Probability
The Science of Uncertainty

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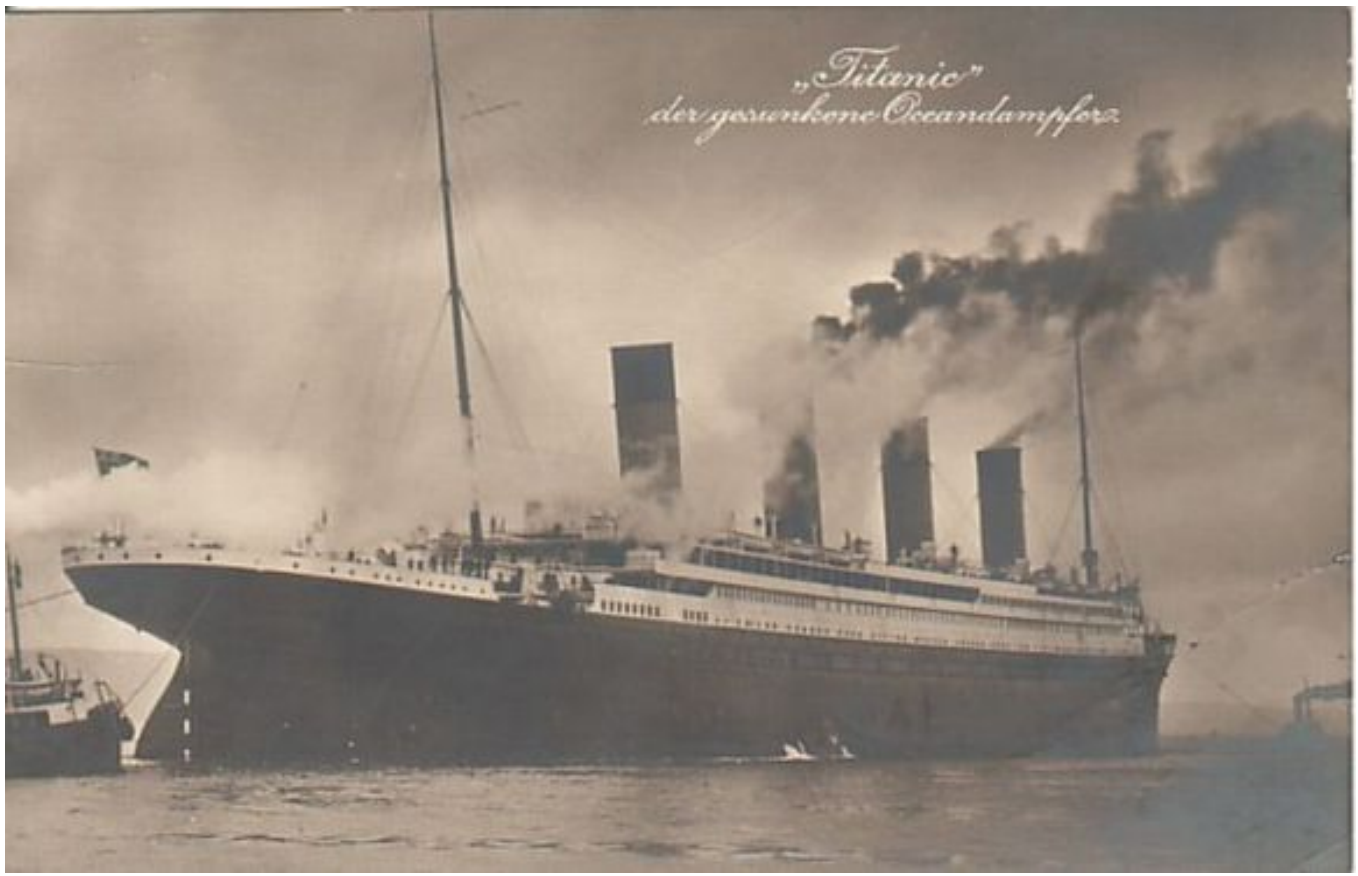
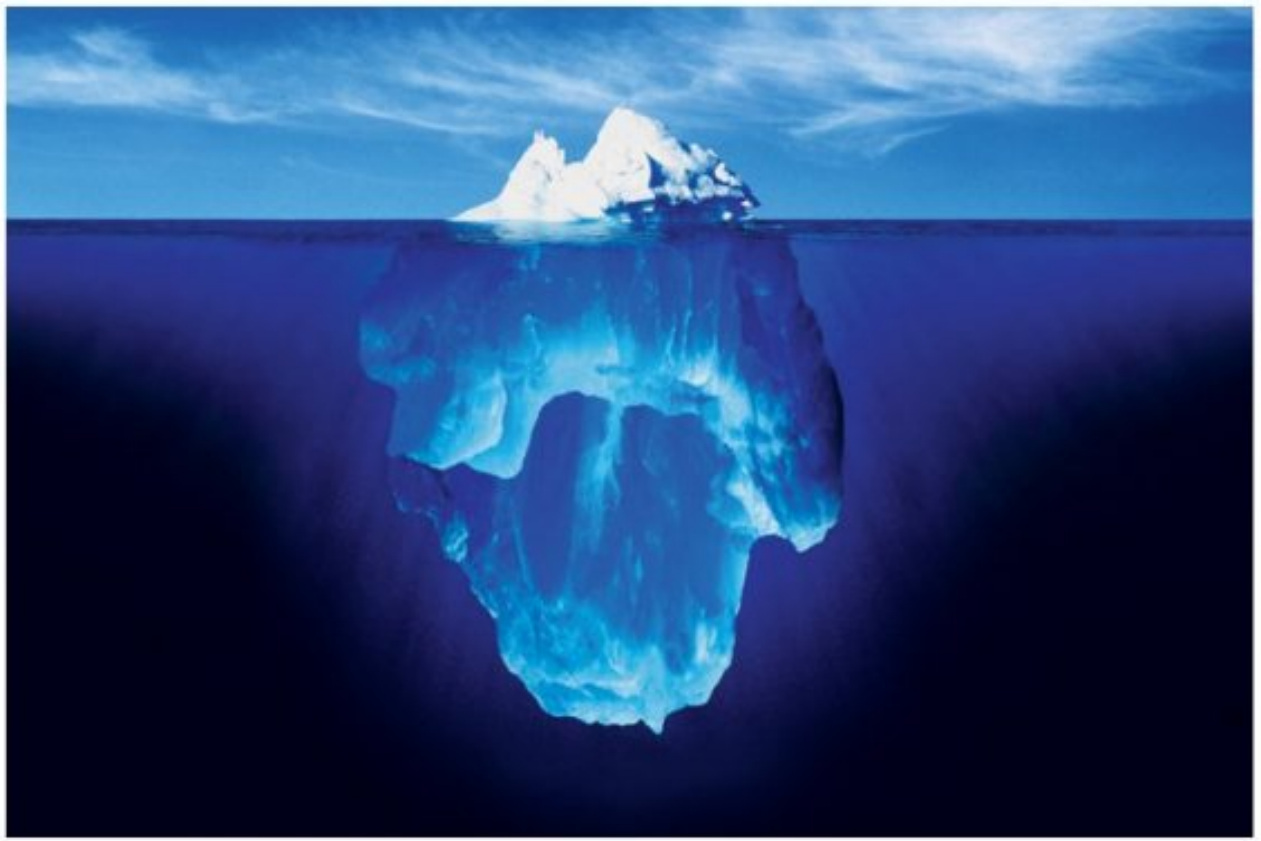
April 2012

Titanic, 1912

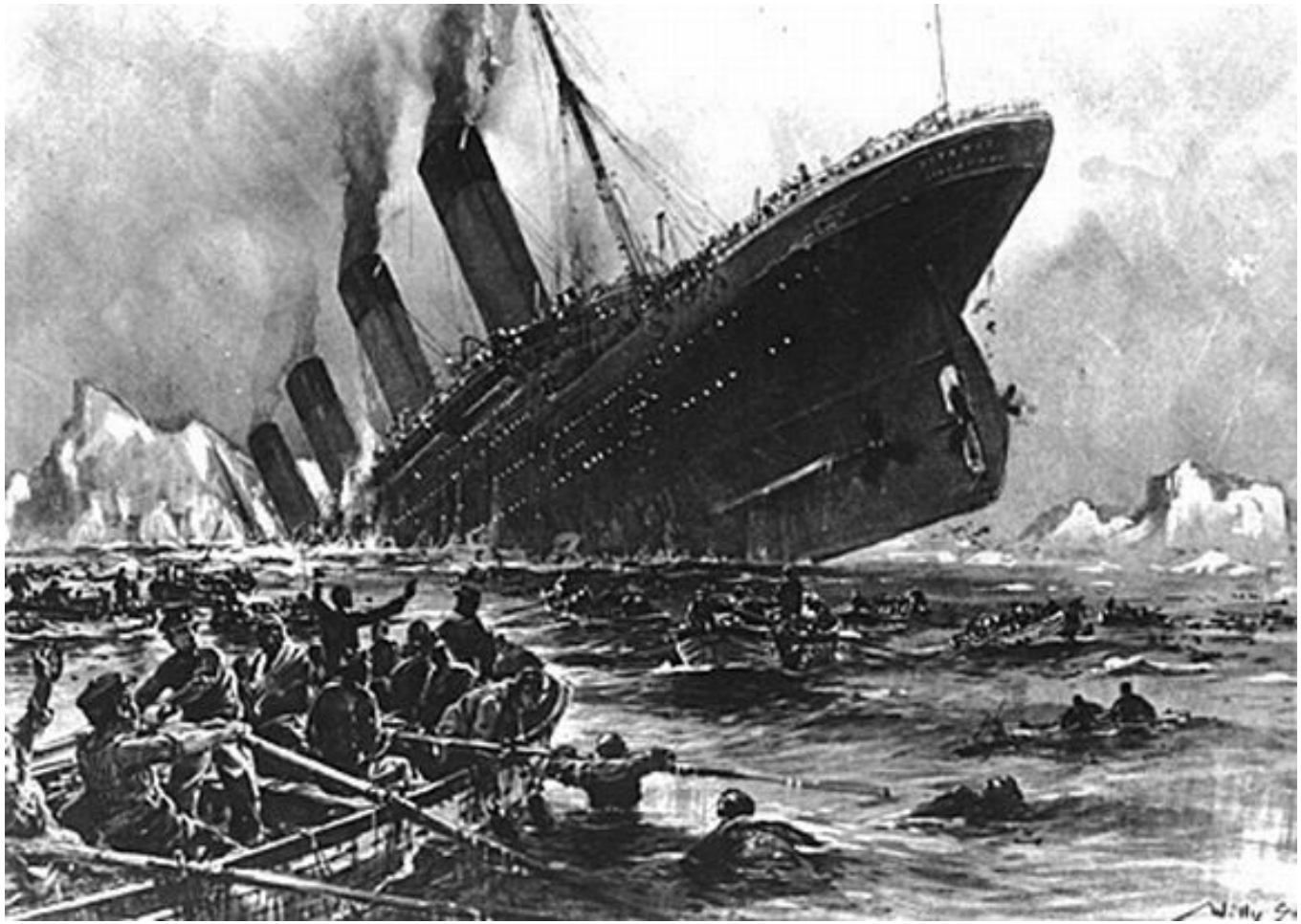


An iceberg



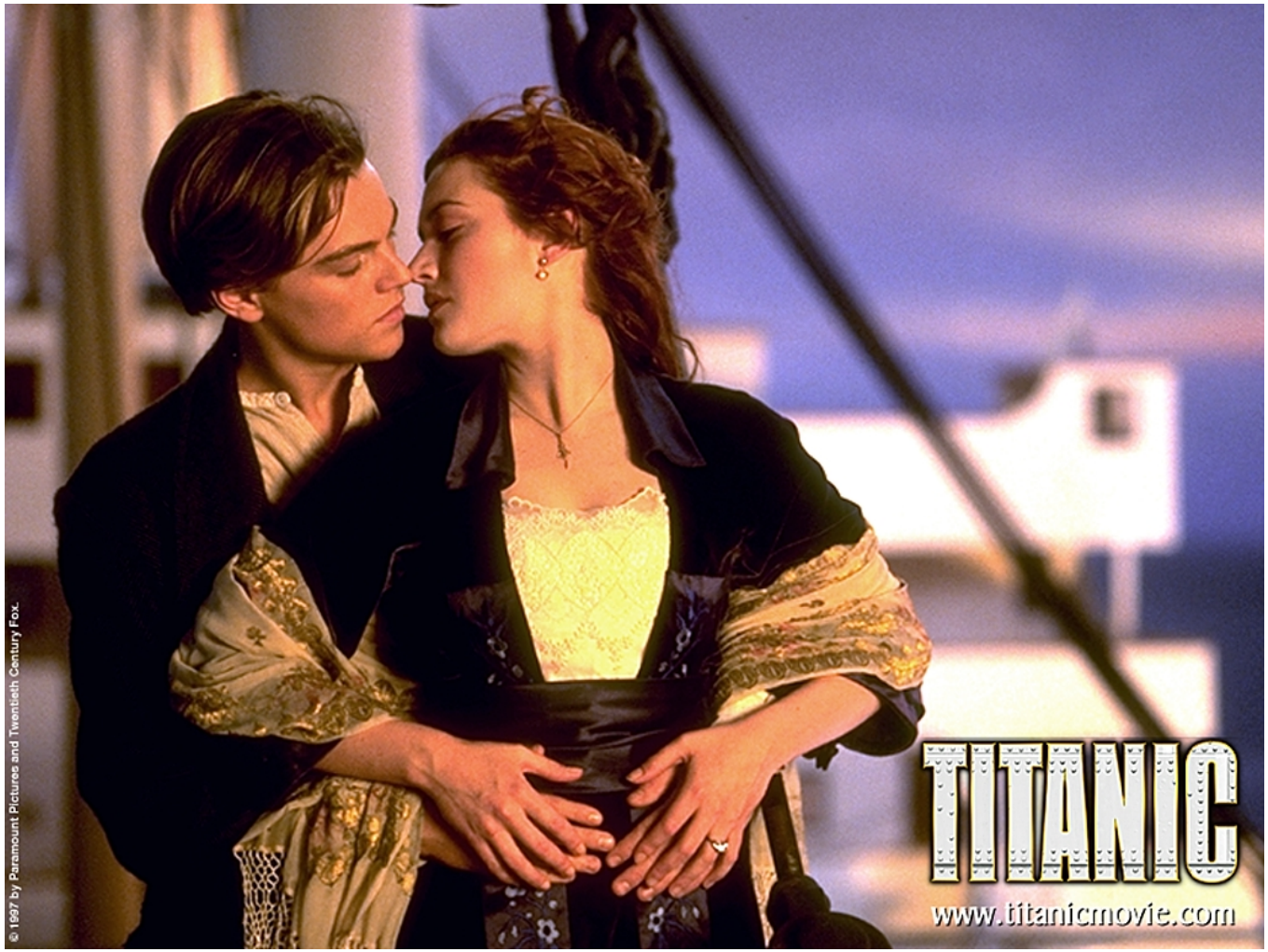
~90% below sea-level

Titanic, 15 April 1912



1,517 (of 2,223) people died

James Cameron gave us the movie



What is the probability of the Titanic sinking?

Guide: a number between 0 and 1,
with 0 meaning 'impossible', and 1 'certain'

Is it 1?

Is it 1/10? 1/1,000,000?

What is the probability
of the Titanic sinking?

- what does the question mean?
- what are the assumptions?
- the ‘covered coin’?

'Facts'

- beam = 30m
- in iceberg 'infested' waters for $\sim$ 100km
- inadequate watch
- there existed icebergs
- about 2250 people on board

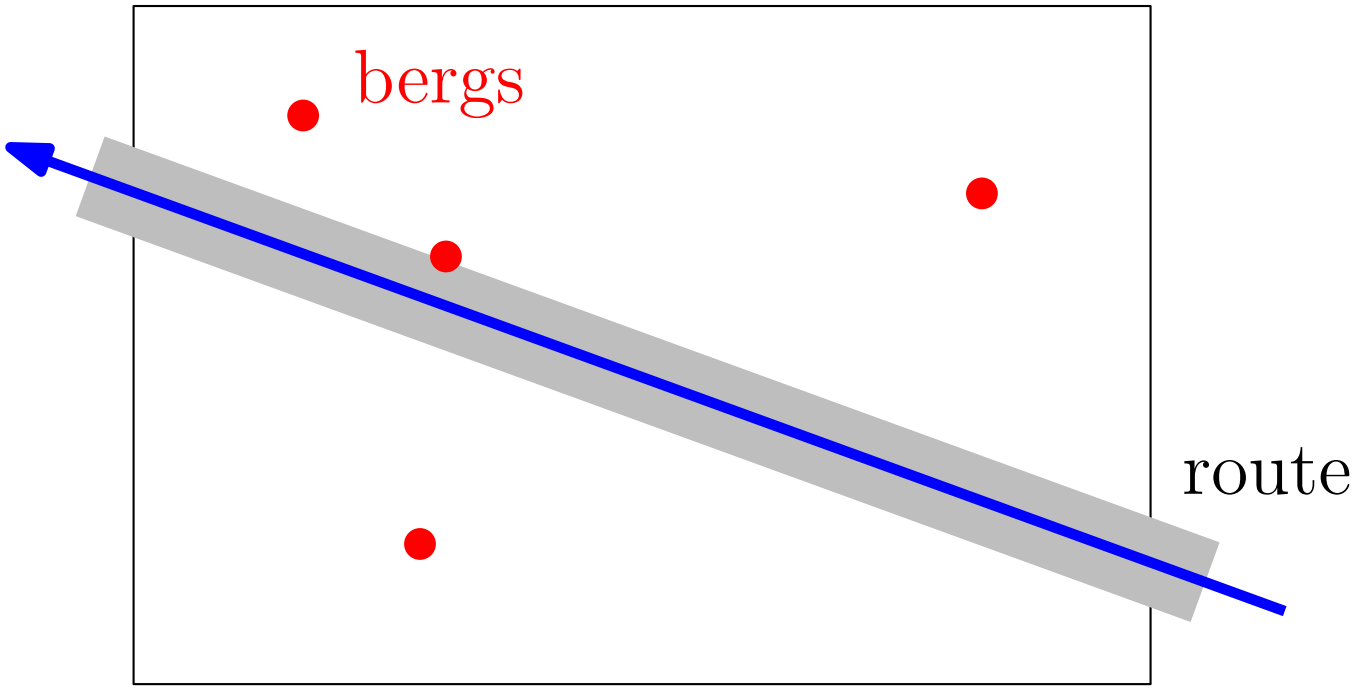
Question and Assumption

Question: What is the probability
of striking a large iceberg?

Assumption: Bergs are distributed at random

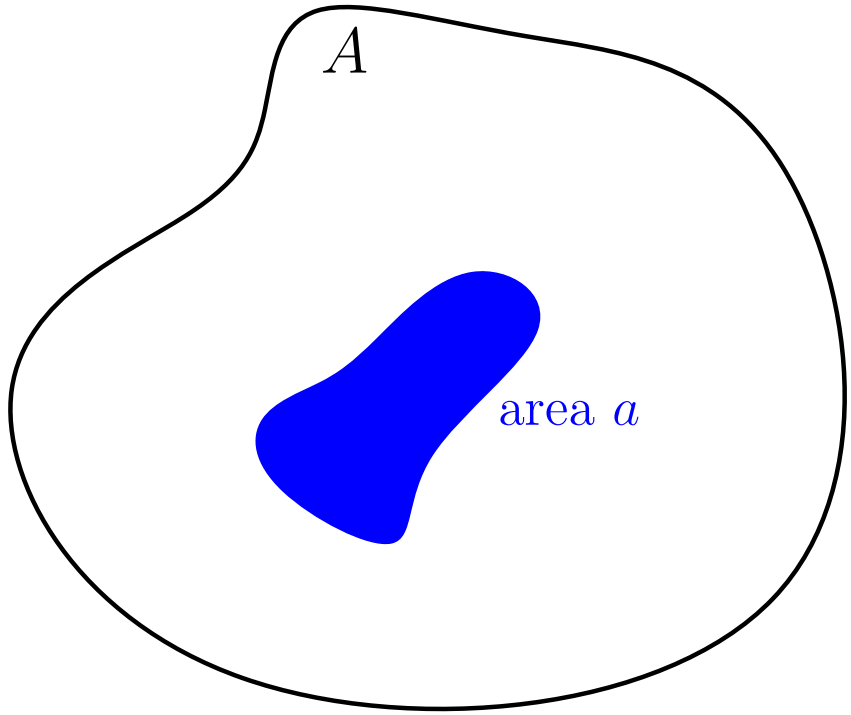
NOT: probability of sinking/rescue etc

Route through icebergs



- how many large icebergs?
- where are they?
- probability there is one intersecting the grey zone?

Poisson process



Drop points into area A at random

Density $= m$ points/unit area

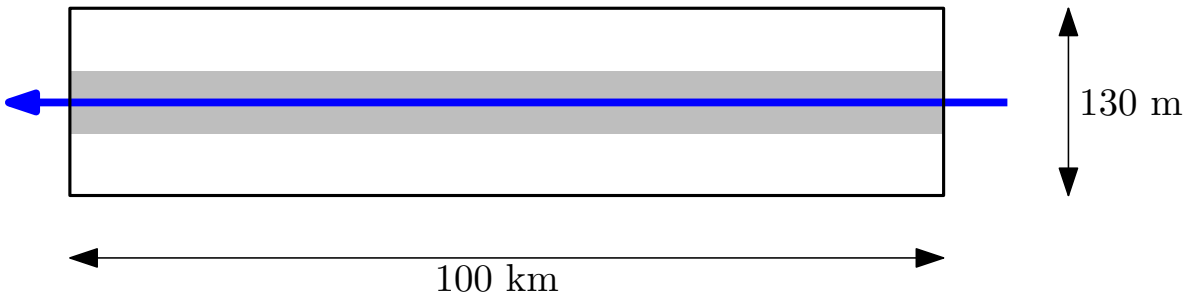
$$P\big(k \text{ points in area } a\big) = \frac{(ma)^k e^{-ma}}{k!}, \quad k = 0, 1, 2, \ldots$$

Siméon Denis Poisson (1781–1840)

"Life is good for only two things, discovering mathematics and teaching mathematics."

The calculation

Consider icebergs with diameter 100 m



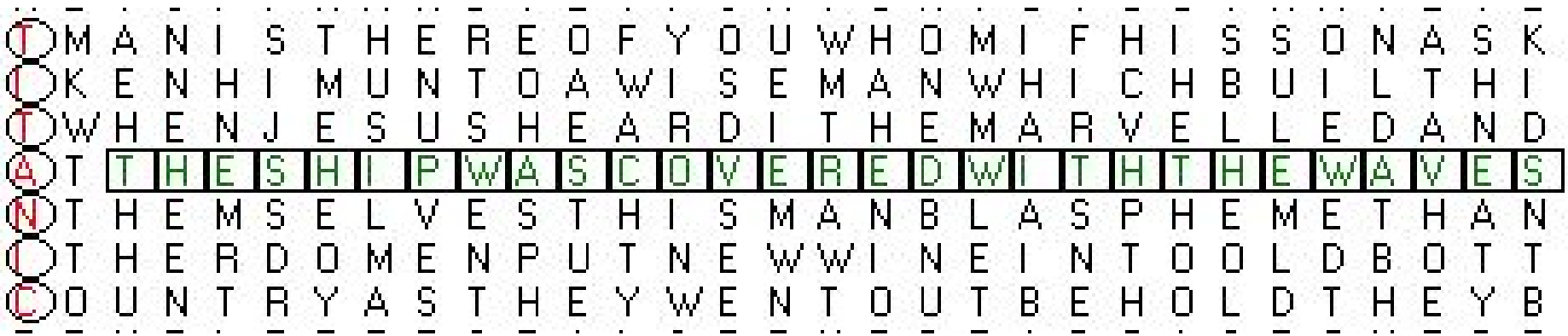
diameter \ #/deg^2	1	10
100m	$\frac{1}{800}$ 3	$\frac{1}{80}$ 30
150m	$\frac{1}{500}$ 5	$\frac{1}{50}$ 50

probability / mean number of fatalities in worst case

Discussion

- modelling assumptions
- input values
- robustness
- meaning of small probabilities
- chance/loss analysis

Titanic disaster foretold!



Bible, New Testament, King James edition

$$1611 < 1912$$

The Bible Codes

(1994) Eliyahu Rips, Doron Witztum

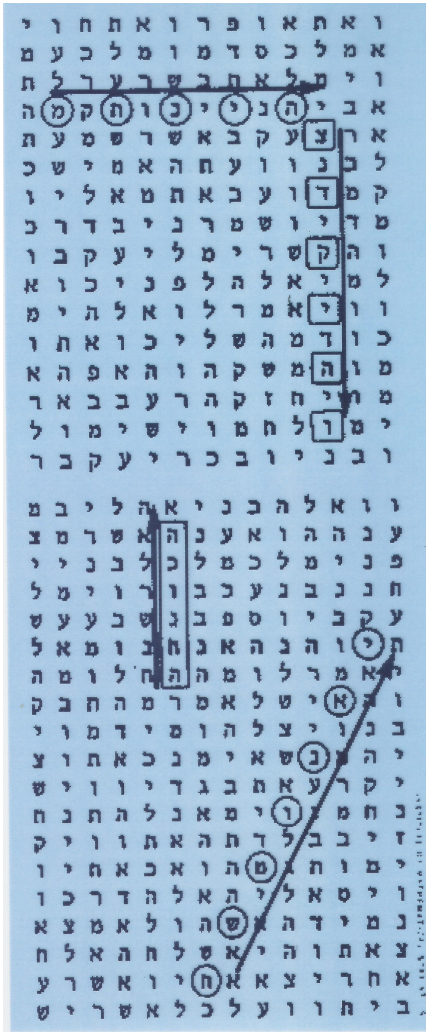
found names etc of 34 rabbis (as yet unborn) in the Torah

(1994) Michael Drosnin

found ASSASSIN WILL ASSASSINATE YITZHAK RABIN

(1995) Rabin assassinated

Source material



spelling?

multiplicities?

The Sun


USE YOUR SOCIAL SECURITY NUMBER TO WIN LOTTO
STARTLING NEW COMPUTER TRANSLATIONS WILL CHANGE YOUR LIFE FOREVER
Sun
HIDDEN BIBLE MESSAGES REVEALED!
• FAMILY • LOVE • HEALTH & how to find lasting peace of mind
• Seven secret signs of Jesus' return to Earth
• Warning of new Sodom & Gomorrah catastrophe
JIMMY HOFFA
WHERE HE REALLY IS TODAY

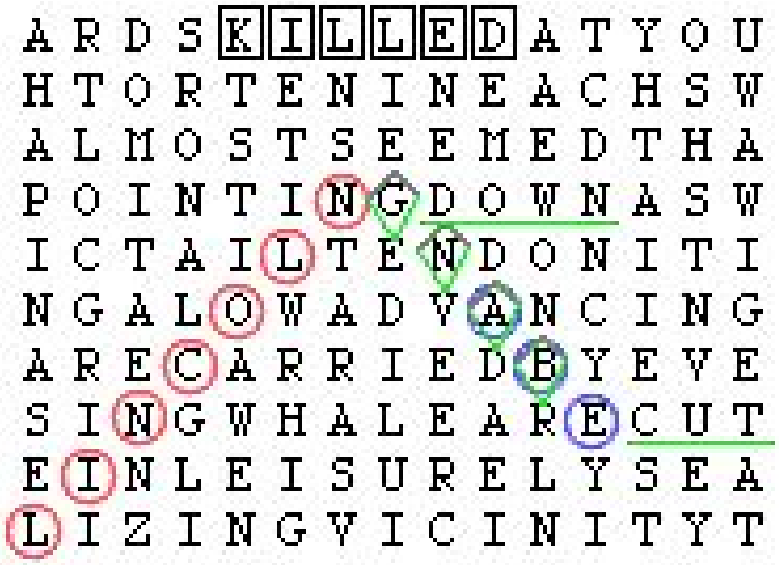
Furore!

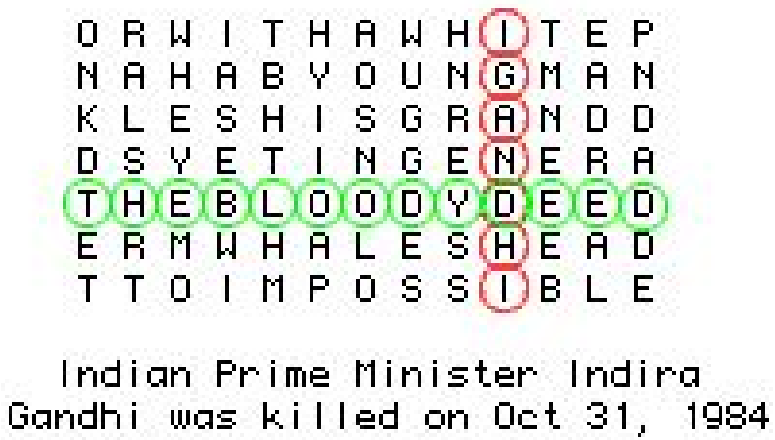
A. The Torah as the home of all truth. All shall be revealed.

B. What are the assumptions, where is the rigour?

Drosnin's challenge

"When my critics find a message about the assassination of a prime minister encrypted in Moby Dick, I'll believe them" [Newsweek, 9 June 1997]





etc, etc [Brendan MacKay]

Energy vs Entropy

Alphabet size A

Word length L

$$\text{1/Energy:} \quad P(\text{specific instance of word}) \approx \left(\frac{1}{A}\right)^L \quad \text{small}$$

$$\text{Entropy:} \quad \text{Number of possible instances } N \quad \text{large}$$

$$\text{mean number of occurrences} \sim N\left(\frac{1}{A}\right)^L$$

If $N \sim A^L$, then occurrence is not surprising

Blind Watchmaker (Paley/Dawkins), Physics

Probability ratio

$$\frac{P(\text{occurs} \mid \text{random})}{P(\text{occurs} \mid \text{intelligent design})} \approx \frac{E(\#\text{ occurrences} \mid \text{random})}{1}$$

$$\approx \frac{NA^{-L}}{1}$$

Relative likelihood under different hypotheses:

DNA profiling, courtroom evidence, false positives

Conditional probability?

Science of probability

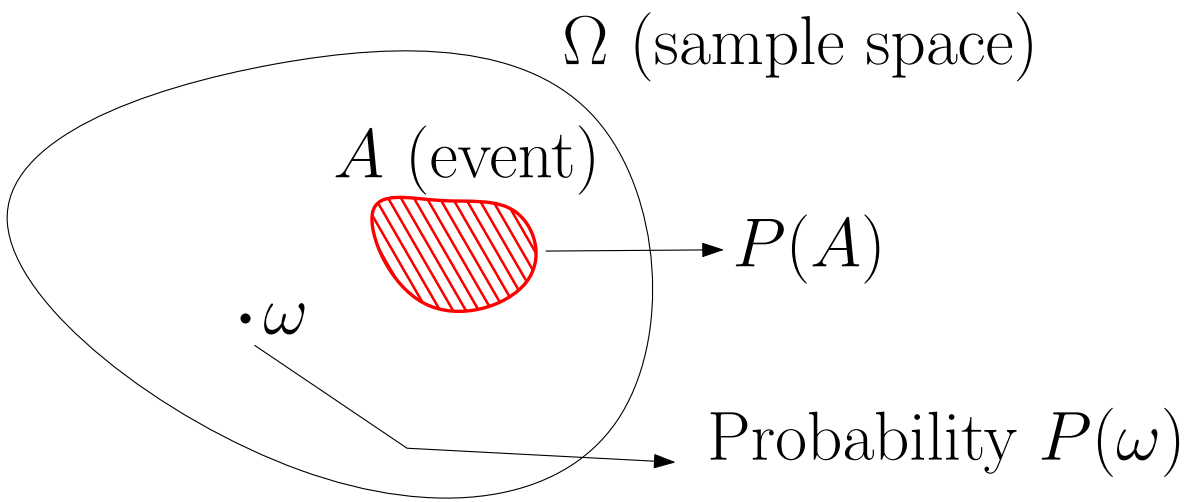
Experiment : coin flip

die throw

horse race

stock market

Outcomes : have probabilities



Example: two coin flips

$$\Omega = \{\mathsf{HH}, \mathsf{HT}, \mathsf{TH}, \mathsf{TT}\}$$
$$P(\omega) = p^2, p(1-p), (1-p)p, (1-p)^2$$

Assumptions: same coin

independence of outcomes

Probabilities

ABSOLUTE

$$P(A) = \text{Probability that } A \text{ occurs}$$

CONDITIONAL

$$P(A \mid B) = \text{Probability of } A \text{ given that } B \text{ occurs}$$

Definition of conditional probability

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

history, Pascal/Fermat

Accumulation of evidence in law

$$I := \text{innocence}$$
$$G := \text{guilt}$$
$$E_1, E_2, \ldots := \text{pieces of evidence } E$$

Likelihood/probability ratio:

$$\frac{P(\text{evidence} \mid G)}{P(\text{evidence} \mid I)} = \lambda$$

"Prosecutor's fallacy": CONVICT if λ is large

$$\lambda \approx 10^6,\ 10^{12},\ 10^{18}?$$

How large?

$$P(G \mid \text{evidence}) = P(\text{evidence} \mid G) \times \frac{P(G)}{P(\text{evidence})}$$

Relevant ratio:

$$\frac{P(G \mid \text{evidence})}{P(I \mid \text{evidence})} = \lambda \times \frac{P(G)}{1 - P(G)}$$

$$\approx \lambda \times \frac{1}{\text{number } N \text{ of possible perpetrators}}$$

Conclusion: λ must be much bigger than N

disputed paternity

Conditional probability

Lewis Carroll 'proved' that, if an urn contains two balls, each either red or blue, then one is red and one is blue.

$$P(\mathrm{RR}) = P(\mathrm{RB}) = P(\mathrm{BR}) = P(\mathrm{BB}) = \tfrac{1}{4}$$

Add a red ball:

$$P(\mathrm{RRR}) = P(\mathrm{RRB}) = P(\mathrm{RBR}) = P(\mathrm{RBB}) = \tfrac{1}{4}$$
$$P(\text{random ball is red}) = 1 \cdot \tfrac{1}{4} + \tfrac{2}{3} \cdot \tfrac{1}{4} + \tfrac{2}{3} \cdot \tfrac{1}{4} + \tfrac{1}{3} \cdot \tfrac{1}{4} = \tfrac{2}{3}$$

Conclusion: Must have RRB, so originally RB.

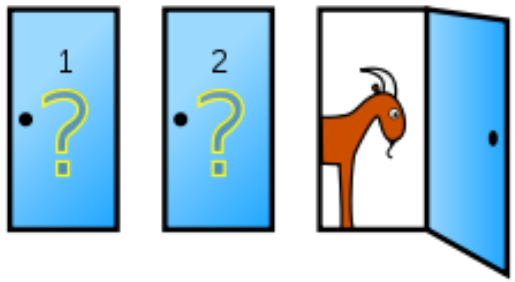
Family planning

A family has two children.

What is the probability that both are boys, given that one is a boy?

An ill-posed question

Prisoner (Monty Hall) problem



Three prisoners, A, B, C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warder knows which one is to be pardoned, but is not allowed to tell that person. Prisoner A asks the warder the identity of one of the others who is going to be executed.

The warder tells A that B is to be executed.

Without information: $P(A \text{ pardoned}) = \frac{1}{3}$

With information: $P(A \text{ pardoned}) = \frac{1}{3}, \ \frac{1}{2}$?

Another ill-posed question

Two 'modern' applications

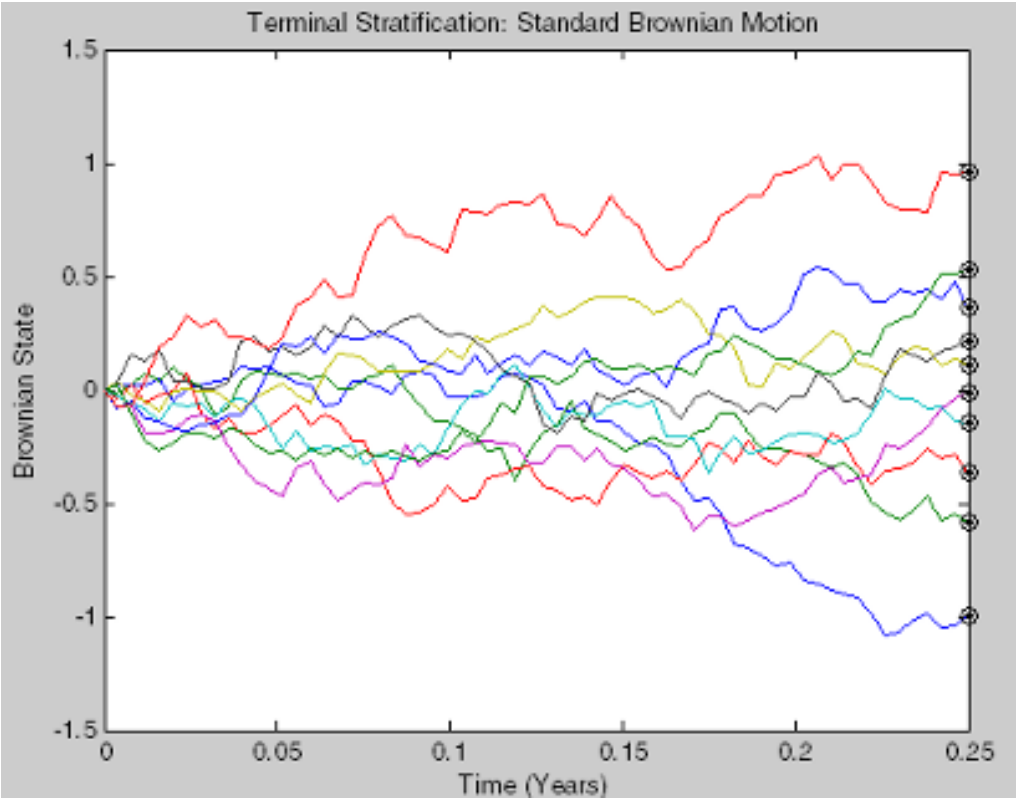
A. Mathematical finance

B. Ising model for magnetism

Brownian motion ($\sim$ 1827)

Robert Brown (1773–1858)

BM in one dimension



continuous, non-differentiable function, Gaussian law

'totally random motion', fractal structure,

modelling tool (finance, Black–Scholes etc)

Bonds and stocks

bonds: M_t at time t

$$M_t = e^{rt}$$

stocks: S_t at time t

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

European stock option: buy one unit of stock at future time T for price K.

Question: What is this option worth?

Assume absence of arbitrage: no risk-free profits are available

Black–Scholes formula

Theorem

The value of the European stock option at time t ($< T$) is the known function $V_t(S_t, r, K, \dots)$

Proof.

Uses the beautiful Cameron–Martin–Girsanov 'change of measure' formula. □

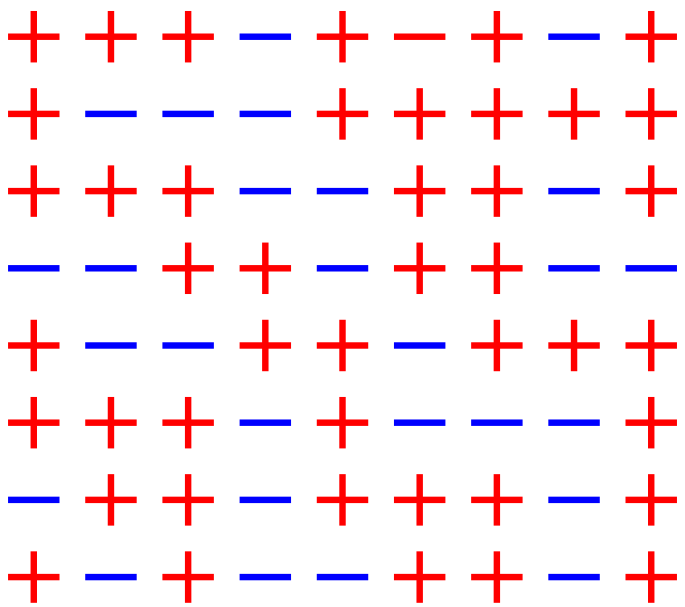
Energy vs entropy in physics

Pierre Curie: Iron retains its magnetism only at temperatures

$$T < T_c$$

Critical temperature: $T_c \approx 770\,\text{deg C}$

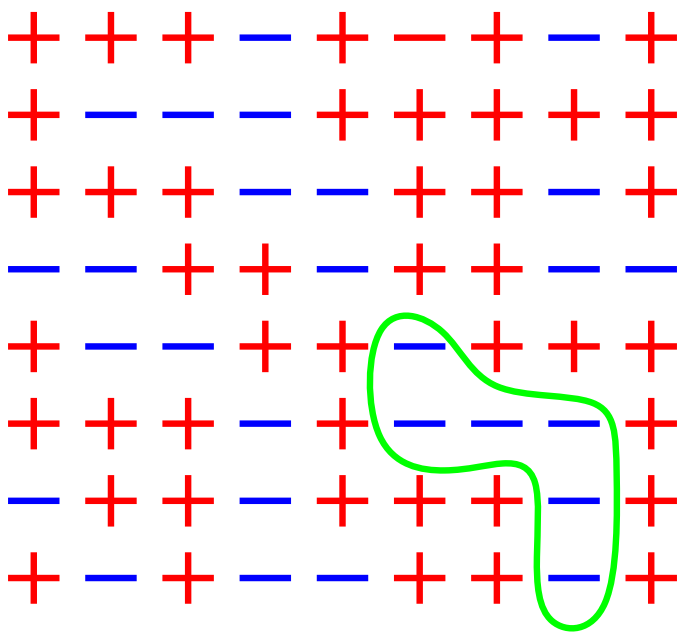
Ising model (1925)



number of molecules $\sim 10^{23} = N$, Avogadro's number

number of configurations $\sim 2^N$, large entropy

Hamiltonian/energy



Hamiltonian: configuration σ has (large) energy $H(\sigma)$
$=$ total length of interfaces

probability of configuration $Ce^{-H(\sigma)/T}$, tiny

Balance entropy/energy, to prove results about the model

Maths of ferromagnetism

Existence of critical point

$T > T_c$: external field is forgotten

$T < T_c$: external field is remembered

Near criticality, $T \approx T_c$: singularity of power-type
critical exponents
universality
confomality in two dimensions

Proofs: (2D) Onsager (1944) ... Smirnov (2011) +
(3D)?

What are dice?

A die is a small polkadotted cube made of ivory and trained like a lawyer to lie on the wrong side (Ambrose Bierce, 1911)