CRITICAL PROBABILITIES FOR SITE AND BOND PERCOLATION MODELS

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Abstract. Any infinite graph $G = (V, E)$ has a site percolation critical probability p_c^{site} and a bond percolation critical probability p_c^{bond}. The well known weak inequality $p_c^{\text{site}} \geq p_c^{\text{bond}}$ is strengthened to strict inequality for a broad category of graphs G, including all the usual finite-dimensional lattices in two and more dimensions. The complementary inequality $p_c^{\text{site}} \leq 1 - (1 - p_c^{\text{bond}})^{\Delta-1}$ is proved also, where Δ denotes the supremum of the vertex degrees of G.

0. Introduction and results

Let $G = (V, E)$ be an infinite connected graph. Our target in this paper is study the relationship between site and bond percolation on G, and particularly to prove inequalities between the two critical probabilities p_c^{site}, p_c^{bond} of these models. To date, the only general inequality of this type appears to be the weak inequality $p_c^{\text{site}} \geq p_c^{\text{bond}}$, valid for all connected graphs G. Our principal purpose here is to extend the coupling arguments used in proving this weak inequality, and to exploit recently developed methods for proving strict inequalities, in order to obtain the inequality $p_c^{\text{site}} > p_c^{\text{bond}}$ for a certain broad category of graphs.

Several difficulties arise in this programme, under two general headings. First, a method is required for utilising the 'strict inequality' methods of Aizenman and Grimmett (1991) in the 'non-static' setting which occurs when studying stochastic couplings of site and bond percolation. Secondly, there are graph-theoretic complications in applying such techniques to general graphs.

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We shall give two theorems about strict inequalities, rather than one only. Our first such result is Theorem 1, which applies specifically to hypercubic lattices in d (≥ 2) dimensions; this will be proved using a relatively straightforward construction. Although this construction may in principle be extended (with some combinatorial complications) to certain other graphs on an *ad hoc* basis, it does not seem to be easy to adapt it to a broad general class of graphs. Therefore, we have separated out in Theorem 2 our more general result, the proof of which requires substantially deeper ideas than that of Theorem 1. Since $\mathbb{Z}^d$ is the major playground for percolation, Theorem 1 and its proof are valuable in their own right.

We present next a description of the two percolation models in question. In the bond percolation model on G, we are provided with ea collection $(X_e : e \in E)$ of independent Bernoulli random variables, each having the same mean p, indexed by the set E of edges (or 'bonds'). If $X_e = 1$, we say that the edge e is *open*; otherwise it is called *closed*. Given any two vertices (or 'sites') x and y, we say that y can be reached from x (and we write $x \leftrightarrow y$) if there exists a path of open edges from x to y. Let 0 denote a specific vertex of G, called the 'origin'. The (random) set of vertices which can be reached from the origin is denoted by C_0:

$$C_0 = \{x \in V : 0 \leftrightarrow x\}.$$

The principal event of interest is that of C_0 being infinite, and we define

$$\theta^{\text{bond}}(p) = \mathbb{P}_p(|C_0| = \infty),$$

where $\mathbb{P}_p$ denotes the appropriate product probability measure on $\{0,1\}^E$. We refer the reader to Durrett (1988) and Grimmett (1989, 1997) for further information and standard results concerning percolation. The 'bond critical probability' $p_c^{\text{bond}} = p_c^{\text{bond}}(G)$ is defined as

$$p_c^{\text{bond}}(G) = \sup\{p : \theta^{\text{bond}}(p) = 0\},$$

so that

$$\theta^{\text{bond}}(p) \begin{cases} = 0 & \text{if } p < p_c^{\text{bond}}, \\ > 0 & \text{if } p > p_c^{\text{bond}}. \end{cases}$$

In the site percolation model, we have instead a collection $(Y_x : x \in V)$ of Bernoulli random variables, each with mean p, indexed by the set of vertices of G.

If $Y_x = 1$, we say that the vertex x is *active*; otherwise we say that x is *inactive*. In this model, we say that vertex y can be reached from vertex x (written $x \leftrightarrow y$) if there exists a path from x to y consisting of active vertices only (in particular, x and y are required to be active). We make similar definitions to those in the bond model, obtaining thus a site percolation probability $\theta^{\text{site}}(p)$, and a site critical probability $p_{\text{c}}^{\text{site}} = p_{\text{c}}^{\text{site}}(G)$.

It is natural to ask whether there exists any relationship between the site and bond percolation models on some fixed graph. It is known that, for any graph G,

$$p_{\text{c}}^{\text{site}}(G) \geq p_{\text{c}}^{\text{bond}}(G); \tag{0.1}$$

see Hammersley (1961) for a statement of this result; see Oxley and Welsh (1979) and Kesten (1982) for proofs. In fact a stronger result is true, namely that for any p and any fixed graph G and starting vertex 0, it is the case that $\theta^{\text{site}}(p) \leq p\theta^{\text{bond}}(p)$; in this sense, percolation does not occur so readily in the site model as in the bond model. This last inequality follows from a fairly obvious coupling argument, which we shall present during the proof of our Lemma 5.

We may ask for conditions under which the inequality in (0.1) is strict. If the graph in question is a tree then necessarily the site and bond critical probabilities are equal. In this case, if we declare the starting vertex 0 to be automatically active for the site model, then the site and bond models are essentially identical, in the sense that C_0 has the same distribution for both models. Furthermore, we may make certain changes to a tree without changing the values of its critical probabilities; one example of such a change is the addition of finitely many edges. One sees in this way that there exist infinite, connected graphs which are not trees, for which the site and bond critical probabilities are equal. However, it is reasonable to suppose that for a broad category of graphs G, including all the standard finite-dimensional lattices in two or more dimensions, the strict inequality

$$p_{\text{c}}^{\text{site}}(G) > p_{\text{c}}^{\text{bond}}(G) \tag{0.2}$$

is valid. No general derivation of this is known. The only cases for which (0.2) is known appear to arise either through special properties of the graph in question (such as self-duality in the case of $\mathbb{Z}^2$, see Higuchi (1982), Tóth (1985)) or via explicit numerical bounds (see Hughes (1996), pp. 182–183). Certain two-dimensional inequalities were proved by Kesten (1982), using somewhat elaborate techniques.

Of greatest interest perhaps is the case when G is the d-dimensional hypercubic lattice with vertex set $\mathbb{Z}^d$ and edge set $\mathbb{E}^d$.

Theorem 1. *Let $d \geq 2$. We have that $p_c^{\text{site}}(\mathbb{Z}^d) > p_c^{\text{bond}}(\mathbb{Z}^d)$.*

The strict inequality of Theorem 1 is valid for a much wider range of graphs than just the hypercubic lattices. As remarked above, these two critical probabilities are equal for trees, but one might reasonably expect them to be distinct for graphs which have, in some appropriate sense, a positive density of cycles. It is not immediately clear what the best way is to make this notion precise for graphs having little or no symmetry, so we restrict our attention for the moment to graphs having a large number of automorphisms.

Let $G = (V, E)$ be an infinite connected graph. We call G *locally finite* if all vertices have finite degree, and we assume henceforth that this holds. We denote by $\Delta(G)$ the supremum of the vertex degrees of G. Let $\text{Aut}(G)$ denote the group of automorphisms of G. This group acts on the vertex set V in the obvious way. We say that G is *finitely transitive* if this group action has only finitely many orbits. Finite transitivity is true of all graphs commonly referred to as 'lattices' and has the additional appeal of being a purely graph-theoretic property: it does not depend on any particular embedding of the graph in Euclidean space.

We call an edge e of G a *bridge* if the removal of e disconnects G; we say that G is *bridgeless* if it contains no bridges. Connected bridgeless graphs with at least three vertices are also known as 2-edge-connected graphs.

Theorem 2. *Let G be an infinite, finitely transitive, connected, locally finite, bridgeless graph. Then either*

(i) $p_c^{\text{site}}(G) = p_c^{\text{bond}}(G) = 1$, *or*
(ii) $0 < p_c^{\text{bond}}(G) < p_c^{\text{site}}(G) < 1$.

Part (i) applies, for example, to an infinite ladder (the product of $\mathbb{Z}$ with K_2), and part (ii) to $\mathbb{Z}^d$ for $d \geq 2$. This theorem is applicable to graphs whose growth functions are bigger than polynomials, and in particular to the Cayley graphs of many groups. Percolation on such graphs has been studied recently by several authors; see the papers and references of Benjamini and Schramm (1996) and Benjamini, Peres, Lyons, and Schramm (1997). A more general version of Theorem 2, which does not require the graph to have any non-trivial automorphisms, is given at Theorem 10.

We make next some remarks concerning strict inequalities. There exist many situations in which a weak inequality between critical points may be established but where the corresponding strict inequality is elusive. There seems to be only one general method for proving such inequalities, namely that described by Aizenman and Grimmett (1991) (see also Menshikov (1987)). Using this approach, Aizenman and Grimmett were able to prove strict inequalities for certain percolation and Ising systems, and Bezuidenhout, Grimmett, and Kesten (1993) proved similar results for general Potts and random-cluster models. The latter results were extended to many-body interactions by Grimmett (1994). Strict inequalities can be strangely difficult to prove, even in situations when the weak inequality is nearly a triviality. The methods used here lead also to quantified versions of the strict inequalities of Theorems 1, 2, and 10, but the bounds obtained thus seem to be of limited interest.

Our third main result is an inequality complementary to that of Theorem 2.

Theorem 3. *Let G be a connected graph satisfying $\Delta = \Delta(G) < \infty$. Then*

$$p_{\mathrm{c}}^{\mathrm{site}} \leq 1 - (1 - p_{\mathrm{c}}^{\mathrm{bond}})^{\Delta - 1}.$$

The inequality of Theorem 3 may well not be the best possible, although we have not been able to improve on it. We deduce from (0.1) and Theorem 3 that, for graphs with bounded degree, $p_{\mathrm{c}}^{\mathrm{site}} < 1$ if and only if $p_{\mathrm{c}}^{\mathrm{bond}} < 1$. It is easy and standard to prove by counting paths that

$$p_{\mathrm{c}}^{\mathrm{site}}, p_{\mathrm{c}}^{\mathrm{bond}} \geq \frac{1}{\Delta - 1}, \tag{0.3}$$

and we deduce in this case that $p_{\mathrm{c}}^{\mathrm{site}}$ is non-trivial (in the sense that it lies *strictly* between 0 and 1) if and only if $p_{\mathrm{c}}^{\mathrm{bond}}$ is non-trivial. (This fact may be established by other arguments also.)

We prove Theorem 1 in Section 1, and Theorem 3 in Section 2. In Section 3, we present a discussion of the problem of proving (0.2) in general. Finally, we prove Theorem 2 in Section 4.

1. Proof of Theorem 1

The basic approach used in our proof of Theorem 1 is the 'enhancement' technology developed by Aizenman and Grimmett (1991). They considered the following situation. Suppose that we are given two percolation processes, one of which is an 'enhancement' of the other. We now ask whether or not the enhanced process has a critical probability which is *strictly* less than that of the original. Aizenman and Grimmett (1991) developed a technique for showing that, subject to certain conditions, such strict inequality is valid. One approach would therefore be to find a way in which a bond model may be viewed as an enhancement (in the sense of Aizenman and Grimmett) of the site model on the same graph. Rather than do this, we propose instead to find an enhancement of the site model which lies 'strictly beneath' the bond model. The required inequality will then follow.

We do not give full details of the arguments of Aizenman and Grimmett (1991), but present instead a summary, as follows. The enhancements considered by them are defined by translation-invariant rules which are applied systematically about the underlying graph. To describe this precisely, let us restrict our attention for the moment to site percolation on $\mathbb{Z}^d$. Given $R > 0$ we define the box

$$B(R) = [-R, R]^d = \{x \in \mathbb{Z}^d : \|x\| \leq R\},$$

where $\|x\| = \max\{|x_i| : 1 \leq i \leq d\}$ for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$. Let $S \subseteq \mathbb{Z}^d$. The set $\Omega_S = \{0,1\}^S$ is called the set of *configurations* on S. We shall think of a configuration ω in either of two ways: as a $0/1$ vector ω indexed by S, or alternatively as the subset $\eta(\omega) = \{x \in S : \omega(x) = 1\}$ of vertices of S which are active under ω. This defines a one–one correspondence between vectors ω and subsets $\eta(\omega)$. We write $\Omega = \Omega_{\mathbb{Z}^d} = \{0,1\}^{\mathbb{Z}^d}$, and $C(R) = \Omega_{B(R)}$, the set of all configurations on $B(R)$. When convenient, we think of $C(R)$ as the set of subsets of $B(R)$.

We now define the relevant type of enhancement. Let R be a positive integer (the *range* of the enhancement), let s satisfy $0 \leq s \leq 1$ (the *enhancement density*), and let $f : C(R) \to C(R)$ (the *enhancement function*). Then, for $\omega \in \Omega$ and $\eta = \eta(\omega)$, we define $f(0, \eta)$ to be the evaluation of f on the restriction of η to $B(R)$, i.e., $f(0, \eta) = f(B(R) \cap \eta)$. We extend this to a definition of $f(x, \eta)$, for each $x \in \mathbb{Z}^d$, by considering the configuration induced by ω on the box centred about x:

$$f(x, \eta) = f\Big(\big(\eta \cap (B(R) + x)\big) - x\Big) + x.$$

The *enhanced configuration* is then obtained from η by activating the enhancement (i.e., declaring all vertices in $f(x, \eta)$ to be active) if and only if a certain coin flip shows heads. More precisely, given $\omega, \xi \in \Omega$, we define the enhanced configuration $\eta^*(\omega, \xi)$ by

$$\eta^*(\omega, \xi) = \eta(\omega) \cup \Bigg\{ \bigcup_{y:\xi(y)=1} f(y, \eta(\omega)) \Bigg\}.$$

Then we introduce probabilities. Let $0 \le p \le 1$ as usual. On the product space $\Omega \times \Omega$, we put the product probability measure $\mathbb{P}_{p,s} = \mathbb{P}_p \times \mathbb{P}_s$, which is to say that

$$\mathbb{P}_{p,s}(A \times B) = \mathbb{P}_p(A)\mathbb{P}_s(B)$$

for any events $A, B \subseteq \Omega$. We now define $\theta(p, s)$ to be the probability that the origin belongs to an infinite cluster of the enhanced configuration η^*, i.e.,

$$\theta(p,s) = \mathbb{P}_{p,s}\Big(\big\{(\omega,\xi) : 0 \text{ lies in an infinite connected subset of } \eta^*(\omega,\xi)\big\}\Big).$$

There is an important categorisation of enhancements. Following Aizenman and Grimmett (1991), we shall say that an enhancement is *essential* if there exists a configuration ω such that: there exists no doubly infinite path in (the graph induced by) $\eta(\omega)$, but there exists a doubly infinite path in $\eta(\omega) \cup f(0, \eta(\omega))$. We shall make use of the following theorem, taken from Aizenman and Grimmett (1991).

Theorem 4. *Suppose $d \ge 2$ and let $s > 0$. For any essential enhancement of the site percolation model on $\mathbb{Z}^d$, there exists a non-empty interval $(\pi(s), p_c^{\text{site}}]$ of values of p on which $\theta(p,s) > 0$.*

This theorem is also valid for bond percolation and with some work a version of it may be proved for certain other graphs (see Grimmett (1994)).

We now specify the required enhancement. Let $e_1, \dots, e_d$ denote the usual unit vectors of $\mathbb{R}^d$: $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, 0, \dots, 0)$, and so on. Given a vertex $y \in \mathbb{Z}^d$ we define four pairwise disjoint sets of vertices close to y as follows:

$$\mathcal{P}_y^1 = \{y + e_i : 1 \le i \le d\}, \qquad \mathcal{P}_y^2 = \{y + e_i + e_j : 1 \le i < j \le d\},$$
$$\mathcal{N}_y^1 = \{y - e_i : 1 \le i \le d\}, \qquad \mathcal{N}_y^2 = \{y - e_i - e_j : 1 \le i < j \le d\}.$$

Thus, $\mathcal{P}_y^1 \cup \mathcal{N}_y^1$ is precisely the set of neighbours of y. Given a configuration ω, we say that y is a *qualifying vertex* if $y \notin \eta(\omega)$ and $\mathcal{P}_y^1 \cup \mathcal{P}_y^2 \cup \mathcal{N}_y^1 \cup \mathcal{N}_y^2 \subseteq \eta(\omega)$. For $\omega, \xi \in \Omega$, our enhanced configuration is

$$\eta^*(\omega,\xi) = \eta(\omega) \cup \Big\{ y : y \text{ is a qualifying vertex and } \xi(y) = 1 \Big\}.$$

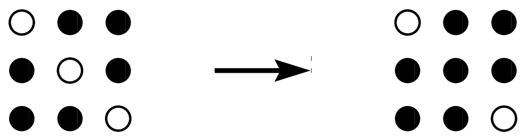

Fig. 1. A representation of the enhancement described, when $d = 2$. Each copy of the configuration on the left is replaced, with probability s, by the configuration on the right. Filled circles indicate active sites. Other sites can be either active or inactive.

It is easy to see that this constitutes an essential enhancement. We shall refer to η^* (or the law it induces on the space of configurations) as simply *enhanced site percolation with parameters* p *and* s, and we write $\theta(p, s)$ to denote the corresponding percolation probability of the enhanced configuration. See Figure 1 for a sketch of the enhancement.

Lemma 5. $\theta(p, p^2) \le \theta^{\mathrm{bond}}(p)$.

Our proof of this lemma begins with a certain 'dynamic coupling' which shows that bond percolation dominates ordinary (i.e., unenhanced) site percolation in the following sense: we start with a bond process, and we construct a site process from it in such a way that any pair of vertices which are connected by a path in the site process were already connected by a path in the bond process. This coupling is not new and, although slightly unwieldy to describe formally, is a natural one which is valid for all graphs. At the second stage of the proof, we will observe that, in the construction of the site process, certain bonds were ignored, and that these bonds may be used to enhance the ensuing site process in the required way. (Since the coupling is 'dynamic', exactly which bonds were ignored depends on the bond configuration.)

Proof of Lemma 5. We begin with a bond percolation process on $\mathbb{Z}^d$: let $(X_e : e \in \mathbb{E}^d)$ be a collection of independent 0/1-valued random variables. Let $(Z_x : x \in \mathbb{Z}^d)$ be a collection of independent 0/1-valued random variables, independent of the X_e, having mean p also. In the first stage of this proof, we construct from these two families a new collection $(Y_x : x \in \mathbb{Z}^d)$ of random variables, which constitutes a site percolation process with density p. This last process will have the property that, for $x, y \in \mathbb{Z}^d$, if y cannot be reached from x in the bond process (X_e), then neither can y be reached from x in the site process (Y_x); this will show that $\theta^{\mathrm{site}}(p) \le \theta^{\mathrm{bond}}(p)$.

Let $e_0, e_1, \ldots$ be an enumeration of the edges of $\mathbb{Z}^d$ and let $x_0, x_1, \ldots$ be an enumeration of its vertices. We wish to define the Y_x in terms of the X_e and the Z_y, and we shall do so by a (possibly transfinite, but definitely countable) recursion. We start with a formal description of the recursion, and then give a somewhat more informal description.

Suppose at some stage that we have defined the set $(Y_x : x \in \mathcal{X})$, where $\mathcal{X}$ is a proper subset of $\mathbb{Z}^d$. (At the start we have $\mathcal{X} = \varnothing$.) Let $\mathcal{Y}$ be the set of vertices not in $\mathcal{X}$ which are adjacent to some currently active vertex (i.e., a vertex $u \in \mathcal{X}$ with $Y_u = 1$). If $\mathcal{Y} = \varnothing$, then let y be the first vertex (in the sense of our enumeration) not in $\mathcal{X}$, and set $Y_y = Z_y$. If $\mathcal{Y} \neq \varnothing$, we let y be the first vertex in $\mathcal{Y}$ and let y' be the first currently active vertex adjacent to it; we then set $Y_y = X_{yy'}$; here, uv denotes the edge joining two neighbours u, v. Repeating this procedure will eventually exhaust all vertices $x \in \mathbb{Z}^d$, and assign values to all the variables Y_x.

This algorithm begins at x_0, and builds up a (possibly infinite) active cluster together with a neighbour set of inactive vertices. When the cluster at x_0 is complete, another vertex is selected as a new starting point, and the process is iterated. Note that this recursion is transfinite, since infinitely many steps are needed in order to build up any infinite active cluster.

We now make two key observations about our construction of the variables Y_x. Firstly, for each vertex x, the probability that $Y_x = 1$, conditional on any information about the values of those Y_y determined prior to the definition of Y_x, is equal to p. Based upon this observation one may prove, with a little care, that the random variables $(Y_x : x \in \mathbb{Z}^d)$ are independent with mean p, which is to say that they constitute a site percolation process on $\mathbb{Z}^d$.

Secondly, it is evident from the manner of the construction that if there exists an active path between two vertices then there exists a (possibly longer) path of open bonds. Therefore we have succeeded in coupling a bond and a site process with the required domination property.

We shall now adapt this construction in order to obtain a suitable coupling of bond percolation with an *enhanced* site percolation process obtained from the Y_x. Before giving a precise definition of the new coupling, we explain the central idea. Suppose that y is a qualifying vertex in the sense of the definition of enhanced site percolation. Then $Y_y = 0$ and $Y_x = 1$ for all $x \in \mathcal{P}_y^1 \cup \mathcal{P}_y^2 \cup \mathcal{N}_y^1 \cup \mathcal{N}_y^2$. Note that all the vertices of $\mathcal{P}_y^1 \cup \mathcal{P}_y^2$ (resp. $\mathcal{N}_y^1 \cup \mathcal{N}_y^2$) must lie in the same site percolation cluster $C_1 = C_1(y)$ (resp. $C_2 = C_2(y)$). If $C_1 = C_2$, then the activation of y makes no difference to the connectivity properties of the graph except at y. If $C_1 \neq C_2$, then activating y effectively joins C_1 and C_2 together. Since $Y_y = 0$, it is the case that at most one edge e incident with y was examined (in the sense that the value of X_e was considered) in the determination of the Y_u. Therefore there exists at least one unexamined edge joining y to $\mathcal{P}_y^1$; let the first such edge in our enumeration be $e = e(y)$. Likewise, there exists a first unexamined edge, $f = f(y)$ say, joining y and $\mathcal{N}_y^1$. We adopt the following enhancement: we declare y to be active if and only if $X_e = X_f = 1$. This has the effect of adding y into the enhanced configuration with probability p^2. Acting thus for all qualifying vertices y yields an enhanced site percolation; the independence of the enhancement at different qualifying vertices follows from the fact that the sequence of all $e(y)$ and $f(y)$ contains no repetitions. Furthermore, the above enhancement cannot join any two vertices which are not already joined by an open path in the bond model: activating y has the effect of connecting y to the clusters $C_1(y)$ and $C_2(y)$ and to no others, and this activation of y occurs only in situations where y is already joined to both of these clusters in the bond model.

It is fairly straightforward to present a formal description of the informal account above. In order to obtain the appropriate enhancement, we require a family $(H_y : y \in \mathbb{Z}^d)$ of independent Bernoulli random variables, having parameter p^2 and independent of the family (Y_x). We only require the H_y for qualifying vertices y, and we may simply set $H_y = X_{e(y)} X_{f(y)}$, where $e(y)$ and $f(y)$ are given as above.

We have now given a coupling of bond percolation and an enhanced site percolation with the property that any two vertices which are in the same cluster of the enhanced site process are in the same cluster of the bond process. In particular, if the cluster containing the origin in the enhanced vertex process is infinite, then the cluster containing the origin in the bond process is infinite also. The required inequality follows. $\square$

Proof of Theorem 1. We use the notation of the proof of Lemma 5. Note first that the enhancement presented there is essential. Let $s = \frac{1}{2} p_c^{\text{site}}$. By Theorem 4, there exists $\pi(s)$ $(< p_c^{\text{site}})$ such that $\theta(p, s) > 0$ for all $p > \pi(s)$. Let $\max\{\pi(s), s\} < p <$

p_c^{site}. Since $p^2 > s^2$, we have that $\theta(p, p^2) \geq \theta(p, s) > 0$. Therefore, by Lemma 5, $\theta^{\text{bond}}(p) > 0$, whence $p_c^{\text{site}} > p \geq p_c^{\text{bond}}$ as required. □

2. Proof of Theorem 3

Let $G = (V, E)$ be an infinite, connected, locally finite graph. We let $\delta(x)$ be the degree of the vertex x, so that $\Delta = \Delta(G) = \sup\{\delta(x) : x \in V\}$. We construct a coupling of a site and bond model on G. As above, we begin with a bond process with density p on G together with an enumeration $x_0, x_1, \ldots$ of V, and we shall construct a site process (or more precisely one component of a site process, together with its boundary). Rather than give all the formalities of the proof, we give an informal account which may easily be made rigorous.

We begin at stage 0 with the first vertex x_0 in the enumeration of V (called the 'origin'), and we declare it to be active if and only if at least one of the bonds incident with it is open; otherwise we declare it to be inactive. If x_0 is inactive, then we stop; otherwise we continue to the next stage. Note that

$$\mathbb{P}_p(x_0 \text{ is active}) = 1 - (1-p)^{\delta(x_0)} \leq 1 - (1-p)^{\Delta}.$$

At each subsequent stage (stage n, say) we consider the vertex, v say, which is the earliest vertex in the enumeration out of those which have not yet been considered and which are adjacent to some vertex which has been considered and declared active; if there are no such vertices then we stop. We declare v to be active if and only if there exists at least one open bond joining v to some vertex which has not yet been considered; otherwise we declare v to be inactive. Writing $\mathcal{F}_n$ for the history of the process up to this stage, we have that

$$\mathbb{P}_p(v \text{ is active} \mid \mathcal{F}_n) = 1 - (1-p)^{\rho} \leq 1 - (1-p)^{\Delta - 1}, \tag{2.1}$$

where ρ is the number of edges of G joining v to vertices not already considered.

In this way we build up precisely one component C of active vertices (which may be finite or infinite); furthermore, every neighbour of every active vertex in C is eventually considered (i.e., declared active or inactive). Note, however, that some vertices which belong to the component containing 0 in the bond process may have been declared inactive in the ensuing site process. Nevertheless, it is easy to see that if the component containing the origin in the bond process is infinite, then so is C. In order to see this, suppose that C is finite. Define its external boundary $\partial_e C$ to be those vertices which are not in C but which are adjacent to a vertex in C; that is, $\partial_e C$ comprises precisely those vertices which have been considered and which have been declared inactive. Then there exists no open edge in the bond process which leads from a vertex in $C \cup \partial_e C$ to a vertex outside $C \cup \partial_e C$. Therefore, the component containing the origin in the bond process is contained in $C \cup \partial_e C$, which is a finite set.

In the construction of C, each vertex other than the origin was declared active with conditional probability at most $1-(1-p)^{\Delta-1}$; cf. (2.1). This implies that the site process is stochastically dominated by an independent site percolation model on G, having density $1-(1-p)^{\Delta-1}$ and conditioned on the origin being active. It follows that

$$\theta^{\text{site}}\big(1-(1-p)^{\Delta-1}\big) \geq \big\{1-(1-p)^{\delta(x_0)}\big\}\theta^{\text{bond}}(p). \tag{2.2}$$

The required inequality follows. □

3. Strict inequality for more general graphs

We discuss next the problem of proving strict inequality for general graphs. Even finitely transitive graphs can contain cycles which do not affect the percolation probabilities. Take, for example, an infinite binary tree, and attach a triangle to each vertex. No infinite self-avoiding path can make use of these additional edges (except, perhaps, at its start), so the percolation probabilities are unaffected. To avoid these inessential parts of a graph, we make a further definition and some associated observations.

We say that a vertex x is *2-connected to infinity* if G has at least two infinite paths which are disjoint except at their common endvertex x. (Note that throughout we use *path* to mean a (finite or infinite, possibly doubly infinite) sequence of *distinct* vertices, each adjacent to the next in the sequence.) We say that G is *2-connected to infinity* if every vertex x has this property. The following lemma gives an useful characterization of vertices which are 2-connected to infinity. The 'only if' part is trivial; the proof of the 'if' part requires just an easy amendment to the proof of the vertex form of Menger's Theorem, otherwise known as the Max-Flow/Min-Cut Theorem (see, for example, Bollobás (1979)), and we omit it.

Lemma 6. *Let G be an infinite, locally finite, connected graph. A vertex x of G is 2-connected to infinity if and only if there exists no vertex y ($\neq x$) with the property that all infinite paths starting at x pass through y.*

Let $\mathrm{Sk}(G)$ denote the subgraph of G induced by the set of all vertices which are 2-connected to infinity; we call $\mathrm{Sk}(G)$ the *skeleton* of G. The following proposition shows that, by replacing G by its skeleton, we can restrict our attention to graphs which are 2-connected to infinity.

Proposition 7. *Let G be an infinite, locally finite, connected graph. Then $\mathrm{Sk}(G)$ is itself a connected (possibly empty) graph which is 2-connected to infinity. Furthermore,*

(i) *if $\mathrm{Sk}(G)$ is empty and $\Delta(G) < \infty$, then $p_c^{\mathrm{bond}}(G) = p_c^{\mathrm{site}}(G) = 1$,*

(ii) *if $\mathrm{Sk}(G)$ is non-empty, then G and $\mathrm{Sk}(G)$ have the same bond and site critical probabilities.*

Proof. Suppose $x \in \mathrm{Sk}(G)$. Then x lies on a doubly infinite path in G, and clearly all the vertices on this path belong to $\mathrm{Sk}(G)$. Hence x lies on a doubly infinite path in $\mathrm{Sk}(G)$. This demonstrates that $\mathrm{Sk}(G)$ is 2-connected to infinity. It is equally easy to see that $\mathrm{Sk}(G)$ must be connected.

Next we introduce some notation. For $x \notin \mathrm{Sk}(G)$, let $f(x)$ denote a vertex y ($\neq x$) with the property that all infinite paths of G, starting at x, pass through y. If there exist more than one such y, then we assign $f(x)$ to be one of these according to some pre-determined rule (according to a fixed ordering of the vertices of G, perhaps). We write B_x for the set of all vertices of G which can be reached from x along paths which do not use $f(x)$.

We prove (ii) first. Suppose that $\mathrm{Sk}(G) \neq \varnothing$, and let $x \in \mathrm{Sk}(G)$. Clearly, any infinite path of $\mathrm{Sk}(G)$ from x is also an infinite path of G. We shall now prove the converse. Let $xx_1x_2\dots$ be an infinite path of G, and suppose that there exists some x_j with $x_j \notin \mathrm{Sk}(G)$. Either $x \in B_{x_j}$ or not. If $x \in B_{x_j}$, then $x \notin \mathrm{Sk}(G)$, a contradiction. If $x \notin B_{x_j}$, then x_j cannot be joined by disjoint paths to x and to infinity, since all such paths must pass through the vertex $f(x_j)$. This is a contradiction, and we deduce that $x_j \in \mathrm{Sk}(G)$ for all j. Therefore, the path $xx_1x_2\dots$ lies in $\mathrm{Sk}(G)$. It follows that x lies in an infinite open (bond or site) path of $\mathrm{Sk}(G)$ if and only if it lies in such a path of G. Claim (ii) follows.

Suppose now that $\mathrm{Sk}(G) = \varnothing$, and let x_0 be a vertex of G. Consider the sequence $x_0, x_1, x_2, \dots$ defined by $x_{j+1} = f(x_j)$. Clearly the x_j are distinct; furthermore, we may find a subsequence $x_0, y_1, y_2, \dots$ no two of which are adjacent. Now, in the site percolation model, if any y_j is inactive then there is no percolation from x_0; it follows that, if $p < 1$, then $\theta^{\mathrm{site}}(p) = 0$, as required. Similarly, in the bond percolation model, if any y_j is isolated (i.e., all incident edges are closed) then there is no percolation from x_0. If $p < 1$ then, since G has bounded degree, the probability that any vertex is isolated is bounded away from zero, and, since no two y_j are adjacent, the corresponding 'isolation events' are independent; hence $\theta^{\mathrm{bond}}(p) = 0$ if $p < 1$, and claim (i) follows. $\square$

Proposition 7 does not require that G be finitely transitive; we remark that, if we make this additional assumption, then it is not hard to show (using ideas related to those above) that $\mathrm{Sk}(G)$ is non-empty. We will not need this fact, however.

A graph which is connected but which is not a tree must contain some cycle; if the graph is finitely transitive then 'equivalent' cycles must occur throughout the graph and so, under every reasonable interpretation, the graph contains a positive density of cycles. Provided that the vertices of these cycles lie in doubly infinite paths, they appear to be of greater assistance to bond percolation than to site percolation.

Conjecture 8. *Let G be an infinite, finitely transitive, connected, locally finite graph whose skeleton* $\mathrm{Sk}(G)$ *is not a tree. Then G satisfies either* (i) *or* (ii) *of Theorem 2.*

Note that the conditions of the conjecture imply that G has bounded degree. Furthermore, if (i) of Conjecture 8 does not hold, then $0 < p_{\mathrm{c}}^{\mathrm{bond}} \le p_{\mathrm{c}}^{\mathrm{site}} < 1$, by (0.1), (0.3), and Theorem 3. Theorem 2 amounts to this conjecture subject to the extra condition that G be bridgeless.

The following corollary of Theorem 2 is immediate by applying Proposition 7.

Corollary 9. *Let G be an infinite, finitely transitive, connected, locally finite graph whose skeleton is bridgeless. Then G satisfies either* (i) *or* (ii) *of Theorem 2.*

It is easy to see that the skeleton of any bridgeless graph is itself bridgeless, and so any graph satisfying the conditions of Theorem 2 is actually covered by Corollary 9. This corollary may be improved substantially. Our proof of Theorem 2

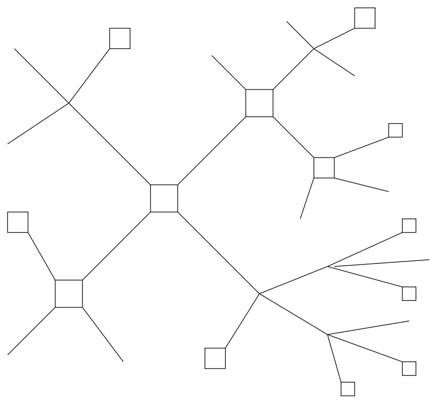

Fig. 2. Part of the graph $T_4^{\square}$ which contains infinite paths of bridges.

does not depend strongly on the assumption of finite transitivity. With only minor modifications, the proof yields the following.

Theorem 10. *Let G be an infinite, connected, locally finite graph, and suppose that the skeleton of G is of bounded degree. Suppose further that there exist constants M and K such that every path of length M in $Sk(G)$ contains an edge which is part of a cycle of length at most K. Then G satisfies either* (i) *or* (ii) *of Theorem 2.*

There remain graphs which satisfy the conditions of Conjecture 8 and yet are not covered by Theorem 10, and we present two such graphs. Our first example is tree-like. We start with the infinite tree in which every vertex has degree four, T_4. We then colour the vertices pink or brown in such a way that every vertex has precisely two pink neighbours and two brown neighbours; there is essentially only one way to do this. We then consider the pink vertices in turn, replacing each vertex with a 4-cycle and connecting each of what were the vertex's neighbours with a different vertex of the 4-cycle. The resulting graph, which we call $T_4^{\square}$, is roughly illustrated in Figure 2.

The graph $T_4^{\square}$ certainly satisfies the conditions of Conjecture 8, yet it contains infinite paths all of whose edges are bridges. A graph such as this, however, can be tackled by considering its partition into blocks, where a *block* is either a bridge or a maximal 2-connected subgraph. When the blocks are themselves finite, as in the case of $T_4^{\square}$, percolation on the graph may be regarded as a multitype branching process, in much the same way as percolation on a homogeneous or periodic tree may be regarded as a branching process. In such cases the percolation probabilities may be calculated explicitly, and methods similar to those of Oxley and Welsh (1979) may be utilised in order to establish general results concerning the equality or inequality of bond and site critical probabilities.

It is not too hard, however, to construct graphs satisfying the conditions of Conjecture 8 which are vertex-transitive and 2-connected to infinity, and contain bridges, and yet have infinitely many infinite blocks. Some such graphs contain doubly infinite paths which cannot be locally modified to contain *any* edge which is part of a cycle. These graphs do not appear to yield either to enhancement technology, or to branching process comparisons, and the corresponding question of strict inequality remains open. Here is an example of such a graph $G = (V, E)$. Let T be the vertex set of a binary tree, and $\mathbb{Z}^2$ the vertex set of the square lattice. The graph G may be defined informally as follows. We start with a binary tree T. At each vertex t of T we 'hang' a copy of $\mathbb{Z}^2$ by identifying t with the origin of this copy. Now, at each vertex of each copy of $\mathbb{Z}^2$, we attach a copy of T by its root, and so on. This amounts to constructing the 'free product' $T * \mathbb{Z}^2$.

More formally we construct G as follows. We write $u \sim v$ to indicate that vertices s and t are neighbours in T or in $\mathbb{Z}^2$. We use O to denote the origin of $\mathbb{Z}^2$ and R to denote the root of T. We define V to be the family of all ordered sequences $(t_0, x_0, \ldots, x_{r-1}, t_r)$ for some $r \geq 0$ with $x_i \in \mathbb{Z}^2 \backslash \{O\}$, $t_i \in T$, and $t_j \neq R$ for $1 \leq j < r$. Two vertices $(t_0, x_0, \ldots, x_{r-1}, t_r)$ and $(u_0, y_0, \ldots, y_{s-1}, u_s)$, with $r \leq s$, are declared adjacent if and only if one of the following holds:

(i) $r = s$, $t_i = u_i$ and $x_i = y_i$ for $i < r - 1$, $t_{r-1} = u_{r-1}$, $x_{r-1} \sim y_{r-1}$ and $t_r = y_r = R$;

(ii) $r = s$, $t_i = u_i$ and $x_i = y_i$ for $i \leq r - 1$, and $t_s \sim u_s$;

(iii) $r = s - 1$, $x_i = y_i$ for $i \leq r - 1$, $t_j = u_j$ for $j \leq r$, $y_s \sim O$ and $u_s = R$.

The edge set of G may be partitioned into the set of bridges (i.e., those edges lying in the one of the copies of T, corresponding to (ii) above) and the set of 'lattice edges' (i.e., those lying in one of the copies of $\mathbb{Z}^2$, corresponding to (i) or (iii)). The difficulty faced by the method of enhancements (see Lemma 11 for the details of how the method operates) is that one can have infinite paths in G which cannot be locally modified to contain any edge which is part of a cycle.

Finally we remark that, although Theorems 2 and 10 give a general class of graphs for which *either* strict inequality holds *or* both critical probabilities equal

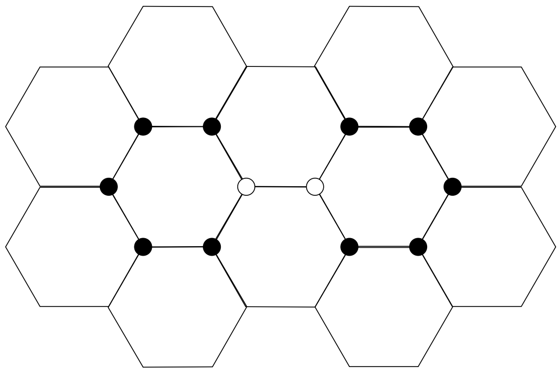

Fig. 3. An enhancement of site percolation on the hexagonal lattice which is dominated by bond percolation. The vertices marked • are active. The two vertices marked ∘ are inactive and with probability $s = p^2$ both become active. The status of other vertices is immaterial.

one, they give no indication of which conclusion is valid for any specific instance. It would certainly be interesting to have a graph-theoretic condition for distinguishing between these two cases in, say, the finitely transitive case. A partial answer to this question is suggested by Benjamini and Schramm (1996). They conjecture that if G is the Cayley graph of an infinite (finitely-generated) group which is not a finite extension of $\mathbb{Z}$ then $p_{\mathrm{c}}^{\mathrm{site}}(G) < 1$.

4. Proof of Theorem 2

It does not seem too hard to adapt the proof of Theorem 1 on a case-by-case basis in order to obtain strict inequality for most familiar lattices. As an illustration, Figure 3 shows an example of a suitable enhancement of site percolation on the hexagonal lattice. Some care is needed in justifying the corresponding versions of Theorem 4 above, and the graph-theoretic arguments have to be individually adapted for each lattice of interest. It seems to be quite another matter to extend the conclusion to a general class of graphs.

Rather than considering an enhancement of site percolation, it turns out to be more profitable to consider a *diminishment* of bond percolation (i.e., a local adjustment to the process which can only decrease the chance of percolation occurring), and to show that this diminished bond percolation dominates site percolation in a suitable sense. It was a little surprising to discover that we could not adapt the argument to find an enhancement of site percolation which is dominated by bond percolation: there appears to be greater flexibility in making local adjustments to the bonds than there is in making adjustments to sites. Diminishments were also used by Holroyd (1998) in a situation where enhancements were difficult to handle.

We begin by describing the diminishment required. Suppose we have a graph $G = (V, E)$ satisfying the conditions of Theorem 2, and recall that 0 denotes the origin of G. We find an integer K such that every edge belongs to some cycle of length at most K, and we denote by $\mathcal{C}$ the set of all cycles having length K or less. Given a bond configuration for G, we say that a particular cycle $x_1 x_2 \dots x_n x_1$ in $\mathcal{C}$ is *correctly configured* if the following conditions all hold:

(i) precisely one of the edges $x_1x_2, x_2x_3, \dots, x_nx_1$ is closed and the others are open;

(ii) exactly one of (a) and (b) following holds (where an edge is said to be *incident* with the cycle if precisely one of its two endvertices is a vertex of the cycle):
(a) precisely two of the edges incident with the cycle, x_iy and x_jz say, are open and all the others are closed, with x_i, x_j, y, z all distinct;
(b) precisely one edge incident with the cycle, x_iy say, is open;

(iii) the origin does not belong to the cycle.

Case (ii)(b) is used only as a technical device for dealing with a dull but slightly awkward part of the proof of the forthcoming Lemma 11.

Roughly speaking, for any correctly configured cycle, our diminishment will, with probability s, declare all the edges of the cycle to be closed. However, if two such cycles have vertices in common, complications arise in the proof of Lemma 12. To obviate this difficulty, we introduce a device for selecting cycles at random, independently of everything else; we will then only consider diminishing those randomly selected cycles. The selection procedure must ensure that any cycle of size at most K is selected with a probability bounded away from 0, and yet no two cycles with any vertex in common are simultaneously selected.

Let $\Xi = \{0,1\}^{\mathcal{C}}$ and let $\Omega = \{0,1\}^{E(G)}$. Let $(\omega, \alpha, \xi) \in \Omega \times \Xi \times \Xi$. We say that a cycle $c \in \mathcal{C}$ is *selected* if: $\alpha(c) = 1$ and, for all $c' \in \mathcal{C}$ which have some vertex in common with c, $\alpha(c') = 0$. We then define the diminished edge-set to be

$$\eta^*(\omega, \alpha, \xi) = \Big\{ e \in E(G) : \omega(e) = 1 \Big\} \Big\backslash \bigcup \Big\{ c \in \mathcal{C} : c \text{ is selected, correctly configured (under } \omega) \text{ and } \xi(c) = 1 \Big\}, \tag{4.1}$$

where the union of a set of cycles is here regarded as meaning the set of all edges contained in at least one of the cycles. Those cycles contributing to this union are said to be 'diminished'.

We now place the product probability measure $\mathbb{P}_{p,s} = \mathbb{P}_p \times \mathbb{P}_{1/2} \times \mathbb{P}_s$ on our space $\Omega \times \Xi \times \Xi$. The probabilities p and s play much the same rôle as before: p is the probability than an edge is open in the undiminished configuration, and s is the diminishment density. The density $\frac{1}{2}$ in the central measure is arbitrary, and any number r strictly between 0 and 1 would suffice. In a numerical study, it would be reasonable to choose r as follows. Let D be the least integer such that any cycle in $\mathcal{C}$ shares a vertex with at most D other cycles, and set $r = (D+1)^{-1}$. The probability that any given cycle is selected is at least $\{1-(D+1)^{-1}\}^D/(D+1) \sim (eD)^{-1}$.

We write $\theta(p,s)$ for the probability that the origin belongs to an infinite cluster in the diminished configuration η^*.

The proof of Theorem 2 now falls naturally into two parts. We must show first that, if conclusion (i) of Theorem 2 does not hold (so that $0 < p_c^{\mathrm{bond}} < 1$), then the above diminishment changes the critical probability (cf. Theorem 4). We must also show that site percolation with parameter p is dominated, in some useful sense, by diminished bond percolation with a suitably chosen non-zero diminishment density s. As in the proof of Theorem 1, the current proof will be complete once we have proved the following two lemmas.

Lemma 11. *Let G be a graph satisfying the conditions of Theorem 2, and such that $p_c^{\mathrm{bond}}(G) < 1$. For any $s > 0$, there exists a non-empty interval $[p_c^{\mathrm{bond}}, \pi(s))$ of values of p on which $\theta(p,s) = 0$.*

Lemma 12. *Let G be a graph satisfying the conditions of Theorem 2. Then*

$$\theta^{\mathrm{site}}(p) \le \theta(p, K^{-1}).$$

Proof of Lemma 11. Let B_n be the set of vertices which can be reached from 0 by a path of length n or less, and let ∂B_n be the set of vertices in B_n having some neighbour outside B_n. We let A_n be the event that the diminished configuration $\eta^*(\omega,\alpha,\xi)$ contains an open path from 0 to some vertex in ∂B_n, and we write $\theta_n(p,s) = \mathbb{P}_{p,s}(A_n)$. We have that $\theta_n(p,s) \downarrow \theta(p,s)$ as $n \to \infty$.

The event A_n is a cylinder event, whence $\theta_n(p,s)$ is a polynomial in p, s, and in particular is differentiable. Our goal is to show that, for all n sufficiently large,

$$-\frac{\partial\theta_n}{\partial s}(p,s) \geq g(p,s)\frac{\partial\theta_n}{\partial p}(p,s), \tag{4.2}$$

for some function $g(p,s)$ which is independent of n, and is continuous and strictly positive on $(0,1)^2$.

Once (4.2) is proved, we argue in the manner of Aizenman and Grimmett (1991) as follows. Let $\epsilon \in (0,s)$. By considering the behaviour of the function $\theta_n(p,s)$ on a small square containing the point (p_c^{bond}, s), one obtains by integrating (4.2) that, for all sufficiently small $\delta > 0$, independent of n (large), we have that

$$\theta_n(p_c^{\text{bond}} + \delta, s) \leq \theta_n(p_c^{\text{bond}} - \delta, s - \epsilon).$$

The right-hand side tends to zero as $n \to \infty$, whence the left-hand side must also do so, as required for the lemma.

We prove (4.2) via Russo's formula. Given a configuration (ω, α, ξ) and an edge e, we define $W^e(\omega,\alpha,\xi)$ to be the configuration obtained from (ω,α,ξ) by setting $\omega(e) = 1$; likewise $W_e(\omega,\alpha,\xi)$ is obtained by setting $\omega(e) = 0$. Similarly, given a cycle $c \in \mathcal{C}$, we define $X^c(\omega,\alpha,\xi)$ (resp. $X_c(\omega,\alpha,\xi)$) to be the configuration obtained by setting $\xi(c) = 1$ (resp. 0). We say that an edge e is $(+)$*pivotal* for an event A if $W_e(\omega,\alpha,\xi) \notin A$ and $W^e(\omega,\alpha,\xi) \in A$; we say that e is $(-)$*pivotal* if the reverse holds, i.e., if $W_e(\omega,\alpha,\xi) \in A$ and $W^e(\omega,\alpha,\xi) \notin A$. Similarly, a cycle c is $(+)$*pivotal* if $X_c(\omega,\alpha,\xi) \notin A$ and $X^c(\omega,\alpha,\xi) \in A$, and is $(-)$*pivotal* if $X_c(\omega,\alpha,\xi) \in A$ and $X^c(\omega,\alpha,\xi) \notin A$. We let $N_\omega^+(A)$ (resp. $N_\omega^-(A)$, $N_\xi^+(A)$, $N_\xi^-(A)$) denote the total number of $(+)$pivotal edges (resp. $(-)$pivotal edges, $(+)$pivotal cycles, $(-)$pivotal cycles).

Let $A = A_n$, and note two points. First, the four random variables just defined are all finite, since A_n depends only on the states of finitely many cycles and edges. Secondly, $N_\xi^+(A_n) = 0$, since switching on a diminishment can never help the connectivity of the graph; however, $N_\omega^-(A_n)$ can be non-zero, since switching off an edge could prevent a cycle from being correctly configured and thereby cause some further edges to be open in the diminished configuration η^*.

Writing $\mathbb{E}_{p,s}$ for expectation, the following formulae follow by applications of Russo's formula (see Grimmett (1989), p. 35):

$$\frac{\partial}{\partial p}\mathbb{P}_{p,s}(A_n) = \mathbb{E}_{p,s}(N_\omega^+(A_n)) - \mathbb{E}_{p,s}(N_\omega^-(A_n)) \tag{4.3}$$

$$\frac{\partial}{\partial s}\mathbb{P}_{p,s}(A_n) = -\mathbb{E}_{p,s}(N_\xi^-(A_n)). \tag{4.4}$$

These equations may be verified either by the usual method of proof, or as follows. Given α, Russo's formula may be applied with the restriction that only selected cycles may be diminished. Next, one averages over α. Since the distribution of α is independent of p and s, the differential operators commute with the expectation, and (4.3)–(4.4) follow.

Inequality (4.2) will follow from (4.3)–(4.4) once we have shown that

$$\mathbb{E}_{p,s}(N_\xi^-(A_n)) \geq g(p,s)\mathbb{E}_{p,s}(N_\omega^+(A_n)), \tag{4.5}$$

for some suitable g, and sufficiently large n; in fact we assume $n \geq K$. The main idea for proving (4.5) is much as in Aizenman and Grimmett (1991): if an edge e is (+)pivotal for A_n, then by making a bounded number of local adjustments to the configuration we are able to create a (−)pivotal cycle within a bounded distance of e. Note that there may exist (+)pivotal edges for A_n which lie outside B_n.

Let e be an edge and let (ω, α, ξ) be a configuration for which e is (+)pivotal for A_n. We suppose for now that $\omega(e) = 0$ so that $(\omega, \alpha, \xi) \notin A_n$ and $W^e(\omega, \alpha, \xi) \in A_n$. Suppose also that e is not within distance $K/2$ of the origin. In order to make local adjustments to the edges, without resulting in complicated changes in whether or not cycles are correctly configured and thereby diminishable, we wish to 'switch off' all nearby diminishments. If, in so doing, e ceases to be pivotal, then we will have found a pivotal cycle along the way. Let $\mathcal{S}_e = \{c_1, c_2, \ldots, c_m\}$ be the set of all cycles which contain some vertex within distance K of e. Let (ω, α, ξ_i) be the configuration obtained from (ω, α, ξ) by setting $\xi(c_j) = 0$ for $1 \leq j \leq i$. Let $I = \min\{i \leq m : (\omega, \alpha, \xi_i) \in A_n\}$, with the usual convention that $\min(\emptyset) = \infty$.

If $I < \infty$ then c_I is (−)pivotal for A_n in (ω, α, ξ_I). Since the configurations (ω, α, ξ_i) are obtained from (ω, α, ξ) by altering a bounded number of state variables, we deduce that there exists a function $h_1(p,s)$, continuous and strictly positive on

$(0,1)^2$, such that

$$\mathbb{P}_{p,s}\Big(e \text{ is } (+)\text{pivotal for } A_n,\ \omega(e)=0,\ I<\infty\Big) \le h_1(p,s)\mathbb{P}_{p,s}(\Pi_e \ge 1), \tag{4.6}$$

where Π_e is the number of cycles which contain some edge within distance K of e and which are $(-)$pivotal for A_n.

Suppose next that $I=\infty$, and denote by (ω,α,ξ') the altered configuration (ω,α,ξ_m). Since $I=\infty$ we have that $(\omega,\alpha,\xi')\notin A_n$. It must also be the case that $W^e(\omega,\alpha,\xi')\in A_n$, implying that e is $(+)$pivotal for A_n in the altered configuration (ω,α,ξ'). Let V_0 be the set of vertices connected to the origin by a path consisting of edges in $\eta^*(\omega,\alpha,\xi')$ and let V_1 be the set of vertices in B_n connected to ∂B_n by such a path. Since $(\omega,\alpha,\xi')\notin A_n$, V_0 and V_1 are disjoint sets; since $W^e(\omega,\alpha,\xi')\in A$ and $\xi(c)=0$ for every cycle c which contains e or is incident with e, we must have $e=v_0v_1$ for some vertices $v_0\in V_0$, $v_1\in V_1$.

Let c_e be a cycle of length at most K which contains the edge e. Let P_0 be a path from 0 to v_0 consisting of edges in $\eta^*(\omega,\alpha,\xi')$; let P_1 be a path from some vertex of ∂B_n to v_1 consisting of edges in $\eta^*(\omega,\alpha,\xi')$. Let w_i be the first vertex of P_i $(i=0,1)$ lying on the cycle c_e. Note that the assumption that e is not within distance $K/2$ of 0 ensures that $w_0\neq 0$; we also assume for now that $w_1\notin\partial B_n$. Let y_i be the vertex on P_i immediately before w_i. We now change the configuration (ω,α,ξ') within distance K of e in such a way that c_e becomes $(-)$pivotal for A_n. We do this by setting

(i) $\omega(e)=0$, and $\omega(f)=1$ for all edges f of c_e other than e;
(ii) $\omega(y_0w_0)=\omega(y_1w_1)=1$;
(iii) $\omega(f)=0$ for any other edge f incident with any vertex of c_e;
(iv) $\alpha(c_e)=1$; $\alpha(c)=0$ for all other cycles which share a vertex with c_e.

Denote the configuration so obtained by (ω',α',ξ'). Note that (i)–(iii) ensure that c_e becomes correctly configured and that (iv) ensures it is selected. Note that any other cycle c which becomes correctly configured, incorrectly configured, selected or deselected by these changes must lie in $\mathcal{S}_e$ and hence $\xi'(c)=0$.

It is not hard to see that c_e is a $(-)$pivotal cycle for the event A_n in the configuration (ω',α',ξ'). To see this, note that $\eta^*(X^{c_e}(\omega',\alpha',\xi'))\subseteq\eta^*(\omega,\alpha,\xi')$ so certainly $X^{c_e}(\omega',\alpha',\xi')\notin A_n$. On the other hand, $\eta^*(\omega',\alpha',\xi')$ contains the edges of P_0 up to v_0, the edges of P_1 up to v_1, and a path from v_0 to v_1 within c_e, whence

$(\omega', \alpha', \xi') \in A_n$.

There are several straightforward ways for dealing with the remaining two cases, namely when e lies within distance $K/2$ of the origin, and when $w_1 \in \partial B_n$, and we consider these cases in turn. When e lies within distance $K/2$ of the origin we let $\mathcal{S}_e$ consist of all cycles containing a vertex within distance $3K/2$ of the origin. We then define I as before, and the case $I < \infty$ is dealt with in the same way as above, with Π_e defined as the number of $(-)$pivotal cycles for A_n within distance $3K/2$ of the origin. If $I = \infty$ then we declare $\omega(e) = 1$ and select a path P from 0 to ∂B_n consisting of edges in $\eta^*(\omega, \alpha, \xi')$. Let e' be the last edge of this path which is incident with both a vertex inside $B_{\lfloor K/2 \rfloor}$ and a vertex outside it; such an edge certainly exists because $n \geq K$. Let Q be any path within $B_{\lfloor K/2 \rfloor}$ from 0 to e'; note that it may not be possible to choose Q to be a portion of P because P may exit $B_{\lfloor K/2 \rfloor}$ several times. We now set $\omega(f) = 1$ for all edges f of Q, we keep $\omega(e') = 1$, and we set $\omega(g) = 0$ for all other edges g incident with a vertex of $B_{\lfloor K/2 \rfloor}$. Now e' is $(+)$pivotal for A_n in the configuration (ω, α, ξ'), and we may proceed just as in the $I = \infty$ case above.

The case when $w_1 \in \partial B_n$ is easier. In this case, we simply do not have a vertex y_1 and we obtain our configuration (ω', α', ξ') exactly as above except that (ii) above is replaced by '$\omega(y_0 w_0) = 1$'. The cycle c_e is correctly configured with (ii)(b) (from the definition at the beginning of this section) holding. It is a $(-)$pivotal cycle exactly as in the case above.

In summary, there exists a function $h_2(p, s)$, continuous and strictly positive on $(0,1)^2$, such that

$$\mathbb{P}_{p,s}\Big(e \text{ is } (+)\text{pivotal for } A_n,\ \omega(e) = 0,\ I = \infty\Big) \leq h_2(p,s)\mathbb{P}_{p,s}(\Pi_e \geq 1), \tag{4.7}$$

for all e. Note that Π_e is defined slightly differently according to whether or not e is within distance $K/2$ of the origin. In any case, however, Π_e is the size of some subset of the set of $(-)$pivotal cycles (for A_n) which contain some edge within distance $2K$ of e.

Combining (4.6) and (4.7), we obtain that

$$\mathbb{P}_{p,s}\Big(e \text{ is } (+)\text{pivotal for } A_n,\ \omega(e) = 0\Big) \leq h_3(p,s)\mathbb{P}_{p,s}(\Pi_e \geq 1),$$

where $h_3 = h_1 + h_2$. Since the pivotality of e is independent of $\omega(e)$, it follows that

$$\mathbb{P}_{p,s}\Big(e \text{ is } (+)\text{pivotal for } A_n\Big) \le \frac{1}{1-p} h_3(p,s)\mathbb{P}_{p,s}(\Pi_e \ge 1).$$

Summing over all edges e we obtain

$$\mathbb{E}_{p,s}(N_\omega^+(A_n)) \le \frac{1}{1-p} h_3(p,s) \sum_e \mathbb{E}_{p,s}(\Pi_e) \le \frac{C}{1-p} h_3(p,s)\mathbb{E}_{p,s}(N_\xi^-(A_n)),$$

where C is an upper bound on the number of cycles of length K or less within distance $2K$ of any one edge. This implies (4.5) as required. □

Proof of Lemma 12. As in the proof of Lemma 5, we make use of the dynamic coupling between site percolation and (undiminished) bond percolation, and we examine where we have room to spare. We begin with some key ideas before moving to the details of the proof, which are more complicated than before.

We call an edge e *examined* if the coupling algorithm takes note of the state of e in the relevant bond model, when constructing the associated site model. Since each examined edge, at the moment of its examination, joins some active vertex to some vertex whose state has not yet been determined, it is the case that the graph of examined edges contains no cycle.

Consider a cycle c which is correctly configured. If it is correctly configured under (ii)(b) of the definition at the beginning of this section, then any infinite path from the origin using only open examined edges cannot use any edge of c; in this case, one may diminish c and still remain 'above' the site model under construction. Suppose now that c is correctly configured under (ii)(a) of the definition; we shall use the notation of that definition. The cycle c consists of two disjoint paths from x_i to x_j. By the observation of the last paragraph, there will be at least one edge of c which the algorithm does not examine. Suppose this edge, and the unique closed edge of c, lie on different x_ix_j paths. Then any infinite path from the origin using only open examined edges cannot use any edge of c. On this event, we may diminish c in the original bond model, and still remain 'above' the site model under construction. Some work is necessary in order to exploit these ideas rigorously.

More formally we proceed as follows. The basic sample space is the product

$$\{0,1\} \times \Omega \times \Xi \times \left\{ \prod_{c \in \mathcal{C}} \{1, 2, \dots, |c|\} \right\} \times \Xi,$$

members of which are quadruples $(\zeta, \omega, \alpha, \pi, \nu)$; here, $|c|$ denotes the number of edges in a cycle c. We put a product probability measure $\mathbb{P} = \mathbb{P}_p \times \mathbb{P}_p \times \mathbb{P}_{1/2} \times \mathbb{Q}_1 \times \mathbb{Q}_2$ on this space, where $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are themselves product measures satisfying

$$\mathbb{Q}_1\big(\{\pi : \pi(c) = j\}\big) = \frac{1}{|c|} \quad \text{for } c \in \mathcal{C} \text{ and } 1 \le j \le |c|,$$

$$\mathbb{Q}_2\big(\{\nu : \nu(c) = 1\}\big) = \frac{|c|}{K} \quad \text{for } c \in \mathcal{C}.$$

For a configuration $(\zeta, \omega, \alpha, \pi, \nu)$ and cycle $c \in \mathcal{C}$, we define

$$\xi(c) = \nu(c) 1_{\{\pi(c)=|c|\}}. \tag{4.8}$$

(Here, 1_A denotes the indicator function of the event A.) Note that the variables $\big(\xi(c) : c \in \mathcal{C}\big)$ are independent of one another and of the variables ζ, ω, α.

We now explain this in words:

- ζ is a 0/1-valued random variable with $\mathbb{P}(\zeta = 1) = p$;
- ω is a configuration of bond percolation with density p;
- α is a set of selection variables, one for each cycle $c \in \mathcal{C}$, with $\mathbb{P}\big(\alpha(c) = 1\big) = \frac{1}{2}$;
- $\pi(c)$ is a random element of $\{1, 2, \dots, |c|\}$;
- ν is a 0/1-valued random variable chosen such that $\mathbb{P}\big(\xi(c) = 1\big) = K^{-1}$;
- all components of the vectors $\zeta, \omega, \alpha, \pi, \nu$ are independent.

We shall use these random variables in order to construct a site model and a diminished bond model. The diminished bond model is exactly that given by (4.1), and its diminishment density is given above as K^{-1}. Next we construct the site model, and then we shall show that the diminished bond model dominates the site model, in the sense that, if 0 lies in an infinite active path of the site model, then it lies in an infinite open path of the bond model.

Suppose we are given the configuration $(\zeta, \omega, \alpha, \pi, \nu)$. We define edge variables $\omega'(e)$ by: $\omega'(e) = \omega(e)$ for all edges e except those belonging to cycles which are selected. In order to define $\omega'(e)$ for edges belonging to selected cycles, we shall adopt an algorithmic approach, as follows.

We start with a fixed enumeration of the vertices and of the edges and, as in the proof of Lemma 5, we examine edges one by one, and we declare vertices to be active or inactive as we proceed. We shall build a cluster at the origin comprising vertices x with $\gamma(x) = 1$, whose external boundary comprises vertices y with $\gamma(y) = 0$. First we set $\gamma(0) = \zeta$. If $\gamma(0) = 0$, we stop; otherwise, we find the earliest edge $e = 0y$ incident with 0, and we declare $\gamma(y) = \omega'(0y)$ (provided $\omega'(0y)$ has been defined). We continue likewise, building up a set of active vertices (i.e., vertices y with $\gamma(y) = 1$) and inactive vertices (with $\gamma(y) = 0$) based on sequential examination of the primed configuration ω', until either of two events occurs. If, at some point, there exists no further unexamined edge uv such that $\gamma(u) = 1$ and $\gamma(v)$ is undetermined, then we stop the process. Alternatively, the algorithm may arrive at a stage when it seeks to examine an edge e lying in a selected cycle, and hence for which $\omega'(e)$ has not yet been defined. Prior to giving the definition of such $\omega'(e)$, we introduce a further concept.

Suppose $c = x_1x_2\dots x_nx_1$ is a correctly configured cycle under part (ii)(a) of the definition at the beginning of this section; we shall use the notation of that definition. Then x_i and x_j are the only vertices of the cycle which are joined by open bonds (in ω) to vertices outside the cycle. (The less interesting case of (ii)(b) may be handled in several different (and easier) ways. For simplicity, we create a nominal x_j ($\neq x_i$) and then behave as if we were in case (a): specifically, we let x_j be the earliest vertex of our cycle (in our given ordering of vertices) other than x_i.)

The cycle c comprises two edge-disjoint paths $A(c)$, $B(c)$ joining x_i to x_j. We call an edge e of c a *completing edge* (for the configuration $(\zeta, \omega, \alpha, \pi, \nu)$) if e is the first edge examined by the algorithm with the following property: after the examination of e, either all edges in $A(c)$ or all edges in $B(c)$ have now been examined. If e is not a completing edge, we simply call e a *non-completing* edge.

We now return to the operation of the algorithm on encountering an edge e lying in a selected cycle c.

(a) If c is not correctly configured, we set $\omega'(e) = \omega(e)$.

(b) If c is correctly configured, we set

$$(4.9) \quad \omega'(e) = \begin{cases} 1_{\{\pi(c)\neq k\}} & \text{if } e \text{ is the } k\text{th non-completing edge of } c \text{ examined,} \\ 1_{\{\pi(c)\neq |c|\}} & \text{if } e \text{ is a completing edge.} \end{cases}$$

We then perform a further step of the algorithm, defining $\gamma(z) = \omega'(e)$, where z is the endvertex of e whose status is to be determined.

We note finally that the $\omega'(e)$ are not generally defined for all edges e. However, they may be extended in the following manner to a full realisation of bond percolation on $\mathbb{Z}^d$. For edges e which do not lie in selected, correctly configured cycles, we set $\omega'(e) = \omega(e)$, as before. For each selected correctly configured cycle, c, we list the edges e of c for which $\omega'(e)$ has not yet been defined, in the order given by the pre-determined enumeration. We label these edges $e_{l+1}, e_{l+2}, \ldots, e_m$, where l is the number of non-completing edges of c examined during the building up of the active cluster at the origin. (So $m = |c|$ if there was no completing edge and $m = |c| - 1$ otherwise.) We treat these edges much as if they were non-completing edges in the previous procedure by setting $\omega'(e_i) = 1_{\{\pi(c) \neq i\}}$. The final result of this procedure is a family $(\omega'(e) : e \in E)$, which agrees with ω except on edges of selected correctly configured cycles; on such cycles, both ω and ω' have precisely one closed edge. If we were to apply the algorithm of the proof of Lemma 5 to the family $(\omega'(e))$, the resulting active cluster at the origin would be precisely the same as that obtained by the algorithm just described.

This terminates the construction of the algorithm applied to the configuration $(\zeta, \omega, \alpha, \pi, \nu)$. Note that although we have defined ω' for all edges, γ has only been defined on the cluster of active vertices at the origin and their neighbours. We claim that the set of active vertices (i.e. vertices y with $\gamma(y) = 1$) has the same distribution as the cluster at 0 of a site percolation model. In order to prove this, it suffices to show that, if we let $f'_1, f'_2, \ldots$ be a listing of the edges examined by the algorithm in building up the active cluster (in order of examination) then, for any $\epsilon_i \in \{0, 1\}$, and any n,

$$\mathbb{P}\big(\omega'(f'_i) = \epsilon_i \text{ for } 1 \leq i \leq n\big) = p^k (1-p)^{n-k} \tag{4.10}$$

where $k = \#\{i : \epsilon_i = 1\}$. Note that (4.10) implies that each examined edge is open with probability p, independently of the states of previous edges examined.

In order to show that (4.10) holds, we compare the algorithm described above, which we shall call Algorithm A, with a more familiar algorithm which we shall call Algorithm B. The probabilities of corresponding events for the two algorithms will be unchanged, but the corresponding version of (4.10) is transparently valid for Algorithm B.

Algorithm B is simply the usual edge testing algorithm, employed in the proof of Lemma 5. It uses the same predetermined ordering of vertices and edges as Algorithm A (it does not matter what this ordering is) but unlike Algorithm A does not behave differently on selected correctly configured cycles. So, at each stage of the algorithm (other than the start) we look for the first edge (in our ordering) uv such that $\gamma(u) = 1$ and $\gamma(v)$ is undetermined, and we set $\gamma(v) = \omega(uv)$.

We let $f_1, f_2, \dots$ be the sequence of edges examined by Algorithm B in building up the cluster at the origin. Since the $(\omega(e) : e \in E(G))$ are mutually independent and independent of ζ, we have that (4.10) holds for Algorithm B (with ω' replaced by ω and f_i' replaced by f_i).

Abbreviating 'selected correctly configured cycle' by 'SCCC', let $\mathcal{F}$ be the σ-field generated by the events $\{e \text{ is in a SCCC}\}$ and $\{\omega(e) = 1 \text{ and } e \text{ is not in a SCCC}\}$, as e ranges over all edges of the graph, together with the single event $\{\zeta = 1\}$. We let

$$\psi(e) = \begin{cases} 2 & \text{if } e \text{ is in a SCCC,} \\ \omega(e) & \text{otherwise,} \end{cases}$$

so that $\mathcal{F}$ is generated by ζ and the $\psi(e)$. For any events A, B, we define the conditional probability

$$\mathbb{P}(A \mid B, \mathcal{F}) = \frac{\mathbb{E}(1_A 1_B \mid \mathcal{F})}{\mathbb{E}(1_B \mid \mathcal{F})},$$

whenever the denominator is non-zero. [Here, $\mathbb{E}$ denotes the expectation operator and 1_A the indicator function of A. Such conditional probabilities are only defined 'almost surely', but we overlook this in the following.]

Let $\epsilon_i \in \{0, 1\}$ for $i \geq 1$, and let $G_i' = \{\omega'(f_i') = \epsilon_i\}$ and $G_i = \{\omega(f_i) = \epsilon_i\}$. We may compute probabilities of the form $\mathbb{P}\big(G_{n+1}' \,\big|\, G_n' \cap G_{n-1}' \cap \cdots \cap G_1', \mathcal{F}\big)$ as follows. If f_{n+1}' lies in a SCCC we write c' for this cycle, we let $I_{c'} = \{i \leq n : f_i' \in c'\}$ and let $k = |c'| - |I_{c'}|$. We then have that

$$\begin{aligned} &\mathbb{P}\big(\omega'(f_{n+1}') = 1 \,\big|\, G_n' \cap G_{n-1}' \cap \cdots \cap G_1', \mathcal{F}\big) \\ &\qquad\qquad = \begin{cases} \psi(f_{n+1}') & \text{if } \psi(f_{n+1}') \in \{0, 1\}, \\ \dfrac{k-1}{k} & \text{if } \psi(f_{n+1}') = 2 \text{ and } \epsilon_i = 1 \text{ for all } i \in I_{c'}, \\ 1 & \text{if } \psi(f_{n+1}') = 2 \text{ and } \epsilon_i = 0 \text{ for some } i \in I_{c'}. \end{cases} \end{aligned}$$

Equality holds here also with ω', f_i', c' replaced by ω, f_i, c, where c is defined where necessary as the SCCC containing f_{n+1}.

We note that $f_{n+1}' = f_{n+1}$ whenever $\omega'(f_i') = \omega(f_i)$ for $1 \leq i \leq n$. Therefore, for any given $\epsilon_1, \epsilon_2, \dots$ such that the conditional probabilities are defined,

$$\mathbb{P}\big(G'_{n+1} \,\big|\, G'_n \cap G'_{n-1} \cap \cdots \cap G'_1, \mathcal{F}\big) = \mathbb{P}\big(G_{n+1} \,\big|\, G_n \cap G_{n-1} \cap \cdots \cap G_1, \mathcal{F}\big) \quad \text{for all } n \geq 1.$$

Hence, by induction on n,

$$\mathbb{P}\big(G'_n \cap G'_{n-1} \cap \cdots \cap G'_1 \,\big|\, \mathcal{F}\big) = \mathbb{P}\big(G_n \cap G_{n-1} \cap \cdots \cap G_1 \,\big|\, \mathcal{F}\big) \quad \text{for all } n \geq 1.$$

Taking expectations, we deduce that

$$\mathbb{P}(G'_n \cap G'_{n-1} \cap \cdots \cap G'_1) = \mathbb{P}(G_n \cap G_{n-1} \cap \cdots \cap G_1),$$

which is equivalent to

$$\mathbb{P}\big(\omega'(f'_i) = \epsilon_i \text{ for } 1 \leq i \leq n\big) = \mathbb{P}\big(\omega(f_i) = \epsilon_i \text{ for } 1 \leq i \leq n\big). \tag{4.11}$$

Since the equivalent of (4.10) holds for Algorithm B, we deduce (4.10) for Algorithm A from (4.11).

We complete the proof of Lemma 12 by showing the required domination. Write $I = \{y : \gamma(y) = 1\}$, and suppose that $|I| = \infty$. There must exist an infinite path P of edges e which were examined by the above algorithm and for which $\omega'(e) = 1$. We shall show that the diminished bond model $\eta^*(\omega, \alpha, \xi)$, given in (4.1), necessarily contains an infinite path of open edges.

First consider edges e of P which lie in no cycle c which is both selected and correctly configured. We have that $\omega(e) = \omega'(e) = 1$, and furthermore that e lies in no diminished cycle. Therefore e lies in $\eta^*(\omega, \alpha, \xi)$.

The path P will generally visit certain selected, correctly configured cycles. We claim that every such cycle is necessarily undiminished, and that the segment of P within such a cycle c may be replaced by a sequence of edges of c which are open in ω. Suppose that P intersects a selected, correctly configured cycle $c = x_1 x_2 \dots x_n x_1$. Note that the origin cannot be in a correctly configured cycle, by part (iii) of the definition given at the beginning of this section. Therefore c must be correctly configured under part (ii)(a) of the definition. Adopting the notation of that definition, the infinite path P must enter the cycle at either x_i or x_j (say x_i), it must move around the cycle to x_j via either $A(c)$ or $B(c)$, and then it must leave the cycle. Hence, either all the edges of $A(c)$ or all the edges of $B(c)$ belong to P.

Therefore, the path contains some completing edge f. By (4.8)–(4.9), if $\xi(c) = 1$, then necessarily $\omega'(f) = 0$, a contradiction. It follows that $\xi(c) = 0$, and that c was not diminished. Since c is correctly configured, it contains a path Q_c joining x_i to x_j, all of whose edges g satisfy $\omega(g) = 1$. We now replace the segment of P within c by the path Q_c. After this has been done for every selected, correctly configured cycle (note that two such cycles cannot intersect), we achieve an infinite open path of the diminished set $\eta^*(\omega, \alpha, \xi)$. The conclusion of Lemma 12 follows. $\square$

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