonable to believe that $P_p(E \cap \partial B(n) \ne \varnothing)$ should decay exponentially as $n \to \infty$, whenever $p < p_{\mathrm{c}}^{\mathrm{ent}}$, and it is an open problem to prove this. The inequalities of Section 3.1 allow a little progress in the direction of estimating the decay rate of $P_p(E \cap \partial B(n) \ne \varnothing)$ as $n \to \infty$, as follows.

For a positive integer k, we write λ_k for the kth iterate of the natural logarithm function. More precisely, let

$$\begin{aligned} \lambda_1(x) &= \log x, \\ \lambda_{k+1}(x) &= \max\bigl\{1, \log \lambda_k(x)\bigr\} \quad \text{for } k \ge 1. \end{aligned}$$

Theorem 4.2. *There exists* $p_0 > 0$ *such that, for* $p \in (0, p_0)$ *and* $k \ge 1$, *there exists* $\alpha_k(p) > 0$ *such that*

$$P_p\bigl(E \cap \partial B(n) \ne \varnothing\bigr) \le \exp\left\{ -\frac{\alpha_k(p)n}{\lambda_k(n)} \right\} \quad \textit{for all large } n. \tag{4.4}$$

We expect that the logarithmic term in (4.4) may be removed, and that the conclusion is valid for all p satisfying $p < p_{\mathrm{c}}^{\mathrm{ent}}$. The proof of Theorem 4.2 exploits versions of Propositions 3.1–3.3, and may be found in [10].

5. Disordered Random-Cluster Models

The problem of proving strict inequality between two critical points occurs frequently in probability theory and statistical mechanics. The fundamental mechanism summarised in Section 3.2 for establishing such inequalities was applied in [2] to percolation and Ising models. This work was extended in [3, 7] to random-cluster models, thereby deriving an attractive methodology for Ising and Potts systems. There has been considerable interest recently in disordered (or 'quenched') systems, in which the interaction function is itself sampled at random from an appropriate ensemble, and it is the purpose of this section to explore strict inequalities for random-cluster models in this setting.

Perhaps the main motivation for the general study of disordered systems is the desire to understand phase transitions in spin glasses (see [18]). Indeed, it is currently unknown whether or not such phase transitions exist. Theorem 5.2 of this section has proved useful to recent work [4] intended to elucidate this question.

Here is a description of a disordered random-cluster model. Let $\mathbf{J} = \{J_e : e \in \mathbb{E}^d\}$ be a family of non-negative random variables governed by a probability measure $\mathbb{P}$; we allow the J_e to take values in the extended half-line $[0, \infty]$, and we define

$$p_e = 1 - e^{-\beta J_e}, \quad e \in \mathbb{E}^d,$$

where $0 < \beta < \infty$. Let q be a real number satisfying $q \geq 1$. The random-cluster measure $\phi_{\mathbf{p},q}$, defined as in Section 2, is a random probability measure.

Let I ($\subseteq \Omega$) be the event that there exists at least one infinite open cluster. Since I is an increasing event, we have by the FKG inequality that $\phi_{\mathbf{p},q}(I)$ is non-decreasing in β. We define the critical point $\beta_c(\mathbf{J})$ by

$$\beta_c(\mathbf{J}) = \sup\{\beta > 0 : \phi_{\mathbf{p},q}(I) = 0\},$$

with the convention that the supremum of the empty set is 0. It is more usual (see [8]) to define the critical point via the event $\{0 \leftrightarrow \infty\}$ rather than via I. Such a definition may be inappropriate whenever the J_e are permitted to take the value 0 with strictly positive probability, since there may exist (with strictly positive $\mathbb{P}$-probability) configurations $\mathbf{J}$ such that $\phi_{\mathbf{p},q}(I) > 0$ while $\phi_{\mathbf{p},q}(0 \leftrightarrow \infty) = 0$. It is not difficult to see however that the two such definitions are equivalent whenever $\mathbb{P}$ satisfies

$$\mathbb{P}\big(\phi_{\mathbf{p},q}(x \leftrightarrow y) > 0\big) = 1 \quad \text{for all } x, y \in \mathbb{Z}^d.$$

Let τ_i, $1 \leq i \leq d$, be the d fundamental lattice shifts of $\mathbb{L}^d$; that is, $\tau_i(x) = x + e_i$ where e_i is a unit vector in the direction of increasing ith coordinate. We recall that the *invariant σ-field* $\mathcal{I}$ of the random field $\mathbf{J} = \{J_e : e \in \mathbb{E}^d\}$ is the σ-field of all events which are invariant under the natural shift operators on Ω

induced by the τ_i. We call a σ-field *trivial* if all events therein have probability either 0 or 1.

Theorem 5.1. *If the family* $\mathbf{J}$ *has trivial invariant* σ*-field, then there exists a constant* $\beta_c = \beta_c(\mathbb{P})$ *satisfying* $0 \le \beta_c \le \infty$ *and*

$$\mathbb{P}\big(\beta_c(\mathbf{J}) = \beta_c\big) = 1.$$

Proof. The quantity $\phi_{\mathbf{p},q}(I)$ is a function of $\mathbf{J}$ which is invariant under lattice shifts. Therefore it is measurable on the invariant σ-field, and is therefore a.s. constant. It follows that $\beta_c(\mathbf{J})$ is a.s. constant as claimed. $\square$

We do not have useful necessary and sufficient conditions for the strict inequalities $0 < \beta_c < \infty$. Instead we note that, when $\mathbf{J}$ has trivial invariant σ-field, then

(a) $\beta_c = \infty$ if the edge set $\{e : J_e > 0\}$ possesses a.s. finite clusters only, and
(b) $\beta_c = 0$ if the edge set $\{e : J_e = \infty\}$ possesses a.s. one or more infinite clusters.

We shall prove that $\beta_c(\mathbb{P})$ is strictly monotone in $\mathbb{P}$, subject to certain conditions. Although the main applications of such a result are currently to situations where the J_e are *independent* random variables (see [4]), we shall consider here a more general setting, as follows.

Let $\mathbf{X} = \{X_e : e \in \mathbb{E}^d\}$ and $\mathbf{Y} = \{Y_e : e \in \mathbb{E}^d\}$ be families of non-negative random variables indexed by $\mathbb{E}^d$ and defined on the same probability space $(\Gamma, \mathcal{G}, \mathbb{P})$. We shall assume henceforth that

$$\mathbb{P}(\mathbf{X} \le \mathbf{Y}) = 1, \tag{5.1}$$

which is to say that $\mathbb{P}(X_e \le Y_e) = 1$ for all e. We propose to compare with one another the two random-cluster models having respective edge interactions $\mathbf{X}$ and $\mathbf{Y}$. It is evident (by the FKG inequality) that

$$\beta_c(\mathbf{X}) \ge \beta_c(\mathbf{Y}). \tag{5.2}$$

In order to prove a strict inequality, we shall require some sort of lower bound for the difference $\mathbf{Y} - \mathbf{X}$. Let $\eta \in \mathbb{E}^d$ and let $\mathbf{k} = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d$ satisfy $k_j \ne 0$ for $1 \le j \le d$. The pair $(\eta, \mathbf{k})$ generates a periodic class

$$\Xi = \Xi(\eta, \mathbf{k}) = \{\eta + \mathbf{m}.\mathbf{k} : \mathbf{m} \in \mathbb{Z}^d\} \tag{5.3}$$

of edges, where $\mathbf{m}.\mathbf{k} = (m_1 k_1, m_2 k_2, \dots, m_d k_d)$.

For $f \in \mathbb{E}^d$, let

$$\mathcal{F}_f = \sigma\big(\{X_e, Y_e : e \in \mathbb{E}^d,\ e \ne f\}\big)$$

denote the σ-field generated by the random variables X_e, Y_e for $e \neq f$. We shall require that there exists η, $\mathbf{k}$, and $\delta > 0$ such that

$$\mathbb{P}(Y_f - X_f \mid \mathcal{F}_f) \geq \delta \quad \text{a.s., for all } f \in \Xi(\eta, \mathbf{k}). \tag{5.4}$$

[The expression $\mathbb{P}(Z \mid \mathcal{F}_f)$ denotes the appropriate conditional expectation of the random variable Z.]

Let $0 \leq s \leq 1$, and set

$$J_e = J_e(s) = X_e + s(Y_e - X_e), \quad e \in \mathbb{E}^d. \tag{5.5}$$

The $J_e(s)$ interpolate between $J_e(0) = X_e$ and $J_e(1) = Y_e$. If the family $\{(X_e, Y_e) : e \in \mathbb{E}^d\}$ has trivial invariant σ-field, then so does $\mathbf{J}(s) = \{J_e(s) : e \in \mathbb{E}^d\}$, whence there exists by Theorem 5.1 a constant $\beta_c = \beta_c(\mathbb{P}, s)$ such that

$$\mathbb{P}\big(\beta_c(\mathbf{J}(s)) = \beta_c(\mathbb{P}, s)\big) = 1.$$

Our target is to identify conditions under which $\beta_c(\mathbb{P}, 0) > \beta_c(\mathbb{P}, 1)$.

Theorem 5.2. *Let $q > 1$. Assume that*:

(i) *$\mathbb{P}$, η, $\mathbf{k}$, δ are such that $\delta > 0$ and (5.4) holds,*

(ii) *there exist reals ρ, σ such that $0 < \rho \leq \sigma < \infty$ and*

$$\mathbb{P}(\rho \leq X_e \leq Y_e \leq \sigma) = 1 \quad \textit{for all } e \in \mathbb{E}^d, \tag{5.6}$$

(iii) *the invariant σ-field of the family $\{(X_e, Y_e) : e \in \mathbb{E}^d\}$ is trivial.*

We have that $\beta_c(\mathbb{P}, 0) > \beta_c(\mathbb{P}, 1)$.

It may be possible to relax condition (ii) while retaining the conclusion of this theorem. A similar result is valid when $q = 1$, but a different argument is needed; see the relevant discussion in [7]. An inequality related to the above theorem, and derived independently of the present paper, will appear in [5].

We begin the proof of Theorem 5.2 with a preliminary lemma. Let $G = (V, E)$ be a finite graph; let $\mathbf{p} = (p_e : e \in E)$ be a vector of numbers in $[0, 1]$, and let $q \geq 1$. We write ϕ_G for the random-cluster measure on $\{0, 1\}^E$ having edge parameters p_e and cluster-weighting factor q; that is,

$$\phi_G(\omega) = \frac{1}{Z_G} \left\{ \prod_{e \in E} p_e^{\omega(e)} (1 - p_e)^{1 - \omega(e)} \right\} q^{k(\omega)}, \quad \omega \in \Omega_E = \{0, 1\}^E,$$

where $k(\omega)$ is the number of open clusters of ω. For $e \in E$, let J_e denote the event $\{\omega(e) = 1\}$. We denote by $G.e$ (respectively $G\backslash e$) the graph obtained from G by contracting (respectively deleting) the edge e.

Lemma 5.3. *Let $e \in E$, $q \ge 1$, and $0 < p_e < 1$.*
(a) *For any event A ($\subseteq \{0,1\}^E$),*

$$\frac{d}{dp_e}\,\phi_G(A) = \frac{\phi_G(J_e)(1-\phi_G(J_e))}{p_e(1-p_e)}\,\Delta_G(A,e) \tag{5.7}$$

where

$$\Delta_G(A,e) = \phi_{G.e}(A) - \phi_{G\backslash e}(A).$$

(b) *We have that*

$$\frac{1}{q} \le \frac{\phi_G(J_e)(1-\phi_G(J_e))}{p_e(1-p_e)} \le q. \tag{5.8}$$

Proof. (a) By Proposition 4 of [3],

$$\frac{d}{dp_e}\,\phi_G(A) = \frac{1}{p_e(1-p_e)}\Big\{\phi_G(A\cap J_e) - \phi_G(A)\phi_G(J_e)\Big\}.$$

Now,

$$\phi_G(A\cap J_e) - \phi_G(A)\phi_G(J_e) = \phi_G(J_e)\phi_G(\overline{J_e})\Big\{\phi_G(A \mid J_e) - \phi_G(A \mid \overline{J_e})\Big\},$$

and (5.7) follows by [8], Theorem 2.3.
(b) It is standard (see [8], equation (3.10)) that

$$\frac{p_e}{p_e + (1-p_e)q} \le \phi_G(J_e) \le p_e$$

and the claim follows easily. □

Proof of Theorem 5.2. Assume the hypotheses of the theorem. Let $0 \le s \le 1$, and define $\mathbf{J}(s) = \{J_e(s) : e \in \mathbb{E}^d\}$ accordingly by (5.5). With $\mathbf{p} = \mathbf{p}(s)$ given by

$$p_e(s) = 1 - \exp\big(-\beta J_e(s)\big),$$

we write $I_{m,n} = \{\partial\Lambda_m \leftrightarrow \partial\Lambda_n\}$, and

$$\theta_{m,n} = \phi_{\Lambda_n,\mathbf{p},q}(I_{m,n}) \quad \text{for } m \le n.$$

We have that

$$\begin{aligned}\frac{\partial\theta_{m,n}}{\partial s} &= \sum_e \frac{\partial\theta_{m,n}}{\partial p_e}\frac{\partial p_e}{\partial s}\\ &\geq \sum_{f\in\Xi}\frac{\partial\theta_{m,n}}{\partial p_f}\frac{\partial p_f}{\partial s}\\ &\geq \sum_{f\in\Xi}\frac{\partial\theta_{m,n}}{\partial p_f}\beta(Y_f - X_f)e^{-\beta\sigma},\end{aligned}\tag{5.9}$$

where $\Xi = \Xi(\eta, \mathbf{k})$. Similarly,

$$\begin{aligned}\frac{\partial\theta_{m,n}}{\partial\beta} &= \sum_e \frac{\partial\theta_{m,n}}{\partial p_e}\frac{\partial p_e}{\partial\beta}\\ &\leq \sum_e \frac{\partial\theta_{m,n}}{\partial p_e}\sigma e^{-\beta\rho}.\end{aligned}\tag{5.10}$$

These two sums may be compared with one another via the forthcoming Lemma 5.4.

Let $e \in \mathbb{E}_{\Lambda_n}$ and let $f = f(e)$ be the edge of $\Xi \cap \mathbb{E}_{\Lambda_{n-1}}$ which is closest to e. [That is, the midpoint of f is closest to the midpoint of e, according to some given norm, say L^∞, on $\mathbb{R}^d$. If two or more such edges f exist, we pick one of them according to some predetermined rule.] We note that, for any given edge $f \in \Xi$, there exist at most $K = d2^d k_1 k_2 \dots k_d$ edges e with $f(e) = f$.

Lemma 5.4. *There exist a positive integer N and a function $\zeta = \zeta(\beta)$, continuous and finite when $0 < \beta < \infty$, such that*

$$\frac{\partial\theta_{m,n}}{\partial p_e} \leq \zeta\frac{\partial\theta_{m,n}}{\partial p_{f(e)}} \quad \textit{for all } e \in \mathbb{E}_{\Lambda_n},\ m \leq n,\ \textit{and } n \geq N.\tag{5.11}$$

Note that (5.11) is an inequality between random variables. The proof of this lemma is given later.

We deduce from (5.10)–(5.11) that

$$\frac{\partial\theta_{m,n}}{\partial\beta} \leq K\zeta\sigma e^{-\beta\rho}\sum_{f\in\Xi}\frac{\partial\theta_{m,n}}{\partial p_f} \quad \text{for } n \geq N.\tag{5.12}$$

By (5.9) and Lemma 5.3,

$$\frac{\partial\theta_{m,n}}{\partial s} \geq \frac{\beta e^{-\beta\sigma}}{q}\sum_{f\in\Xi}\Delta_{m,n}(f)(Y_f - X_f),\tag{5.13}$$

where

$$\Delta_{m,n}(f) = \phi_{\Lambda_n \cdot f,\mathbf{p},q}(I_{m,n}) - \phi_{\Lambda_n\setminus f,\mathbf{p},q}(I_{m,n}).$$

Note that $\Delta_{m,n}(f)$ does not depend on the random variable $p_f(s)$, and is therefore $\mathcal{F}_f$-measurable. It follows that

$$\begin{aligned}
\mathbb{P}\left(\frac{\partial \theta_{m,n}}{\partial s}\right) &\ge \frac{\beta e^{-\beta\sigma}}{q} \sum_{f\in\Xi} \mathbb{P}\Big(\Delta_{m,n}(f)(Y_f - X_f)\Big) \\
&= \frac{\beta e^{-\beta\sigma}}{q} \sum_{f\in\Xi} \mathbb{P}\Big(\Delta_{m,n}(f)\mathbb{P}(Y_f - X_f \mid \mathcal{F}_f)\Big) \\
&\ge \frac{\beta e^{-\beta\sigma}}{q} \sum_{f\in\Xi} \delta\, \mathbb{P}\Big(\Delta_{m,n}(f)\Big) && \text{by (5.4)} \\
&\ge \frac{\beta\delta e^{-\beta\sigma}}{q^2 K\zeta\sigma e^{-\beta\rho}}\, \mathbb{P}\left(\frac{\partial \theta_{m,n}}{\partial \beta}\right) && \text{for } n \ge N,
\end{aligned}$$

where we have used Lemma 5.3 and (5.12) at the last step.

In summary, there exists $\zeta'(\beta)$, continuous and finite when $0 < \beta < \infty$, such that

$$\mathbb{P}\left(\frac{\partial \theta_{m,n}}{\partial \beta}\right) \le \zeta' \mathbb{P}\left(\frac{\partial \theta_{m,n}}{\partial s}\right) \quad \text{for } n \ge N.$$

It follows that $\Gamma_{m,n} = \mathbb{P}(\theta_{m,n})$ satisfies

$$\frac{\partial \Gamma_{m,n}}{\partial \beta} \le \zeta' \frac{\partial \Gamma_{m,n}}{\partial s} \quad \text{for } n \ge N.$$

Now,

$$\begin{aligned}
\Gamma_{m,n} &\to \mathbb{P}\Big(\phi_{\mathbf{p}(s),q}(\partial\Lambda_m \leftrightarrow \infty)\Big) && \text{as } n \to \infty \\
&\to \mathbb{P}\Big(\phi_{\mathbf{p}(s),q}(I)\Big) && \text{as } m \to \infty,
\end{aligned}$$

by the dominated convergence theorem. Furthermore, by Theorem 5.1,

$$\mathbb{P}\Big(\phi_{\mathbf{p}(s),q}(I)\Big) = \begin{cases} 0 & \text{if } \beta < \beta_c(\mathbb{P}, s) \\ 1 & \text{if } \beta > \beta_c(\mathbb{P}, s), \end{cases}$$

where we have used assumption (iii) of the theorem. It follows as in [2, 9] (see also Section 3.2 of the current article) that $\beta_c(\mathbb{P}, s)$ is strictly decreasing in s, which implies the claim of the theorem. □

Proof of Lemma 5.4. This is very similar to the proof of Theorem 1 of [3], and we therefore omit many of the details. The first step is to express the two derivatives in (5.11) in terms of two coupled Markov processes R_t, S_t on the common state space $\Omega^1_{\Lambda_n}$, satisfying $R_t \le S_t$, and whose respective equilibrium distributions are $\phi_{\Lambda_n,\mathbf{p},q}(\cdot)$ and $\phi_{\Lambda_n,\mathbf{p},q}(\cdot \mid I_{m,n})$; such representations are easily derived as in (5.9)–(5.10) from Proposition 5 of [3], which states that

$$\frac{\partial\theta_{m,n}}{\partial p_e} = \frac{\theta_{m,n}}{p_e(1-p_e)} \lim_{t\to\infty} \big\{P(R_t(e)=0,\ S_t(e)=1)\big\}. \tag{5.14}$$

[Here, P denotes the appropriate probability measure for the processes R, S.] Utilising the argument of [3], particularly the proof of inequality (4.4) there, one may obtain in the following way the required (5.11). The only difference of significance arises in the definition of the events V^1, V^2, V^3 of [3].

Let $e \in \mathbb{E}_{\Lambda_n}$ and let $f = f(e)$; let u, v be the endvertices of e. We may assume that e has at least one endvertex, u say, belonging to Λ_{n-1}; if, on the contrary, $u, v \in \Lambda_n \setminus \Lambda_{n-1}$ then it is a consequence of our assumption of wired boundary conditions that

$$\frac{\partial\theta_{m,n}}{\partial p_e} = 0,$$

and inequality (5.11) is trivial in this case.

Let C be a circuit of edges in $\mathbb{E}_{\Lambda_n}$ containing both e and f. The set $C\setminus\{e,f\}$ is the union of two paths π_u, π_v, where π_u (respectively π_v) is the path containing u (respectively v). Let $C_n(e)$ be a shortest such circuit with the property that π_u contains no vertices in $\Lambda_n \setminus \Lambda_{n-1}$. The following statement constitutes an easy piece of graph theory. There exists a constant M, depending only on d and $\mathbf{k}$, such that $C_n(e)$ exists, and furthermore every vertex therein belongs to $e + \Lambda_M$.

Let $\langle B_e\rangle$ denote the collection of all edges of $\mathbb{E}^d$ having both endvertices in $(e+\Lambda_{M+1})\cap\Lambda_n$. Assume that the event $V_t = \{R_t(e)=0,\ S_t(e)=1\}$ occurs. We define the following further events V^1, V^2, V^3:

(i) V^1 is the event that: during the time-interval $(t, t+1]$, all edges in $\langle B_e\rangle$ which are present in R_t are removed, and no edges in $\langle B_e\rangle$ are added to R; e remains present in S,

(ii) V^2 is the event that: during $(t+1, t+2]$, all edges in $C_n(e)\setminus\{e\}$ are added to R, but no other edges in $\langle B_e\rangle$ are added to R; e remains present in S,

(iii) V^3 is the event that: during $(t+2, t+3]$, the edge f is removed from R but not from S.

We note that

$$V_t \cap V^1 \cap V^2 \cap V^3 \subseteq \big\{R_{t+3}(f)=0,\ S_{t+3}(f)=1\big\}.$$

It may be shown as in [3] that

$$P(V^1 \cap V^2 \cap V^3 \mid V_t) \ge \nu(\beta)$$

for some $\nu(\beta)$ which is continuous and strictly positive on $(0, \infty)$. With the above definitions of V^1, V^2, V^3, the proof of [3] goes through in the present situation, and yields (5.11) by way of (5.14). □

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Statistical Laboratory, DPMMS, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, United Kingdom
`http://www.statslab.cam.ac.uk/~grg/`