ON INFLUENCE AND COMPROMISE IN TWO-TIER VOTING SYSTEMS

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Abstract. We examine two aspects of the mathematical basis for two-tier voting systems, including that of the Council of the European Union. The Penrose square-root system is based around the concept of equality of influence of the voters across the Union. There are at least two distinct definitions of influence in current use in probability theory, namely, *absolute* and *conditional influence*. These are in agreement when the underlying random variables are independent, but not generally. We review their possible implications for two-tier voting systems, especially in the context of the so-called collective bias model. We show that the two square-root laws of Penrose are unified through the use of conditional influence.

In an elaboration of the square-root system, Słomczyński and Życzkowski have proposed an exact value for the quota to be achieved in a successful vote of the Council, and they have presented numerical and theoretical evidence in its support. We indicate some numerical and mathematical issues arising in the use of a Gaussian (or normal) approximation in this context. We discuss certain aspects of the relationship between theoreticians and politicians in the design of a two-tier voting system, and we reach the conclusion that the choice of quota in the square-root system is primarily an issue for politicians.

1. Introduction and background

The square-root voting system of Penrose [28] is prominent in discussion of two-tier voting systems in general, and in specific of that of the Council of the European Union (see, for example, [21, 36]). The challenge is to devise a system for pooling the views of a number of Member States with varying population sizes. What weight w_j should be assigned to the opinion of State j, having a population of size N_j? The Penrose system amounts to the proposal $w_j \propto \sqrt{N_j}$. The essence of Penrose's argument is the observation that the number S of heads shown in N_j independent, unbiased coin tosses satisfies

$$\mathbb{E}\left|S - \tfrac{1}{2}N_j\right| \sim \sqrt{2N_j/\pi}. \tag{1.1}$$

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(Here and later, $\mathbb{E}$ denotes expectation, and $\mathbb{P}$ denotes probability.)

Penrose [28] discussed also the concept of the 'power' (termed 'influence' in the current work, after [4, 31]) of an individual voter within a given election or vote. He noted that, in a vote within a State containing N_j individuals, this power has order

$$\alpha \sim \sqrt{2/(\pi N_j)} \tag{1.2}$$

(see also Banzhaf [2]). Later authors have linked (1.1) and (1.2) by proposing a weight w_j' for State j such that the product $\alpha w_j'$ does not depend on population-size, that is, $w_j' \propto \sqrt{N_j}$, in agreement with (1.1). This argument appears to assume that: (i) in a population with size N_j and individual power α, the collective power is αN_j, and (ii) $1/\alpha$ has, generically, the same order as $\mathbb{E}\left|S - \frac{1}{2}N_j\right|$. The first assumption here is open to discussion, and the second is, in general, false. Proposition 2.6, below, explains the true relationship between (1.1) and (1.2) in the context of general probability distributions.

The mathematical basis of the preceding discussion has been developed by a number of authors including Słomczyński and Życzkowski [33, 34, 35, 36]. In a method since dubbed the 'Jagiellonian Compromise' (JagCom), the latter have proposed the use of square-root weights together with a particular value q^* for the quota q. Writing $N_1, N_2, \dots, N_s$ for the populations of the s States of the Union, under the JagCom a motion is passed if and only if

$$S(J) - S(\overline{J}) > q^* W \quad \text{where} \quad q^* := \frac{\sqrt{N}}{W},\ W = \sum_{j=1}^{s} \sqrt{N_j},\ N = \sum_{j=1}^{s} N_j. \tag{1.3}$$

Here, J is the set of States voting in favour of the motion, $\overline{J}$ is the set voting against, and

$$S(K) := \sum_{j \in K} \sqrt{N_j}, \qquad K \subseteq \{1, 2, \dots, s\}.$$

The value $q = q^*$ given in (1.3) is supported by a heuristic argument based on approximation by a Gaussian distribution. Although no proper justification of this approximation is yet available (see Section 4.2 of the current work), its conclusions gain some support using exact numerical methods (See Section 4.3).

Certain assumptions appear to be necessary for the above analyses, and the purpose of the current article is to examine some of these. There are four areas that receive special attention:

(a) the underlying model in which individuals vote according to an unbiased coin toss, independently of other voters [Section 2.3],

(b) an alternative interpretation of the concept of 'voting power' or 'influence' [Section 2.2],

(c) the assumptions of mathematical smoothness under which the Gaussian approximation is suitable for finite populations [Section 4.2],
(d) some implications of exact computations of voting powers in the Council of the European Union [Section 4.3].

Numerous earlier authors have of course considered some of these issues, namely (a), (c), and (d). We mention [21, 23, 25, 30, 35], with apologies to those authors whose work is not listed explicitly. We hope that some novel issues are illuminated in this work.

In Section 2, we introduce the notions of the absolute and the conditional influences of an individual in an election. The absolute influence is that considered by Penrose and later authors; the conditional influence is sometimes considered more appropriate in situations where individuals' votes are *dependent* random variables. The two quantities are equal in the independent case, but not generally so.

Salient features of two-tier voting systems are summarised in Section 3, with special attention to the work of Kirsch [20, 21] and Słomczyński and Życzkowski [33, 34, 35, 36]. This is followed by a discussion in Section 4 of the influences of the weighted States within the Council, and of the use and potential misuse of the normal approximation in estimating certain related probabilities. The closing Section 5 contains some reflections on the JagCom, and in particular the following conclusions.

1. Despite some fragility in the assumptions about voting patterns used to justify the square-root weights of the JagCom, we offer no superior alternative here.
2. The justification for the proposed JagCom quota q^* is numerical rather than mathematical. However, the numerics provide only equivocal guidance which does not eliminate other values of the quota, including the simpler value $q = 0$. The final choice of quota is best informed by *political* input, supported by theoretical analysis.

2. Absolute and conditional influence

2.1. **The history and context of influence.** The concept of 'influence' is central in the probability theory of disordered systems. Consider a system that comprises m sub-systems. These could be, for example, individual voters in an election, nodes in a disordered medium (as in the percolation model), or particles in a model for the ferromagnet (such as the Ising/Potts models). In studying the behaviour of the collective system, it is often key to understand the effect of a variation within a given sub-system. That is, what is the probability that a change in a given sub-system has a substantial effect on the collective system?

The quantification of influence is long recognised as being central to the understanding of complex random systems. For example, influence in voting systems was studied by Ben-Or and Linial [4] in work that played an important role in stimulating a systematic theory of influence and sharp threshold with many applications in random systems (see [19] for a review). In percolation theory, the influence of a node is the probability that the node is pivotal for a given global event (see (2.3) for the definition of pivotality). Estimates for influence are key to most of the principal results for percolation (see [15], for example). In these two areas above, the sub-systems are generally taken to be *independent* random variables. This is, however, not so for a number of important processes of statistical mechanics including the Ising and Potts models, in which the sub-systems are dependent but usually positively correlated. For such systems, 'influence' requires a new definition, and this is provided in [13, 14] in the context of the Ising and random-cluster models (see [16]).

The origins of influence are rather older than the above work, and go back at least to the work of Penrose [28] in 1946 and possibly the reliability literature surveyed by Barlow and Proschan [3] in 1965. The two definitions of influence, referred to above, are presented next in the context of a voting system (we shall use the standard terminology of probability theory and the theory of interacting systems).

2.2. **Definitions of absolute and conditional influences.** There is a population P containing m individuals, and a vote is taken between two possible outcomes, labelled $+1$ and -1. Each individual votes either $+1$ or -1. We write $X(i)$ for the vote of person i, and we assume the $X(i)$ are random variables. The *vote-vector* $X = (X(1), X(2), \dots, X(m))$ takes values in the so-called 'configuration space' $\Sigma = \{-1, 1\}^m$. There exists a predetermined subset $A \subseteq \Sigma$, and the vote is declared to *pass* if and only if $X \in A$. It is normal to consider sets A which are *increasing* in that

$$\sigma \in A,\ \sigma \le \sigma' \quad \Rightarrow \quad \sigma' \in A. \tag{2.1}$$

The inequality $\sigma \le \sigma'$ refers to the natural partial order on Σ given by

$$\sigma \le \sigma' \quad \text{if and only if} \quad \sigma(i) \le \sigma'(i) \text{ for all } i \in P.$$

For concreteness, *we assume henceforth that A is an increasing subset of Σ.*

For $i \in P$ and a configuration $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(m)) \in \Sigma$, we define the vectors σ^i, σ_i by

$$\sigma^i(j) = \begin{cases} 1 & \text{if } j = i, \\ \sigma(j) & \text{otherwise,} \end{cases} \qquad \sigma_i(j) = \begin{cases} -1 & \text{if } j = i, \\ \sigma(j) & \text{otherwise.} \end{cases} \tag{2.2}$$

That is, σ^i (respectively, σ_i) agrees with σ except at i, with i's vote set to 1 (respectively, -1). Individual i is called *pivotal* if the outcome of the vote changes when s/he changes opinion (the words *decisive* and *critical* are sometimes used in the voting literature). More formally, i is called *pivotal* for the configuration σ if

$$\sigma_i \notin A, \qquad \sigma^i \in A. \tag{2.3}$$

In all situations considered in this paper, the individual votes $X(i)$ are assumed to be identically distributed and symmetric in that

$$\mathbb{P}(X(i) = 1) = \mathbb{P}(X(i) = -1) = \tfrac{1}{2}, \tag{2.4}$$

where $\mathbb{P}$ denotes the probability measure that governs the vote-vector X. Assumptions of independence will be introduced where appropriate.

Definition 2.1. *We say that the vector X is* symmetric *if*

(i) *X and $-X$ have the same distributions, and*
(ii) *for all $i \neq j$ there exists a permutation π of $\{1, 2, \ldots, m\}$ with $\pi_i = j$ such that X and $X(\pi)$ have the same distribution, where $X(\pi)$ denotes the permuted vector $(X_{\pi_1}, X_{\pi_2}, \ldots, X_{\pi_m})$.*

Example 2.2 (Circular voting)**.** *Condition (ii) above is weaker than requiring that X be exchangeable (see* [18, p. 324]*). Here is a simple one-dimensional example of a random vector that is symmetric but not exchangeable. Suppose the m (≥ 4) voters are distributed evenly around a circular table. Let $Z_1, Z_2, \ldots, Z_m$ be the outcomes of m independent tosses of a fair coin that shows the values ± 1. Let $X(i)$ be the majority value of Z_{i-1}, Z_i, Z_{i+1}, with the convention that $Z_{m+k} = Z_k$ for all k (and a similar convention for the $X(i)$). It may be checked that $X(i)$ and $X(j)$ are independent if and only if i and j are distance 3 or more away from each other. The joint distribution of X is invariant under the rotation $i \mapsto i + 1$ modulo m, and is hence symmetric.*

Similar examples may be constructed in two and higher dimensions. In models that incorporate a spatial element in the relationships between individual voters, it may be argued that symmetry is a reasonable assumption where exchangeability is not.

Definition 2.3.

(a) *The* absolute influence *of voter i is*

$$\begin{aligned}\alpha(i) &:= \mathbb{P}(X^i \in A) - \mathbb{P}(X_i \in A) \\ &= \mathbb{P}(X_i \notin A,\ X^i \in A).\end{aligned}$$

(b) *The* conditional influence *of voter i is*

$$\kappa(i) := \mathbb{P}(A \mid X(i) = 1) - \mathbb{P}(A \mid X(i) = -1).$$

When $\mathbb{P}$ is a product measure (that is, the $X(i)$ are independent), it may be seen that $\alpha(i) = \kappa(i)$, and the common value α is termed simply *influence* by Russo [31] and Ben-Or and Linial [4]. Equality does not generally hold when $\mathbb{P}$ is not a product measure. The above concept of 'conditional influence' seems to have been identified in [13], where it was shown to be the correct adaptation of absolute influence in proofs of sharp-threshold theorems for certain families of *dependent* measures arising in stochastic geometry and statistical physics.

Remark 2.4 (Success probability)**.** *The* success probability $\eta(i)$ *of voter* i *is the probability that* i *votes on the winning side, that is,*

$$\eta(i) := \mathbb{P}\big(A \cap \{X(i) = 1\}\big) + \mathbb{P}\big(\overline{A} \cap \{X(i) = -1\}\big),$$

where $\overline{A}$ *is the complement of* A*. See, for example,* [7, 24]*. Friedrich Pukelsheim has pointed out that, if the* $X(i)$ *satisfy* (2.4)*, the conditional influence is related to the success probability by the relation* $\eta(i) = \frac{1}{2}(1 + \kappa(i))$*. This relation is, in fact, the key step in the proof of the forthcoming Proposition 2.6.*

2.3. **Examples of influences.** There follow three examples of calculations of absolute and conditional influences. For convenience, we assume $m = 2r + 1$ is an odd number, and take A to be the majority event, that is, $A = \{\sigma : \sum_i \sigma(i) > 0\}$. It is clear that A is an increasing set. We shall take the $X(i)$ to be Bernoulli random variables *with a shared parameter* u *which may itself be random*. Thus, the $X(i)$ are not generally independent.

Let U be a random variable taking values in the interval $(0, 1)$. Conditional on the event $U = u$, the $X(i)$ are defined to be independent random variables with

$$X(i) = \begin{cases} 1 & \text{with probability } u, \\ -1 & \text{with probability } 1 - u. \end{cases} \tag{2.5}$$

If U has a symmetric distribution (in that U and $1 - U$ are equally distributed), then the ensuing vote-vector X is symmetric (and, indeed, exchangeable), and this is called the 'collective bias' model by Kirsch [20, 21] (see also [22]). Here are three examples of collective bias in which the absolute and conditional influences vary greatly.

1. *Independent voting.* Let $\mathbb{P}(U = \frac{1}{2}) = 1$. The $X(i)$ are independent, unbiased Bernoulli variables, and

$$\alpha(i) = \kappa(i) = \binom{2r}{r}\left(\frac{1}{2}\right)^{2r} \sim \frac{1}{\sqrt{\pi r}} = \sqrt{\frac{2}{\pi(m-1)}} \qquad \text{as } m \to \infty. \tag{2.6}$$

2. *Uniform bias.* Let U be uniform on the interval $(0, 1)$. Then

$$\alpha(i) = \int_0^1 \binom{2r}{r} u^r (1-u)^r \, du = \frac{1}{m}, \qquad \kappa(i) = \frac{1}{2} + \mathrm{o}(1). \tag{2.7}$$

3. *Polarised bias.* Let $\mathbb{P}(U = \frac{1}{3}) = \mathbb{P}(U = \frac{2}{3}) = \frac{1}{2}$. There exists $\gamma > 0$ such that

$$\alpha(i) = \mathrm{o}(e^{-\gamma m}), \qquad \kappa(i) = \frac{1}{3} + \mathrm{o}(1). \tag{2.8}$$

We remind the reader that $f(m) = \mathrm{o}(g(m))$ means $f(m)/g(m) \to 0$ as $m \to \infty$. Cases 2 and 3 are exemplars of more general situations in which the distribution of

U has an absolutely continuous component on a neighbourhood of $\frac{1}{2}$, and U is almost surely bounded away from $\frac{1}{2}$, respectively.

Remark 2.5. *Only in the case of independence does the absolute influence have the order of the square root $1/\sqrt{m}$. In the two other situations, the absolute influence is as small as $1/m$ and $e^{-\gamma m}$, respectively.*

Correlations are easily computed in the collective bias model of (2.5). For example, for $i \neq j$, the covariance ρ between $X(i)$ and $X(j)$ satisfies

$$\begin{aligned}
\rho &= \mathbb{E}\big[\mathbb{E}(X(i)X(j) \mid U)\big] \\
&= \mathbb{E}\big[\mathbb{E}(X(i) \mid U)\mathbb{E}(X(j) \mid U)\big] && \text{by conditional independence} \\
&= \mathbb{E}[(2U-1)^2] && \text{by (2.5)} \\
&= 4\operatorname{var}(U).
\end{aligned}$$

In particular,

$$\rho = \begin{cases} \frac{1}{3} & \text{for uniform bias,} \\ \frac{1}{9} & \text{for the above polarised bias.} \end{cases}$$

Further discussion of the relationship between absolute and conditional influence may be found in [13, Sect. 2]. A review of influence for product measures is found at [19], see also [17, Sect. 4.5]. Uniform bias was discussed in [30], and polarised bias in [13].

2.4. **Two square-root laws unified.** We present next an elementary application of conditional influence. We will see its relevance in the discussion of the Penrose square-root law in Remark 3.3.

Proposition 2.6. *Let $m = 2r + 1$ be odd. Assume that X and $-X$ have the same distributions. Then $S = \sum_{i=1}^{m} X(i)$ satisfies*

$$\mathbb{E}|S| = \sum_{i=1}^{m} \kappa(i).$$

If X is symmetric, then $\kappa = \kappa(i)$ is constant and $\mathbb{E}|S| = m\kappa$.

The question arises of deciding the 'correct' definition of influence in the voting context. The answer may depend on the context of the question.

(a) If we are trying to capture the probability that an individual can, as a theoretical exercise in free will, affect the outcome of a vote, then we might favour absolute influence.
(b) If we view the opinion of an individual as being representative of the opinions of the entire population, this perhaps favours conditional influence.

(c) This issue is connected to the interpretation of 'chance' or 'randomness' in the voting model. Some authors have discussed the proposal that the views of voters may not themselves be considered random, but it is rather the *proposals* that are random (see, for example, [11, p. 38] and [20, p. 360]). This interesting suggestion poses some philosophical challenges.

This section closes with a note. An application of Proposition 2.6 to the two square-root laws of Penrose is summarised in Remark 3.3, where it is pointed out that his "two" square-root laws are in reality only one, so long as one uses *conditional* rather than *absolute* influences. When voting is truly independent, the distinction is nominal only. Seen in the light of Remark 2.4, Proposition 2.6 supports the thesis that, for general probability measures, the success probability is a more central quantity than the absolute influence.

Proof of Proposition 2.6. Let 1_A denote the indicator function of an event A. Then, since X and $-X$ have the same distribution,

$$\begin{aligned}
\mathbb{E}|S| &= \mathbb{E}(S1_{S>0}) - \mathbb{E}(S1_{S<0}) \\
&= 2\mathbb{E}(S1_{S>0}) \\
&= 2\sum_{i=1}^{m} \mathbb{E}(X_i 1_{S>0}) \\
&= \sum_{i=1}^{m} \big[\mathbb{P}(S>0 \mid X_i = 1) - \mathbb{P}(S>0 \mid X_i = -1)\big] \\
&= \sum_{i=1}^{m} \kappa(i).
\end{aligned}$$

Subject to symmetry, the constantness of $\kappa(i)$ holds by choosing suitable permutations of $\{1, 2, \ldots, m\}$. $\square$

3. Two-tier voting

3.1. **The two-tier voting structure.** We assume there exist s States with respective populations $N_1, N_2, \ldots, N_s$ (which we take for simplicity to be odd numbers). States are each allowed one representative on the Council. Each State is assumed to conduct a ballot on a given issue, and the vote of voter i in State j is denoted $X_j(i) \in \{-1, 1\}$. The outcome of the vote in state j is taken to be

$$\chi_j := \begin{cases} 1 & \text{if } S_j := \sum_{i=1}^{N_j} X_j(i) > 0, \\ -1 & \text{otherwise.} \end{cases} \tag{3.1}$$

That is, $\frac{1}{2}(1 + \chi_j)$ is the indicator function of the event that $S_j > 0$.

Assumption 3.1 ([20]). *We assume the vectors $X_j = (X_j(i) : i = 1, 2, \dots, N_j)$, $j = 1, 2, \dots, s$, are independent, which is to say that the votes of different States are independent. We make no assumption for the moment about the voters of any given State beyond that, for given j, the vectors X_j and $-X_j$ have the same distribution.*

To the State j is assigned a *weight* $w_j > 0$, and we write $W = \sum_j w_j$ for the aggregate weight of the States. The representative of state j votes χ_j, and the weighted sum

$$V := \sum_{j=1}^{s} w_j \chi_j,$$

is calculated. The motion is said to *pass* if

$$V > qW, \tag{3.2}$$

and to *fail* otherwise, where q is a predetermined *quota* (this is not quite the quota of [35], but rather that of [20], see also (1.3)). This voting system depends on the weights $w = (w_j)$ and the quota q, and we refer to it as the (w, q) system.

Question 3.2. *How should the weights w_j and the quota q be chosen?*

We summarise two approaches.

3.2. **Penrose/Kirsch and least squares** [20, 28]**.** Penrose has argued that, within any given state, the strength of a vote is proportional to the "edge", that is, the difference $N_F - N_A$ where N_F is the number voting for the successful outcome and N_A is the number voting against. Now, $N_F - N_A = \mathbb{E}|S_j|$, where S_j is given in (3.1). In the case of independent voters (see (1.1)), this has the order $\sqrt{N_j}$, and this motivates the proposal that

$$w_j = \sqrt{N_j}.$$

This calculation follows in one line by Proposition 2.6 when X is symmetric: we have that $\mathbb{E}|S_j| = N_j \kappa_j$, where $\kappa_j \sim \sqrt{C/N_j}$ by (2.6). In this sense, Penrose's "two square-root laws" are unified as one (see [11]).

Remark 3.3. *Assuming only that the jth vote-vector X_j is symmetric (that is, with no assumption of independence), Proposition 2.6 yields that $\mathbb{E}|S_j| = N_j \kappa_j$. In the language of Penrose, the mean "edge" differs from the* conditional influence *by the constant multiple N_j. Thus, in the context of general distributions, conditional influence takes precedence over absolute influence.*

Kirsch [20] has proposed choosing the w_j in such a way as to minimise the mean sum of squared errors

$$T := \mathbb{E}\left(\left[\sum_{j=1}^{s} (S_j - w_j \chi_j)\right]^2\right).$$

A quick proof of the following proposition is given at the end of the subsection.

Proposition 3.4 ([20, Thm 2.1]). *The quantity T is minimised when $w_j = \mathbb{E}|S_j|$ for $j = 1, 2, \dots, s$.*

Thus, Kirsch's least-squares principle leads to the Penrose solution $w_j = \mathbb{E}|S_j|$, which we call the *majority rule*. As explained by Kirsch, this motivates the choice

$$w_j = \begin{cases} \sqrt{N_j} & \text{if there is no long-range order,} \\ N_j & \text{if there is long-range order,} \end{cases} \tag{3.3}$$

where 'long-range order' is interpreted in the sense of statistical mechanics as the non-decay of correlations. For example, Case 1 of Section 2 has no long-range order, but Cases 2 and 3 possess long-range order. See also Proposition 2.6.

Kirsch proposed studying the voting problem via the analogy of a ferromagnetic model, such as the classical Ising model. He concentrated in [20] on the so-called Curie–Weiss (or mean-field) model, in which each vertex v of the complete graph has a random spin σ_v taking values in $\{-1, 1\}$ according to a certain probability distribution dictated by the so-called Ising model. The analysis is especially simple in this so-called 'mean-field' case since the complete graph has the maximum of symmetry. Similar results are, however, fairly immediate for finite-dimensional systems also, as follows. For concreteness, let $d \geq 1$ and let $\mathbb{T}_n$ be the d-dimensional torus obtained from the square grid $\{0, 1, \dots, n\}^d$ with periodic boundary conditions. Let β_{c} be the critical value of the inverse-temperature β of the Ising model on $\mathbb{Z}^d$ (we refer to [16, 17] for explanations of the model and notation). Interpreting σ_v as the vote of an individual placed at the vertex v, the aggregate vote

$$S = \sum_{v \in \mathbb{T}_n} \sigma_v$$

satisfies

$$\mathbb{E}|S| \approx \begin{cases} n^{d/2} & \text{if } \beta < \beta_{\mathrm{c}}, \\ n^d & \text{if } \beta > \beta_{\mathrm{c}}. \end{cases} \tag{3.4}$$

The bibliography associated with the Ising model and its ramifications is extended and complex, and is directed mostly at the corresponding infinite-volume problem defined on the entire d-dimensional space $\mathbb{Z}^d$. Some of the above claims for periodic boundary conditions are well known, and others may be derived from classical results. The relevant literature includes [1, 9, 27].

Proof of Proposition 3.4. By Assumption 3.1,

$$\begin{aligned} T &= \operatorname{var}\left(\sum_j (S_j - w_j\chi_j)\right) && \text{since } \mathbb{E}(S_j) = \mathbb{E}(\chi_j) = 0 \\ &= \sum_{j=1}^{s} \operatorname{var}(S_j - w_j\chi_j) && \text{since the } X_j \text{ are independent.} \end{aligned}$$

By calculus, the last summand is a minimum when $w_j = \mathbb{E}(S_j\chi_j)$ as claimed. □

3.3. **Słomczyński/Życzkowski and influence** [34, 35, 36]**.** Let us concentrate on the situation in which the entire vote-set $(X_j(i) : i = 1, 2, \dots, N_j,\ j = 1, 2, \dots, s)$ is an independent family of random variables. By independence, the absolute and conditional influences (within States) are equal. The influence $\alpha_j := \alpha_j(i)$ of a member of State j is (asymptotically as $N_j \to \infty$) proportional to $1/\sqrt{N_j}$, by (2.6). According to the Penrose square-root law (2.6) for influence, with $w_j = \sqrt{N_j}$ one achieves a product $\alpha_j w_j$ that is (asymptotically) constant across the States. This may be seen as evidence that the voting system with this set of weights is 'fair' across the union of the States.

How does one calculate the so-called 'total influence' of a given voter in the (w, q) system? A given voter is pivotal overall if s/he is pivotal within the relevant State vote, and furthermore the outcome of that vote is pivotal in the Council's vote. By Assumption 3.1, the *total influence* I_j of voter i in State j is the product

$$I_j = \alpha_j\beta_j, \tag{3.5}$$

where $\beta_j = \beta_j(w, q)$ is the influence of State j in the Council's vote. (See [11, p. 67].)

The total influences I_j of (3.5) need not agree with the products of the previous paragraph, since the ratios β_j/w_j are in general non-constant across the States. A number of authors including Słomczyński and Życzkowski [36] have developed the following approach.

1. Allocate to State j the weight $w_j = \sqrt{N_j}$.
2. Calculate or estimate the State-influences β_j as functions of (w, q).
3. Identify a quota q such that β_j is an approximately linear function of w_j.
4. The ensuing products $I_j = \alpha_j\beta_j$ are approximately constant across States.

They have proposed choosing the quota q in (3.2) in such a way that, for the given weights (w_j), the sum of squared differences

$$T := \sum_{j=1}^{s} (\overline{w}_j - \overline{\beta}_j)^2$$

is a minimum, where $\overline{w}_j$ and $\overline{\beta}_j$ are the normalised influences and weights, respectively (see Table 1). They present numerical, empirical, and theoretical evidence that this is often achieved when q is near

$$q^* := \frac{\sqrt{N}}{\sum_j \sqrt{N_j}}, \qquad \text{where } N = \sum_{j=1}^{s} N_j. \tag{3.6}$$

The theoretical foundation for this proposal lies in: (i) approximating β_j by a Gaussian integral, and (ii) picking q such that the integrand is close to linear in w_j. The latter step is achieved by finding the point at which the $N(\mu, \sigma)$ Gaussian density function has an inflection, and is thus locally closest to being locally linear. This inflection is easily found by calculus to be at $q := \mu \pm \sigma$, and this leads to the formula (3.6).

In summary, they argue that, when $w_j = \sqrt{N_j}$ and $q = q^*$, the $\beta_j = \beta_j(w, q)$ are close to the w_j, and hence the total influences $I_j = \alpha_j \beta_j$ are close to the products $\alpha_j w_j$. Finally, since $\alpha_j \sim C/\sqrt{N_j}$ and $w_j = \sqrt{N_j}$, the last product is constant across the States.

The above procedure is termed the Jagiellonian Compromise (or JagCom). We note that the weights w_j are chosen first, and then the quota q according to a minimisation algorithm. It may instead be preferable to choose the parameters (w, q) in such a way that the deviation in the total influences I_j is minimised. We discuss in Section 4 some aspects of the derivation of the quota q^* in (3.6).

4. 'Total influences' in a two-tier system

4.1. **Total influences.** A mathematical derivation of the JagCom quota q^*, (3.6), seems to require certain approximations which we discuss next. The first issue is to identify the target of the analysis. Let I_j be the total influence of a member of State j, as in (3.5). One extreme way of achieving the near-equality of the I_j is to set the quota q on the left side of (1.3) to be either $-\epsilon + \sum_j \sqrt{N_j}$ or its negation, where $\epsilon > 0$ is small. If we insist on such unanimity, we achieve

$$I_j = \alpha_j \left(\frac{1}{2}\right)^s \sim \frac{C}{\sqrt{N_j}} \left(\frac{1}{2}\right)^s.$$

For large s, these influences are nearly equal, indeed nearly equal to 0. Their ratios however can be as large as $\sqrt{N_{\max}/N_{\min}}$, where $N_{\max}$ (respectively, $N_{\min}$) is the maximum (respectively, minimum) State population size. An alternative target is that the ratios I_j/I_k be as close to unity as possible, and a secondary target might be that the total influences are as large as possible. We consider this next.

Consider a vote of the Council in which each State k has a preassigned weight $w_k > 0$. Let $j \in \{1, 2, \ldots, s\}$. By (3.2), State j is pivotal for the outcome if: the set

J of States (other than j) voting for the motion is such that

$$w_J + w_j - w_{\overline{J}} > qW, \qquad w_J - w_j - w_{\overline{J}} \le qW, \tag{4.1}$$

where $J \subseteq \{1, 2, \dots, s\} \setminus \{j\}$, $\overline{J} = \{1, 2, \dots, s\} \setminus (J \cup \{j\})$, and

$$w_K := \sum_{k \in K} w_k, \qquad K \subseteq \{1, 2, \dots, s\}.$$

Inequalities (4.1) may be written in the form $qW - w_j < Z_j \le qW + w_j$ where

$$Z_j = w_J - w_{\overline{J}} = \sum_{k \ne j} w_k \chi_k, \qquad j \in \{1, 2, \dots, s\}, \tag{4.2}$$

and $(\chi_k : k = 1, 2, \dots, s)$ is a family of independent Bernoulli random variables with

$$\mathbb{P}(\chi_k = 1) = \mathbb{P}(\chi_k = -1) = \tfrac{1}{2}.$$

Therefore, State j is pivotal in the Council with probability

$$\begin{aligned} \beta_j &:= \mathbb{P}\big(qW - w_j < Z_j \le qW + w_j\big) \\ &= F_{Z_j}(qW + w_j) - F_{Z_j}(qW - w_j), \end{aligned} \tag{4.3}$$

where F_{Z_j} is the distribution function of Z_j. (Similar formulae appear in [35, App.].)

4.2. **The argument via the Berry–Esseen bound.** It is tempting to argue roughly as follows. We approximate the distribution of Z_j by the Gaussian distribution with mean and variance given by

$$\mu = \mathbb{E}(Z_j) = 0, \qquad \sigma_j^2 = \operatorname{var}(Z_j) = \sum_{k \ne j} w_k^2. \tag{4.4}$$

Motivated by the local central limit theorem for non-identically distributed random variables (see [18, p. 195] and [12], or otherwise), we aspire to an approximation of (4.3) of the form

$$\begin{aligned} \beta_j &\approx \int_{qW - w_j}^{qW + w_j} \phi_{\sigma_j}(z)\, dz \approx 2w_j \phi_{\sigma_j}(qW) \\ &= \frac{2w_j}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{(qW)^2}{2\sigma_j^2}\right), \end{aligned} \tag{4.5}$$

where ϕ_σ is the density function of the $N(0, \sigma^2)$ Gaussian distribution. This leads to the following approximation for the total influence of a voter in State j:

$$I_j = \alpha_j \beta_j \simeq \frac{C}{\sqrt{N_j}} \frac{2w_j}{\sqrt{\Delta^2 - w_j^2}} \exp\left(-\frac{1}{2} \cdot \frac{(qW)^2}{(\Delta^2 - w_j^2)}\right),$$

where $C > 0$ is an absolute constant, and

$$\Delta^2 = \sum_{k=1}^{s} w_k^2.$$

Let $\delta = N_{\max}/N$, and set $w_j = \sqrt{N_j}$.

(a) If we set $q = q^*$ as in (3.6), we obtain the approximate inequalities

$$\frac{2C}{\Delta} e^{-1/[2(1-\delta)]} \preceq I_j = \alpha_j \beta_j \preceq \frac{2C}{\Delta\sqrt{1-\delta}} e^{-1/2}, \qquad j = 1, 2, \dots, s, \tag{4.6}$$

with $\Delta = \sqrt{N}$. (The symbol $\preceq$ is used in order to indicate that the inequalities are based on the unproven approximation (4.5).) These bounds are independent of the choice of j, and are increasingly close to one another in the limit as $\delta \to 0$.

(b) If, instead, we set $q = 0$, we obtain the inequalities (4.6) with the exponential terms deleted.

The exact numerical values of the β_j are calculated in Section 4.3 for the particular case of the 27 Member States of the European Union post-Brexit.

The above analysis depends on two Assumptions:

1. the normal (or Gaussian) approximation (4.5) is reasonable,
2. the ratio $\delta = N_{\max}/N$ is small.

Assumption 2 is unavoidable in some form, and its use within (4.6) is quantified therein. We therefore concentrate henceforth on Assumption 1. The approximation of (4.5) is a statement about a finite population, and thus one needs a rate of convergence in the central limit theorem. The classical such result is as follows.

Theorem 4.1 (Berry–Esseen [5, 10, 32])**.** *There exists $C \in [0.4906, 0.5600]$ such that the following holds. Let $X_1, X_2, \dots, X_s$ be independent random variables with*

$$\mathbb{E}(X_i) = 0, \quad \mathbb{E}(X_i^2) = t_i^2 > 0, \quad \mathbb{E}(|X_i|^3) = \gamma_i < \infty,$$

and write

$$\sigma^2 = \sum_{i=1}^{s} t_i^2, \qquad S = \frac{1}{\sigma}\sum_{i=1}^{s} X_i.$$

Then

$$\sup_{z\in\mathbb{R}} \big|\mathbb{P}(S \le z) - \Phi(z)\big| \le \frac{C}{\sigma^3}\sum_i \gamma_i,$$

where Φ is the distribution function of the $N(0,1)$ distribution.

Applying this to the random variable Z_j of (4.2), we obtain

$$\sup_{z\in\mathbb{R}}\big|F_{Z_j}(z)-\Phi_{\sigma_j}(z)\big| = \sup_{z\in\mathbb{R}}\big|\mathbb{P}\left(Z_j/\sigma_j \le z\right)-\Phi(z)\big| \le C\frac{\sum_{k\neq j} w_k^3}{\left(\sum_{k\neq j} w_k^2\right)^{3/2}}, \tag{4.7}$$

where σ_j^2 is given in (4.4), and Φ_σ is the distribution function of the $N(0,\sigma^2)$ distribution. Therefore, by (4.3) (see (4.5)),

$$\left|\beta_j - \int_{qW-w_j}^{qW+w_j} \phi_{\sigma_j}(z)\,dz\right| \le 2C\frac{\sum_{k\neq j} w_k^3}{\left(\sum_{k\neq j} w_k^2\right)^{3/2}}, \tag{4.8}$$

where $2C \le 1.12$.

Example 4.2. *Suppose $s=27$ and the State populations $N_1, N_2, \dots, N_{27}$ are the QMV2017 figures for the Member States of the EU, as in* [29, Table 1]. *We write $N_1 > N_2 > \cdots > N_{27}$, so that $N_{\max} = N_1$, and we choose $w_j = \sqrt{N_j}$ and $q = q^*$ with q^* as in (3.6).*

The integral on the left side of (4.8) may be expressed as

$$\int_{(\sqrt{N}-w_j)/\sqrt{N-N_j}}^{(\sqrt{N}+w_j)/\sqrt{N-N_j}} \phi(z)\,dz, \tag{4.9}$$

and its numerical value decreases monotonically from 0.207 (when $j=1$) to 0.015 (when $j=27$). The Berry–Esseen bound on the right side of (4.8) takes the value 0.332 (when $j=1$), 0.349 (when $j=5$), and 0.334 (when $j=27$), and is monotone on each of the two intervals $j\in[1,5]$ and $j\in[5,27]$. The bounds are too large to yield useful information about the β_j, and thus they cannot be estimated using the Berry–Esseen bound. In contrast, the values of the integral in (4.9) are notably close to the exact values given in Table 1.

A similar analysis is valid when $q=0$, and this case dominates all other possible choices of $q\in\mathbb{R}$. Indeed, with $q=0$, the right side of (4.8) is strictly less than the integral if and only if $j=1$, and this is therefore the unique case for which (4.8) rules out the possibility that $\beta_j=0$.

The above decimals are given to three significant digits. The calculations have been performed using Microsoft Excel.

We emphasise that the above observations do not invalidate the JagCom. Preferable to the Berry–Esseen bound would be sufficiently precise rate of convergence in the local central limit theorem for discrete, non-identically distributed random variables. We are unaware of such a result.

We conclude this subsection as follows. *No mathematical proof is known of the optimality of the choice* (3.6) *of the quota* q^* *in the JagCom.* Even if a rate can be proved in the appropriate central limit theorem, it is unlikely to be sufficiently tight to justify the choice q^* (see also the end of Section 4.3).

4.3. **The argument via numerical methods.** Once one has accepted the thesis that voters are independent and unbiased, there is a clear and transparent logic to the choice of weights $w_j = \sqrt{N_j}$. Attention then turns to the choice of quota q. It was shown in Section 4.2 that the mathematical argument of Słomczyński and Życzkowski [35], while neat, is at best incomplete. The numerical evidence of [33], in favour of $q = q^*$, retains some persuasive power. Similar numerical work has been carried out for the current article using QMV2017 population data taken from [29], with the results reported in Table 1. These results are exact rather than being based on simulation.

Table 1 lends some support to the choice $q = q^*$.

(a) The ratios of normalised influences $\overline{\beta}_j$ to normalised weights $\overline{w}_j$ are very close to 1 when $q = q^*$.
(b) Further calculations show that the sum of squared differences $T = \sum_j (\overline{w}_j - \overline{\beta}_j)^2$, considered as a function of $q = 0, \frac{1}{2}q^*, q^*, \frac{3}{2}q^*$, is a minimum when $q = q^*$. (More refined calculations are possible.)

We note, however, the following.

(i) The choice $q = q^*$ lacks transparency. In contrast, the choice $q = 0$ is simple and easy to explain.
(ii) The ratios $\overline{\beta}_j / \overline{w}_j$ are also close to 1 when $q = 0$. They are not quite so perfect as when $q = q^*$, but the differences are minor.
(iii) The sum T is similarly close to 0 when $q = 0$, albeit not so close as when $q = q^*$.
(iv) The influences β_j are largest when $q = 0$. (See also [35, App.].)

In summary, the numerics are best when $q = q^*$, but the improvements relative to the more transparent choice of $q = 0$ are minor. The numerical differences between these two cases (and indeed other reasonable values of q) are so small that they are unlikely to be separated by any technical analysis of the type of Section 4.2. We conclude that, *on the basis of the theoretical and numerical evidence, there is no convincing evidence that any one value of the quota is materially preferable to any other.*[1]

[1] Large positive or negative values are evidently poor, but we consider here only values q such that $qW/\sqrt{N}$ has order 1. Other choices for q have been considered in, for example, [6, 8].

Member State		weights		$q = 0$			$q = q^*$		
j		w_j	$\overline{w}_j$	β_j	$\overline{\beta}_j$	$\overline{\beta}_j/\overline{w}_j$	β_j	$\overline{\beta}_j$	$\overline{\beta}_j/\overline{w}_j$
1	Germany	9.059	9.963	0.357	10.414	1.045	0.211	9.937	0.997
2	France	8.165	8.979	0.317	9.239	1.029	0.191	8.984	1.001
3	Italy	7.830	8.611	0.302	8.816	1.024	0.183	8.619	1.001
4	Spain	6.815	7.495	0.260	7.575	1.011	0.159	7.507	1.002
5	Poland	6.162	6.777	0.233	6.802	1.004	0.144	6.787	1.001
6	Romania	4.445	4.888	0.166	4.839	0.990	0.104	4.891	1.001
7	Netherlands	4.152	4.566	0.155	4.512	0.988	0.097	4.568	1.000
8	Belgium	3.360	3.695	0.125	3.636	0.984	0.078	3.696	1.000
9	Greece	3.285	3.613	0.122	3.554	0.984	0.077	3.613	1.000
10	Czech Rep.	3.232	3.554	0.120	3.495	0.983	0.075	3.554	1.000
11	Portugal	3.216	3.537	0.119	3.478	0.983	0.075	3.537	1.000
12	Sweden	3.162	3.477	0.117	3.418	0.983	0.074	3.477	1.000
13	Hungary	3.135	3.448	0.116	3.389	0.983	0.073	3.447	1.000
14	Austria	2.952	3.246	0.109	3.189	0.982	0.069	3.246	1.000
15	Bulgaria	2.675	2.942	0.099	2.886	0.981	0.062	2.941	1.000
16	Denmark	2.388	2.626	0.088	2.574	0.980	0.056	2.625	1.000
17	Finland	2.338	2.571	0.086	2.520	0.980	0.055	2.570	1.000
18	Slovakia	2.326	2.558	0.086	2.507	0.980	0.054	2.557	1.000
19	Ireland	2.160	2.375	0.080	2.327	0.980	0.050	2.374	1.000
20	Croatia	2.047	2.251	0.076	2.204	0.979	0.048	2.250	1.000
21	Lithuania	1.700	1.870	0.063	1.829	0.978	0.040	1.868	0.999
22	Slovenia	1.437	1.580	0.053	1.545	0.978	0.034	1.579	0.999
23	Latvia	1.403	1.543	0.052	1.508	0.977	0.033	1.542	0.999
24	Estonia	1.147	1.261	0.042	1.233	0.978	0.027	1.260	0.999
25	Cyprus	0.921	1.013	0.034	0.990	0.977	0.021	1.012	0.999
26	Luxembourg	0.759	0.835	0.028	0.815	0.976	0.018	0.834	0.999
27	Malta	0.659	0.725	0.024	0.708	0.977	0.015	0.724	0.999
	Totals	90.930	100	3.429	100		2.123	100	

TABLE 1. Member State j has weight $w_j = \sqrt{N_j}$ and normalised weight $\overline{w}_j = 100w_j/W$, where $W = \sum_j w_j$. Two values of the quota q are considered, namely, $q = 0$ and $q = q^*$ (see (3.6)). For each, the influences β_j have been computed, and the normalised influences $\overline{\beta}_j = 100\beta_j/B$ are given above, where $B = \sum_j \beta_j$. The ratios $\overline{\beta}_j/\overline{w}_j$ are presented alongside the $\overline{\beta}_j$. The ratios lie in the interval $[0.976, 1.045]$ when $q = 0$, and in the interval $[0.997, 1.002]$ when $q = q^*$. It turns out that the β_j are in quite close agreement with the integrals of (4.9).

5. Some remarks on the Jagiellonian Compromise

Theoreticians propose, politicians dispose (and certain Presidents of the United States have historically played on both teams). Members of each group have interests and potential conflicts. The theoretician earns respect through honest assessment of the virtues (or not) of, and principles underlying, a particular proposal. They hope that politicians will accord fair weight and balance to principled proposals irrespective of personal advantage. While theoreticians are usually free of conflicts arising out of employment within a politically aligned organization, politicians are usually heavily conflicted (see, for example, [29]).

Communication between the two groups can be challenging. The use of language such as "local limit theorems" and "Berry–Esseen bound" has a tendency to create barriers. Such methodology is however key to proper study of the two-tier voting system of Sections 3–4, and practitioners have worked diligently to communicate its relevance.

The JagCom proposes the use of square-root weights $w_j = \sqrt{N_j}$ with a specific choice of the quota q. The square-root weights of equation (1.1) and Proposition 3.4 may be justified if: (i) there is no bias, and (ii) there is no "long-range order" (in the language of statistical mechanics). To the current author, each of these assumptions seems perfectionist. Issues before the Council may be systematically more popular in some States than in others, and such bias risks undermining either or both of the above two assumptions. The 'collective bias' model of Kirsch and others (see Section 2) is both more flexible and more empirical, at some cost to the square-root laws for influence and majority (see [22]). That said, no concrete proposal to displace square-root weights is made in the current work.

The identification of the 'exact' quota q^* of (3.6) hinges on the above assumptions, in combination with numerical data and the Gaussian approximation of Section 4.2. The last is unproven and numerically unreliable in the current context of the QMV2017 population data of the States of the EU. In their favour, the proposed square-root weights and the exact quota q^* have been derived via a set of principles that can be stated unambiguously and analysed rigorously, and which are robust with respect to changes in population data.

If the ratios $\overline{\beta}_j/\overline{w}_j$ in Table 1 are close to 1, then the total influences $I_j = \alpha_j\beta_j$ of (3.5) are almost constant across Member States. As indicated in the shaded columns of the table, this holds for both $q = 0$ and $q = q^*$ (they are nearly perfect when $q = q^*$, and very close for other values of q). *One may deduce that, from a practical point of view, there is little to choose between different values of q. This may be a situation in which political considerations may have the final word.*

Overall, the details of the JagCom rely on a number of assumptions that appear fragile. This potential weakness needs to be acknowledged when making the case for the JagCom. The JagCom is a valid proposal for the two-tier voting system of the Council of the EU, whose finer details may profit from input by politicians in choosing a system judged to serve well the needs of the nearly 500 million residents of the 27 Member States of the European Union. Our closing quote (Machover [26, Abs.] accords a balanced responsibility to both theoreticians and politicians: "This is essentially a political matter; but a political decision ought to be made in a theoretically enlightened way."

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