Logarithmic Fluctuations From Circularity

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Harry Kesten's 80th Birthday Conference

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Talk Outline

- Part 1: Logarithmic fluctuations
- Part 2: Limiting shapes
- Part 3: Integrality wreaks havoc

- Part 1: Joint work with David Jerison and Scott Sheffield.
- Parts 2 & 3: Joint work with Anne Fey and Yuval Peres.

Part 1: Logarithmic Fluctuations

Internal DLA

- Start with n particles at the origin in the square grid $\mathbb{Z}^2$.
- Each particle in turn performs a simple random walk until it finds an unoccupied site, stays there.

From random walk to growth model

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$$A(n+1) = A(n) \cup \{X^n(\tau^n)\}$$

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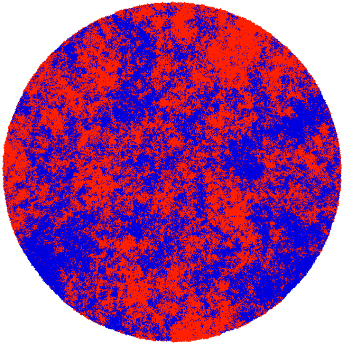
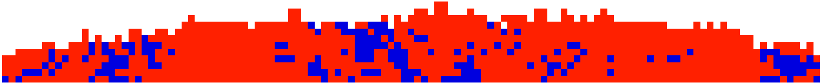
$$A(n+1) = A(n) \cup \{X^n(\tau^n)\}$$

where $X^1, X^2, \ldots$ are independent random walks, and

$$\tau^n = \min\{t \mid X^n(t) \notin A(n)\}.$$



Internal DLA cluster in $\mathbb{Z}^2$.



Closeup of the boundary.

Questions

- Limiting shape
- Fluctuations

Meakin & Deutch, J. Chem. Phys. 1986

- "It is also of some fundamental significance to know just how smooth a surface formed by diffusion limited processes may be."

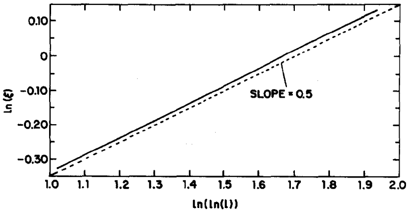


FIG. 2. Dependence of the variance of the surface height (ξ) on the strip width l for two-dimensional (square lattice) diffusion limited annihilation in the long time ($\bar{h} \gg l$) limit.

- "Initially, we plotted $\ln(\xi)$ vs $\ln(\ell)$ but the resulting plots were quite noticably curved. Figure 2 shows the dependence of $\ln(\xi)$ on $\ln[\ln(\ell)]$."

History of the Problem

- **Diaconis-Fulton 1991**: Addition operation on subsets of $\mathbb{Z}^d$.
- **Lawler-Bramson-Griffeath 1992**: w.p.1,

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- **Lawler 1995**: w.p.1,

$$\mathbf{B}_{r-r^{1/3}\log^2 r} \subset A(\pi r^2) \subset \mathbf{B}_{r+r^{1/3}\log^4 r} \quad \text{eventually.}$$

"A more interesting question... is whether the errors are $o(n^\alpha)$ for some $\alpha < 1/3$."

Logarithmic Fluctuations Theorem

Jerison - L. - Sheffield 2010: with probability 1,

$$\mathbf{B}_{r-C\log r} \subset A(\pi r^2) \subset \mathbf{B}_{r+C\log r} \quad \text{eventually.}$$

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Asselah - Gaudillière 2010 independently obtained

$$\mathbf{B}_{r-C\log r} \subset A(\pi r^2) \subset \mathbf{B}_{r+C\log^2 r} \quad \text{eventually.}$$

Logarithmic Fluctuations in Higher Dimensions

In dimension $d \geq 3$, let ω_d be the volume of the unit ball in $\mathbb{R}^d$. Then with probability 1,

$$\mathbf{B}_{r - C\sqrt{\log r}} \subset A(\omega_d r^d) \subset \mathbf{B}_{r + C\sqrt{\log r}} \quad \text{eventually}$$

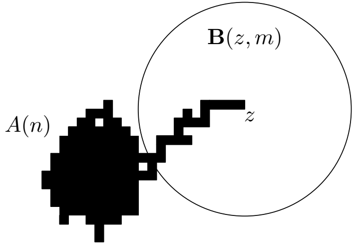
for a constant C depending only on d.

(Jerison - L. - Sheffield 2010; Asselah - Gaudillière 2010)

Elements of the proof

- Thin tentacles are unlikely.
- Martingales to detect fluctuations from circularity.
- "Self-improvement"

Thin tentacles are unlikely



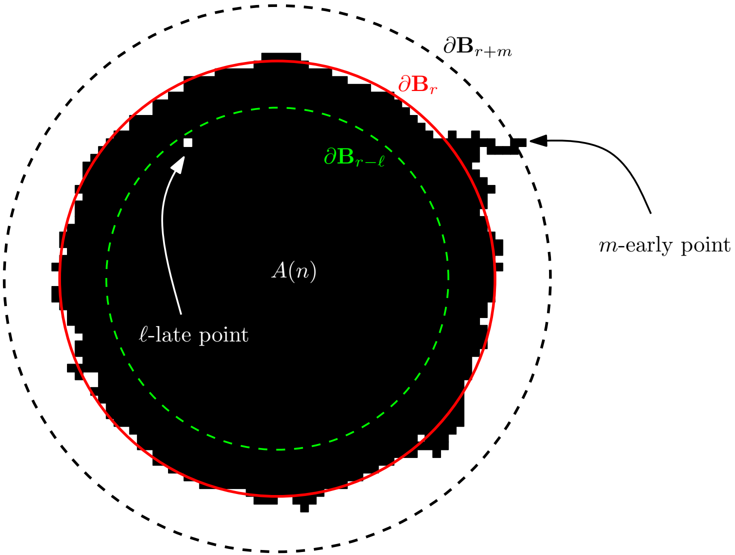
A thin tentacle.

Lemma. If $0 \notin \mathbf{B}(z,m)$, then

$$\mathbb{P}\left\{z \in A(n),\ \#(A(n) \cap \mathbf{B}(z,m)) \leq bm^d\right\} \leq \begin{cases} Ce^{-cm^2/\log m}, & d = 2 \\ Ce^{-cm^2}, & d \geq 3 \end{cases}$$

for constants $b, c, C > 0$ depending only on the dimension d.

Early and late points in $A(n)$, for $n = \pi r^2$



Early and late points

Definition 1. z is an m-early point if:

$$z \in A(n), \quad n < \pi(|z| - m)^2$$

Definition 2. z is an ℓ-late point if:

$$z \notin A(n), \quad n > \pi(|z| + \ell)^2$$

$\mathcal{E}_m[n] =$ event that some point in $A(n)$ is m-early

$\mathcal{L}_\ell[n] =$ event that some point in $\mathbf{B}_{\sqrt{n/\pi}-\ell}$ is ℓ-late

Structure of the argument: Self-improvement

LEMMA 1. No ℓ-late points implies no m-early points:
If $m \geq C\ell$, then

$$\mathbb{P}(\mathcal{E}_m[n] \cap \mathcal{L}_\ell[n]^c) < n^{-10}.$$

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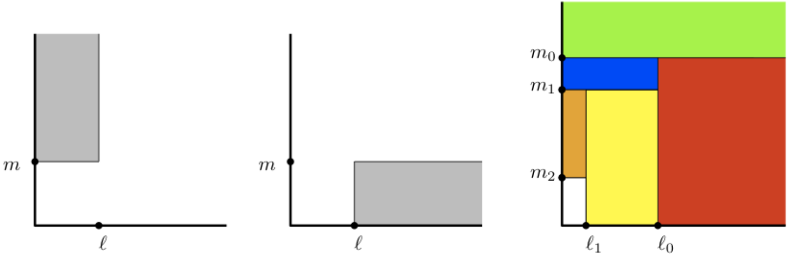
LEMMA 2. No m-early points implies no ℓ-late points:
If $\ell \geq \sqrt{C(\log n)m}$, then

$$\mathbb{P}(\mathcal{L}_\ell[n] \cap \mathcal{E}_m[n]^c) < n^{-10}.$$

Iterate, $\ell \mapsto \sqrt{C(\log n)C\ell}$, which is decreasing until

$$\ell = C^2 \log n.$$

Iterating Lemmas 1 and 2



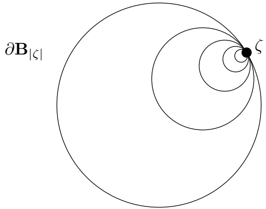
- Fix n and let ℓ, m be the maximal lateness and earliness occurring by time n. Iterate starting from $m_0 = n$:
- (ℓ, m) unlikely to belong to a vertical rectangle by Lemma 1.
- (ℓ, m) unlikely to belong to a horizontal rectangle by Lemma 2.

Early and late point detector

To detect early points near $\zeta \in \mathbb{Z}^2$, we use the martingale

$$M_\zeta(n) = \sum_{z \in \widetilde{A}(n)} \left(H_\zeta(z) - H_\zeta(0)\right)$$

where H_ζ is a discrete harmonic function approximating $\mathrm{Re}\left(\frac{\zeta/|\zeta|}{\zeta - z}\right)$.



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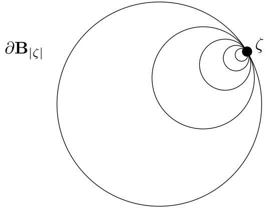

The fine print:

- Discrete harmonicity fails at three points $z = \zeta, \zeta+1, \zeta+1+i$.
- Modified growth process $\widetilde{A}(n)$ stops at $\partial B_{|\zeta|}(0)$.

Time change of Brownian motion

- To get a *continuous time* martingale, we use Brownian motions on the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$ instead of random walks.
- Then there is a standard Brownian motion B_ζ such that

$$M_\zeta(t) = B_\zeta(s_\zeta(t))$$

where

$$s_\zeta(t) = \lim \sum_{i=1}^{N} (M(t_i) - M(t_{i-1}))^2$$

is the quadratic variation of M_ζ.

LEMMA 1. No ℓ-late implies no $m = C\ell$-early

Event $Q[z,k]$:

- $z \in A(k) \setminus A(k-1)$.
- z is m-early ($z \in A(\pi r^2)$ for $r = |z| - m$).
- $\mathcal{E}_m[k-1]^c$: No previous point is m-early.
- $\mathcal{L}_\ell[n]^c$: No point is ℓ-late.

We will use M_ζ for $\zeta = (1 + 4m/r)z$ to show for $0 < k \le n$,

$$\mathbb{P}(Q[z,k]) < n^{-20}.$$

Main idea: Early but no late would be a large deviation!

- Recall there is a Brownian motion B_ζ such that

$$M_\zeta(n) = B_\zeta(s_\zeta(n)).$$

- On the event $Q[z,k]$

$$\mathbb{P}\left(M_\zeta(k) > c_0 m\right) > 1 - n^{-20} \tag{1}$$

and

$$\mathbb{P}\left(s_\zeta(k) < 100 \log n\right) > 1 - n^{-20}. \tag{2}$$

- On the other hand, $(s = 100 \log n)$

$$\mathbb{P}\left(\sup_{s' \in [0,s]} B_\zeta(s') \geq s\right) \leq e^{-s/2} = n^{-50}.$$

Proof of (1)

On the event $Q[z,k]$

$$\mathbb{P}(M_\zeta(k) > c_0 m) > 1 - n^{-20}.$$

- Since $z \in A(k)$ and thin tentacles are unlikely, we have with high probability,

$$\#(A(k) \cap B(z,m)) \geq bm^2.$$

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- For each of these bm^2 points, the value of H_ζ is order $1/m$, so their total contribution to $M_\zeta(k)$ is order m.
- No ℓ-late points means that points elsewhere cannot compensate.

Proof of (2): Controlling the Quadratic Variation

On the event $Q[z,k]$

$$\mathbb{P}(s_\zeta(k) < 100 \log n) > 1 - n^{-20}.$$

- Lemma: There are independent standard Brownian motions $B^1, B^2, \ldots$ such that

$$s_\zeta(i+1) - s_\zeta(i) \le \tau_i$$

where τ_i is the first exit time of B^i from the interval (a_i, b_i).

$$a_i = \min_{z \in \partial \tilde{A}(i)} H_\zeta(z) - H_\zeta(0)$$

$$b_i = \max_{z \in \partial \tilde{A}(i)} H_\zeta(z) - H_\zeta(0).$$

Proof of (2): Controlling the Quadratic Variation

On the event $Q[z,k]$

$$\mathbb{P}(s_\zeta(k) < 100 \log n) > 1 - n^{-20}.$$

- By independence of the τ_i,

$$\mathbb{E}e^{s_\zeta(k)} \leq \mathbb{E}e^{(\tau_1 + \cdots + \tau_k)} = (\mathbb{E}e^{\tau_1}) \cdots (\mathbb{E}e^{\tau_k}).$$

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- Easy to estimate a_i, and use the fact that no previous point is m-early to bound b_i. Conclude that

$$\mathbb{E}\left[e^{s_\zeta(k)} 1_Q\right] \leq n^{50}.$$

What changes in higher dimensions?

- In dimension $d \geq 3$ the quadratic variation $s_\zeta(n)$ is constant order instead of $\log n$.
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- So the fluctuations are instead dominated by thin tentacles, which can grow to length $\sqrt{\log n}$.
- Still open: prove matching lower bounds on the fluctuations of order $\log n$ in dimension 2 and $\sqrt{\log n}$ in dimensions $d \geq 3$.

Part 2: Limiting Shapes

Internal DLA with Multiple Sources

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- Get a **random set** of km occupied sites in $\mathbb{Z}^d$.
- The distribution of this random set does not depend on the order of the walks (**Diaconis-Fulton '91**).

Questions

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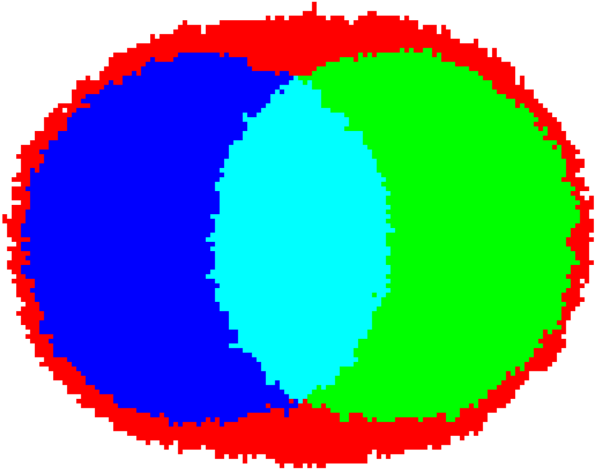
- Recall from part 1: If $k = 1$, then the limiting shape is a ball in $\mathbb{R}^d$. (**Lawler-Bramson-Griffeath '92**)



Two-source internal DLA cluster built from overlapping single-source clusters.

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- To form $A + B$, let $C_0 = A \cup B$ and

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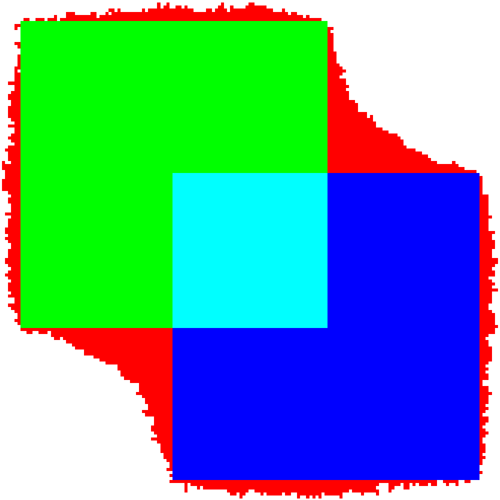
- Define $A + B = C_k$.
- **Abeilan property**: the law of $A + B$ does not depend on the ordering of $y_1, \dots, y_k$.



Diaconis-Fulton sum of two squares in $\mathbb{Z}^2$ overlapping in a smaller square.

Divisible Sandpile

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- As $t \to \infty$, get a limiting region $A \oplus B \subset \mathbb{Z}^d$ of sites with mass 1.
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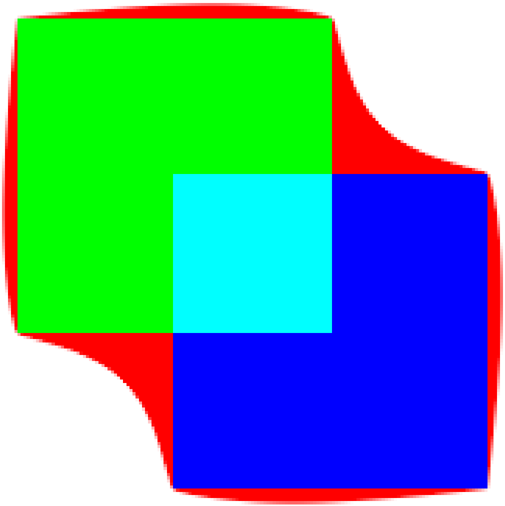
- Abelian property: $A \oplus B$ does not depend on the choices.



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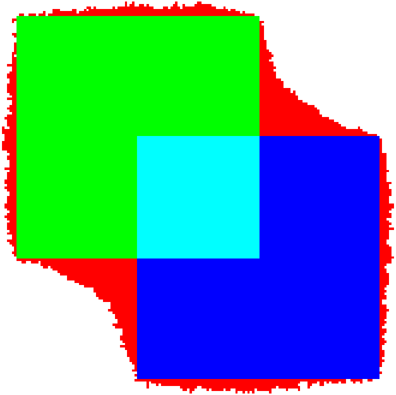

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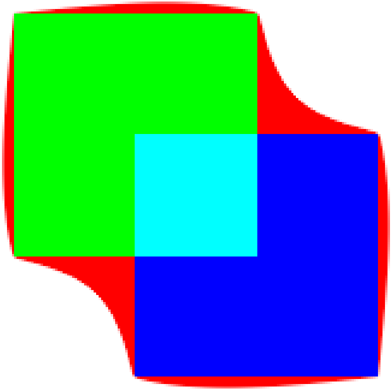

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- Boundary condition: $u = 0$ on $\partial(A \oplus B)$.
- Need additional information to determine the domain $A \oplus B$.

Free Boundary Problem

- Unknown function u, unknown domain $D = \{u > 0\}$.

$$u \geq 0$$
$$\Delta u \leq 1 - 1_A - 1_B$$

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$$
\begin{aligned}
& u \geq 0 \\
& \Delta u \leq 1 - 1_A - 1_B \\
& u(\Delta u - 1 + 1_A + 1_B) = 0.
\end{aligned}
$$

The Obstacle Problem

- Given $A, B \subset \mathbb{Z}^d$, we define the "obstacle:"

$$\gamma(x) = -|x|^2 - \sum_{y \in A} g(x,y) - \sum_{y \in B} g(x,y),$$

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(In $\mathbb{Z}^2$, we use the negative of the potential kernel instead.)

- Let $s(x) = \inf\{\phi(x) \mid \phi \text{ is superharmonic on } \mathbb{Z}^d \text{ and } \phi \geq \gamma\}$.

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where g is the Green function for simple random walk

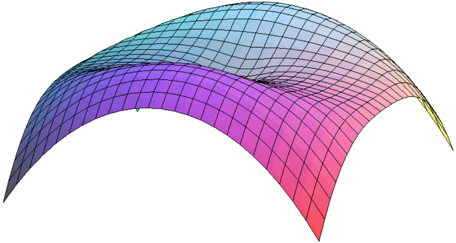
$$g(x,y) = \mathbb{E}_x \#\{k | X_k = y\}.$$

(In $\mathbb{Z}^2$, we use the negative of the potential kernel instead.)

- Let $s(x) = \inf\{\phi(x) \mid \phi \text{ is superharmonic on } \mathbb{Z}^d \text{ and } \phi \geq \gamma\}$.

- Then the $\boxed{\text{odometer function} = s - \gamma.}$

- Obstacle for two overlapping disks A and B:



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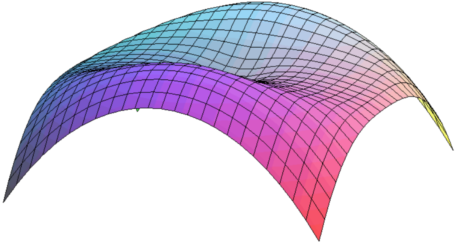

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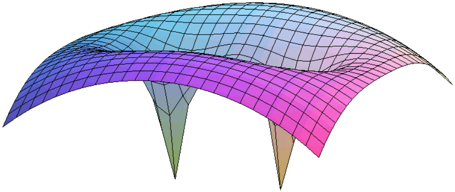
- Obstacle for two point sources x_1 and x_2:



$$\gamma(x) = -|x|^2 - g(x,x_1) - g(x,x_2)$$

The Smash Sum of Two Domains in $\mathbb{R}^d$

- $A, B \subset \mathbb{R}^d$ bounded open sets such that $\partial A, \partial B$ have zero d-dimensional Lebesgue measure.

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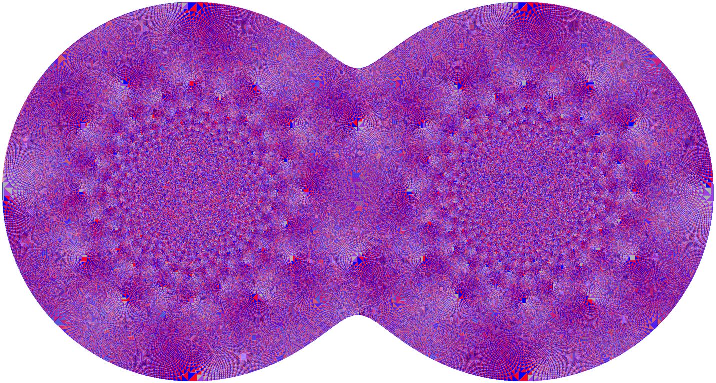

The smash sum

$$A \oplus B = A \cup B \cup \{s > \gamma\}$$

of two overlapping disks $A, B \subset \mathbb{R}^2$.

Properties of the Smash Sum

- $A \cup B \subset A \oplus B$.
- Associativity: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.
- Volume conservation: $\mathrm{vol}(A \oplus B) = \mathrm{vol}(A) + \mathrm{vol}(B)$.

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- **Quadrature identity**: If h is an integrable superharmonic function on $A \oplus B$, then

$$\int_{A \oplus B} h(x)dx \leq \int_A h(x)dx + \int_B h(x)dx.$$

Scaling Limit of the Discrete Models

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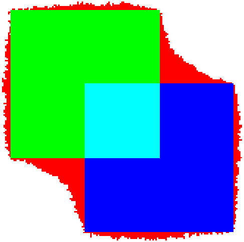
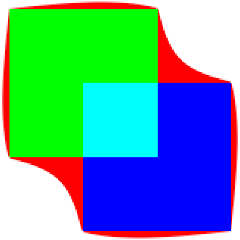
  - Convergence is in the sense of ε-neighborhoods: for all $\varepsilon > 0$

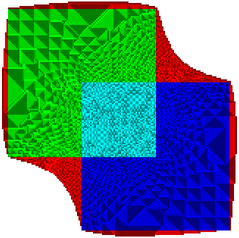
$$(A \oplus B)_{\varepsilon}^{::} \subset D_n, R_n, I_n \subset (A \oplus B)^{\varepsilon ::} \qquad \text{for all sufficiently large } n.$$



Internal DLA



Divisible Sandpile



Rotor-Router Model

Part 3: Integrality wreaks havoc

The Abelian Sandpile as a Growth Model

- Start with a pile of n chips at the origin in $\mathbb{Z}^d$.
- Each site $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ has $2d$ neighbors

$$x \pm e_i, \qquad i = 1, \dots, d.$$

- Any site with at least $2d$ chips is unstable, and topples by sending one chip to each neighbor.

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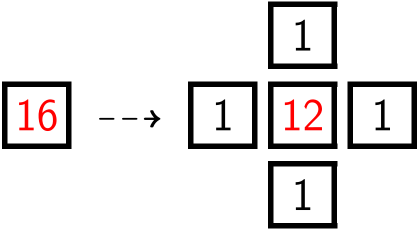
- This may create further unstable sites, which also topple.
- Continue until there are no more unstable sites.

Toppling to Stabilize A Sandpile

- Example: n=16 chips in $\mathbb{Z}^2$.
- Sites with 4 or more chips are unstable.



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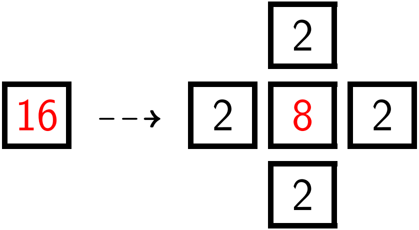

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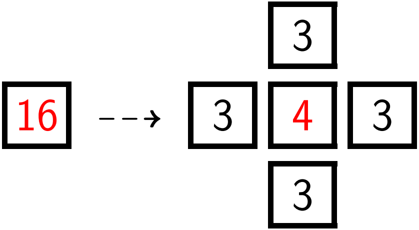

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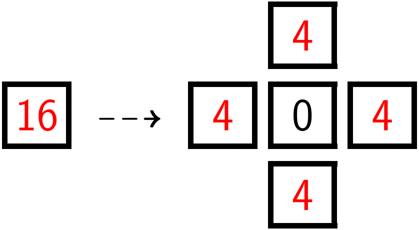

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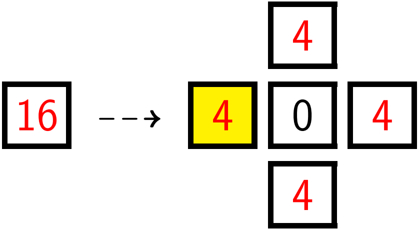

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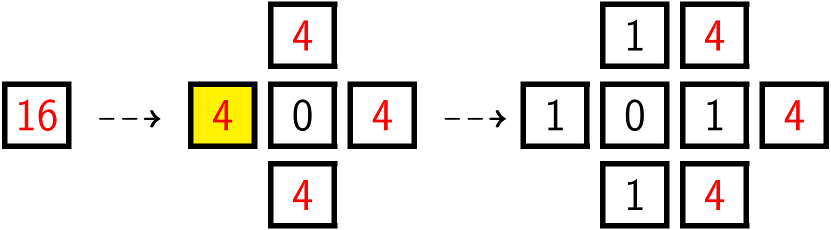

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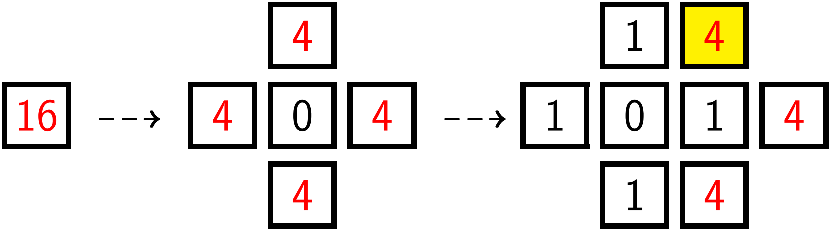

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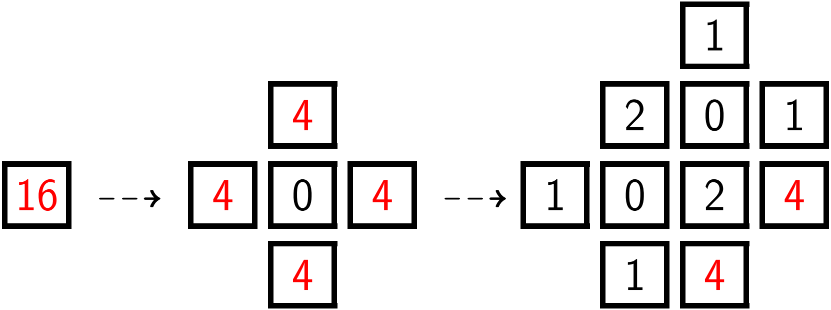

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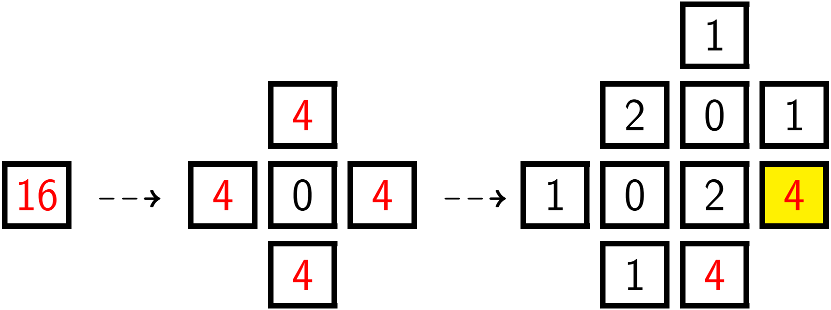

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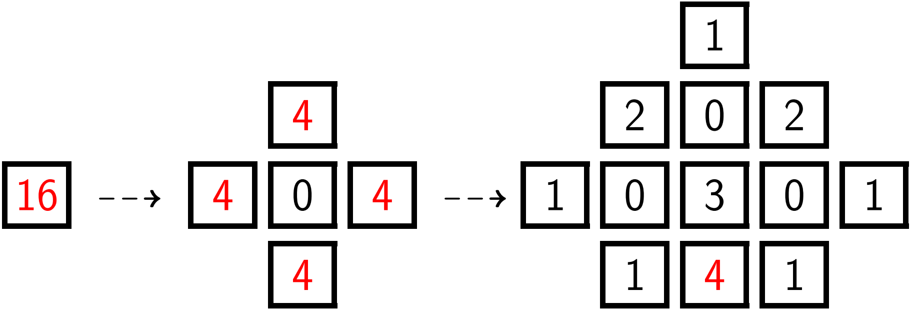

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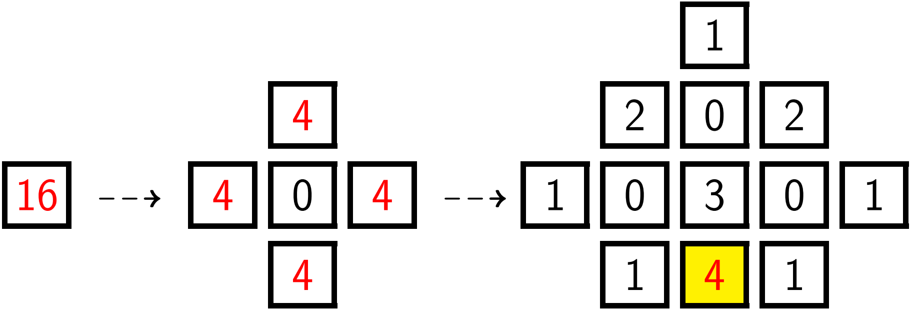

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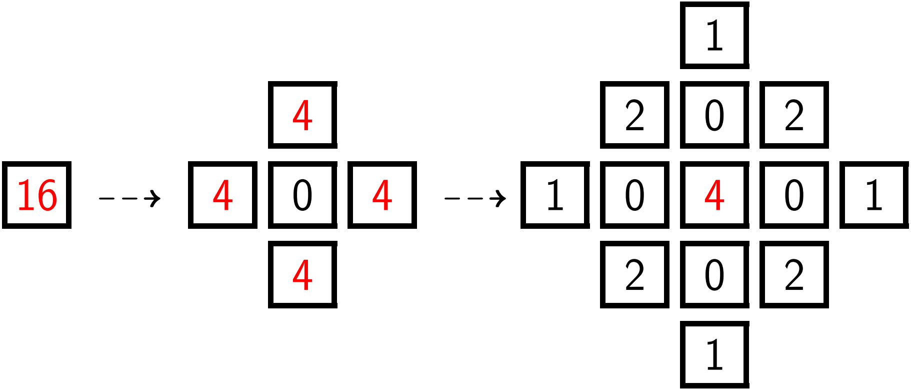

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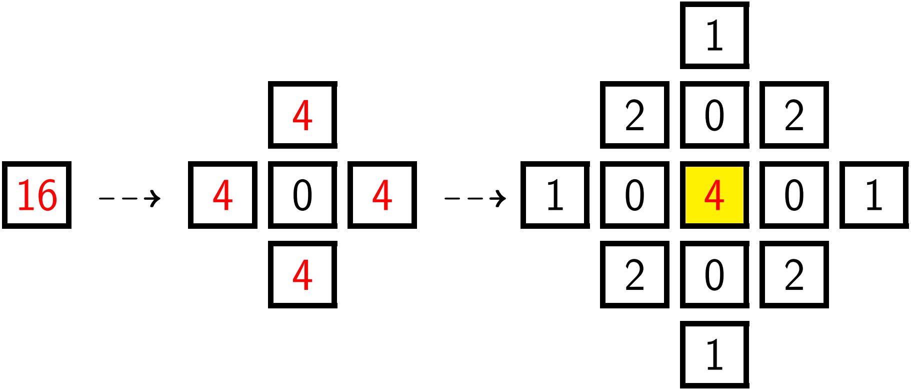

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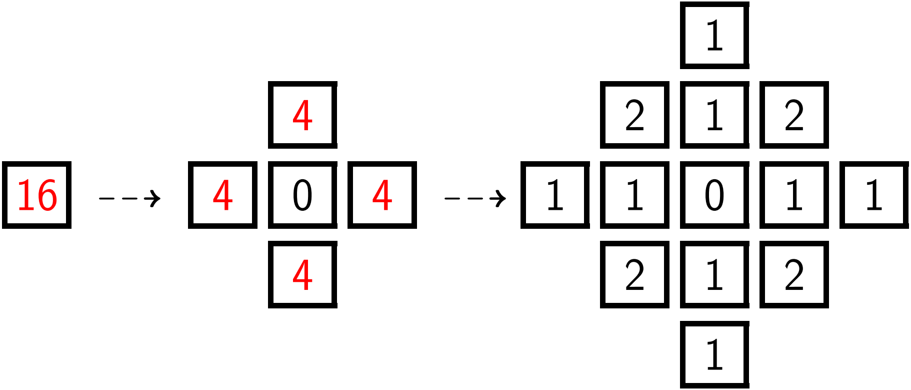
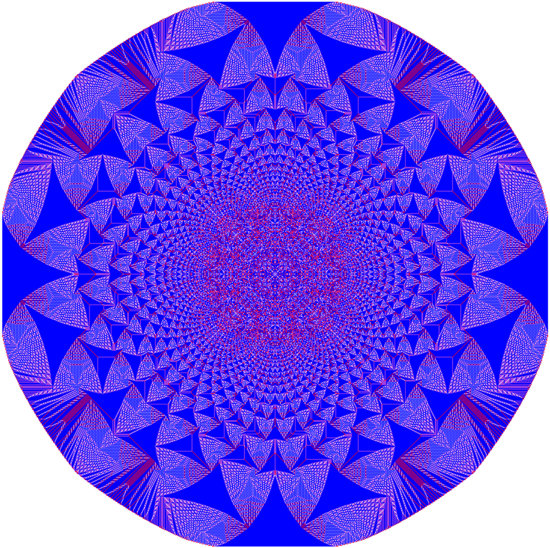


Stable.

Abelian Property

- The final stable configuration does not depend on the order of topplings.
- Neither does the number of times a given vertex topples.

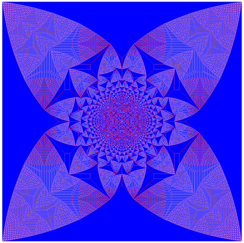
Sandpile of $1,000,000$ chips in $\mathbb{Z}^2$



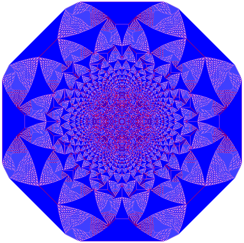
Growth on a General Background

- Let each site $x \in \mathbb{Z}^d$ start with $\sigma(x)$ chips. $(\sigma(x) \leq 2d - 1)$
- We call σ the background configuration.
- Place n additional chips at the origin.
- Let $S_{n,\sigma}$ be the set of sites that topple.

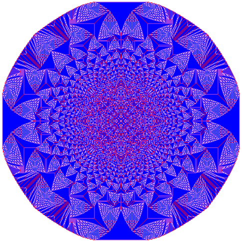
Constant Background $\sigma \equiv h$



$h = 2$



$h = 1$



$h = 0$

What about background $h = 3$?

3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	4	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3

3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	4	3	3	3
3	3	4	0	4	3	3
3	3	3	4	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3

3	3	3	3	3	3	3
3	3	3	4	3	3	3
3	3	5	0	5	3	3
3	4	0	4	0	4	3
3	3	5	0	5	3	3
3	3	3	4	3	3	3
3	3	3	3	3	3	3

3	3	3	4	3	3	3
3	3	5	0	5	3	3
3	5	1	4	1	5	3
4	0	4	0	4	0	4
3	5	1	4	1	5	3
3	3	5	0	5	3	3
3	3	3	4	3	3	3

3	3	5	0	5	3	3
3	5	1	4	1	5	3
5	1	5	0	5	1	5
0	4	0	4	0	4	0
5	1	5	0	5	1	5
3	5	1	4	1	5	3
3	3	5	0	5	3	3

... Never stops toppling!

3	5	1	4	1	5	3
5	1	5	0	5	1	5
1	5	1	4	1	5	1
4	0	4	0	4	0	4
1	5	1	4	1	5	1
5	1	5	0	5	1	5
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. . . Never stops toppling!

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$$\begin{aligned}\Delta u(x) &= \sum_{y \sim x} u(y) - 2d\, u(x) \\ &= \text{chips received} - \text{chips emitted} \\ &= \tau^{\circ}(x) - \tau(x)\end{aligned}$$

where τ is the initial unstable chip configuration
and τ° is the final stable configuration.

Stabilizing Functions

- Given a chip configuration τ on $\mathbb{Z}^d$ and a function $u_1 : \mathbb{Z}^d \to \mathbb{Z}$, call u_1 stabilizing for τ if

$$\tau + \Delta u_1 \leq 2d - 1.$$

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- If u_1 and u_2 are stabilizing for τ, then

$$\begin{aligned} \tau + \Delta \min(u_1, u_2) &\leq \tau + \max(\Delta u_1, \Delta u_2) \\ &= \max(\tau + \Delta u_1, \tau + \Delta u_2) \\ &\leq 2d - 1 \end{aligned}$$

so $\min(u_1, u_2)$ is also stabilizing for τ.

Least Action Principle

- Let τ be a chip configuration on $\mathbb{Z}^d$ that stabilizes after finitely many topplings, and let u be its odometer function.
- Least Action Principle:

If $u_1 : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ is stabilizing for τ, then $u \leq u_1$.

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- So the odometer is minimal among all nonnegative stabilizing functions:

$$u(x) = \min\{u_1(x) \,|\, u_1 \geq 0 \text{ is stabilizing for } \tau\}.$$

- Interpretation: "Sandpiles are lazy."

Obstacle Problem with an Integrality Condition

- **Lemma**. The abelian sandpile odometer function is given by

$$u = s - \gamma$$

where

$$s(x) = \min \left\{ f(x) \;\middle|\; \begin{array}{c} f : \mathbb{Z}^d \to \mathbb{R} \text{ is superharmonic} \\ \text{and } f - \gamma \text{ is } \mathbb{Z}_{\geq 0}\text{-valued} \end{array} \right\}.$$

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- The obstacle γ is given by

$$\gamma(x) = -\frac{(2d-1)|x|^2 + n \cdot g(o,x)}{2d}$$

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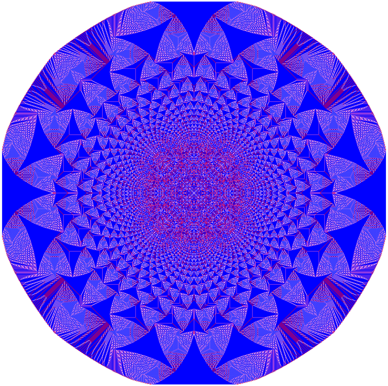
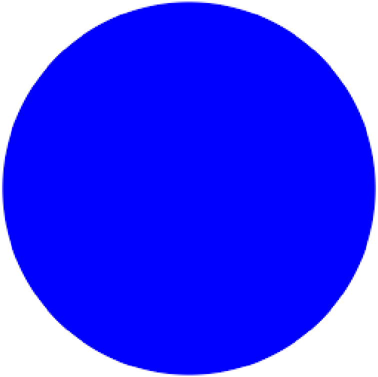


Abelian sandpile
(Integrality constraint)



Divisible sandpile
(No integrality constraint)

Sandpile growth rates

- Let $S_{n,d,h}$ be the set of sites in $\mathbb{Z}^d$ that topple, if $n+h$ chips start at the origin and h chips start at every other site in $\mathbb{Z}^d$.

Theorem (Fey-L.-Peres) If $h \leq 2d-2$, then

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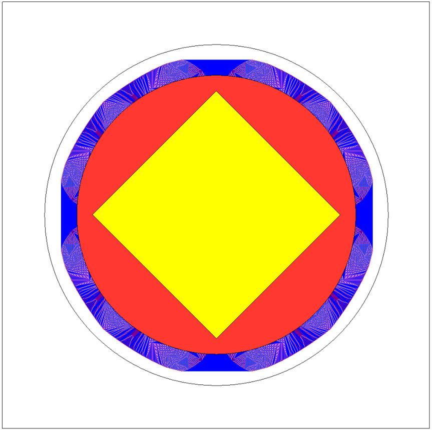
- Extends earlier work of **Fey-Redig** and **Le Borgne-Rossin**.

Bounds for the Abelian Sandpile Shape



$$(\text{Disk of area } n/3) \subset S_n \subset (\text{Disk of area } n/2)$$

A Few Extra Chips Produce An Explosion

- Let $(\beta(x))_{x\in\mathbb{Z}^d}$ be independent Bernoulli random variables

$$\beta(x) = \begin{cases} 1 & \text{with probability } \varepsilon \\ 0 & \text{with probability } 1-\varepsilon. \end{cases}$$

- **Theorem** (Fey-L.-Peres) For any $\varepsilon > 0$, with probability 1, the background $2d-2+\beta$ on $\mathbb{Z}^d$ is explosive.

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- **Theorem** (Fey-L.-Peres) For any $\varepsilon > 0$, with probability 1, the background $2d-2+\beta$ on $\mathbb{Z}^d$ is explosive.
  - i.e., for large enough n, adding n chips at the origin causes every site in $\mathbb{Z}^d$ to topple infinitely many times.

A Few Extra Chips Produce An Explosion

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  - i.e., for large enough n, adding n chips at the origin causes every site in $\mathbb{Z}^d$ to topple infinitely many times.
- Same is true if the extra chips start on an arbitrarily sparse lattice $L \subset \mathbb{Z}^d$, provided L meets every coordinate plane $\{x_i = k\}$.

How to Prove An Explosion

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- **Claim**: If every site in $\mathbb{Z}^d$ topples at least once, then every site topples infinitely often.
- Otherwise, let x be the first site to finish toppling.
- Each neighbor of x topples at least one more time, so x receives at least $2d$ additional chips.
- So x must topple again. $\Rightarrow\Leftarrow$

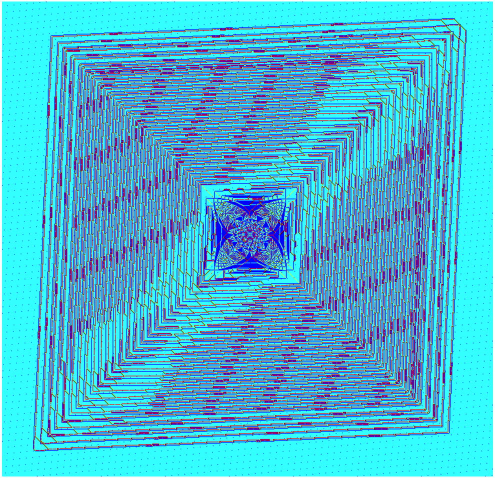
Straley's Argument for Bootstrap Percolation

- Let E_k be the event that each face of the cube Q_k starts with at least one extra chip. Then

$$\mathbb{P}(E_k^c) \le 2d(1-\varepsilon)^k.$$

- By Borel-Cantelli, with probability 1 almost all E_k occur.

An Explosion In Progress



- Sites colored black are unstable. All sites in $\mathbb{Z}^2$ will topple infinitely often!

A Mystery: Scale Invariance

- Big sandpiles look like scaled up small sandpiles!
- Let $\sigma_n(x)$ be the final number of chips at x in the sandpile of n particles on $\mathbb{Z}^d$.

A Mystery: Scale Invariance

- Big sandpiles look like scaled up small sandpiles!
- Let $\sigma_n(x)$ be the final number of chips at x in the sandpile of n particles on $\mathbb{Z}^d$.
- Squint your eyes: for $x \in \mathbb{R}^d$ let

$$f_n(x) = \frac{1}{a_n^2} \sum_{\substack{y \in \mathbb{Z}^d \\ ||y - \sqrt{n}x|| \leq a_n}} \sigma_n(y).$$

where a_n is a sequence of integers such that

$$a_n \uparrow \infty \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \downarrow 0.$$

Scale Invariance Conjecture

- **Conjecture**: There is a sequence a_n and a function $f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ which is locally constant on an open dense set, such that $f_n \to f$ at all continuity points of f.

Scale Invariance Conjecture

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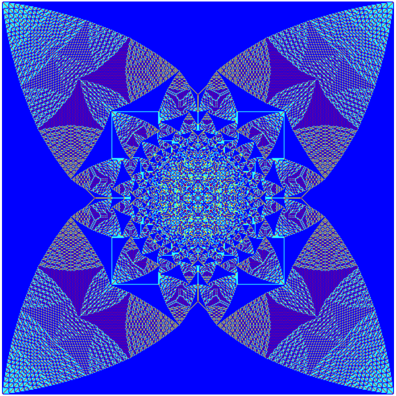
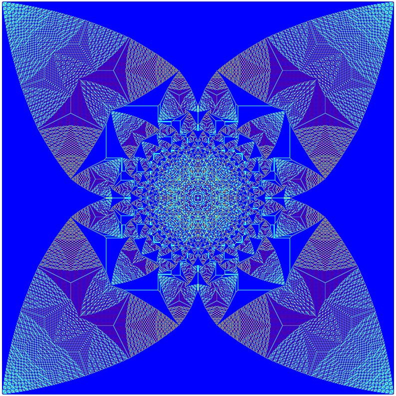
- Now partly proved! **Pegden and Smart** (arXiv:1105.0111) show existence of a weak-$*$ limit for f_n!

Two Sandpiles of Different Sizes

$n = 100,000$

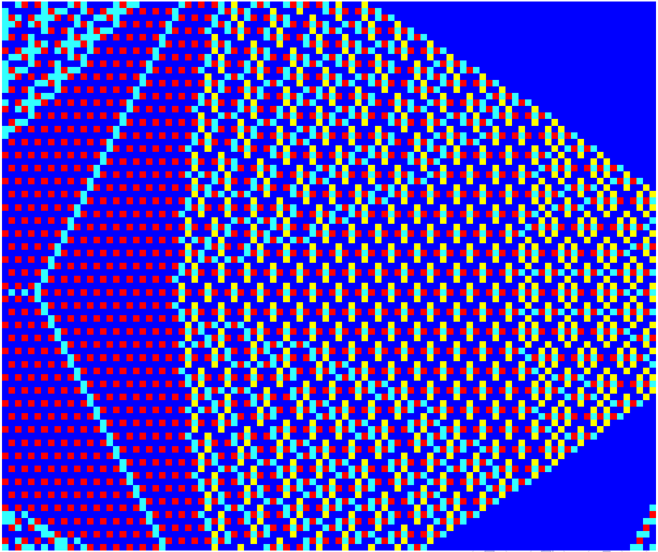


$n = 200,000$



(scaled down by $\sqrt{2}$)

Locally constant "steps" of f correspond to periodic patterns:



A Mystery: Dimensional Reduction

- Our argument used simple properties of one-dimensional sandpiles to bound the diameter of higher-dimensional sandpiles.
- Deepak Dhar pointed out that there seems to be a deeper relationship between sandpiles in d and $d-1$ dimensions...

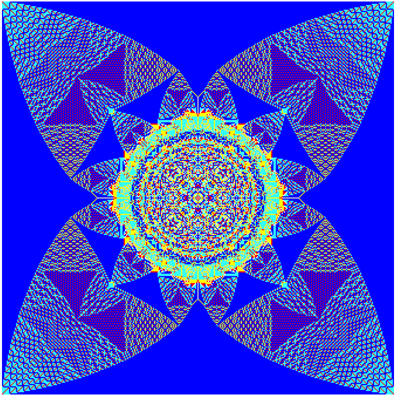
Dimensional Reduction Conjecture

- $\sigma_{n,d}$: sandpile of n chips on background $h = 2d - 2$ in $\mathbb{Z}^d$.
- **Conjecture**: For any n there exists m such that

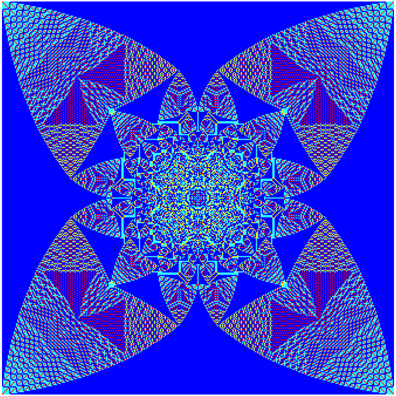
$$\sigma_{n,d}(x_1, \ldots, x_{d-1}, 0) = 2 + \sigma_{m,d-1}(x_1, \ldots, x_{d-1})$$

for almost all x sufficiently far from the origin.

A Two-Dimensional Slice of A Three-Dimensional Sandpile

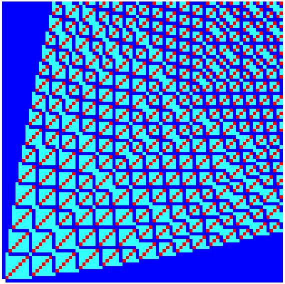


$d = 3$ (slice through origin)
$h = 4$
$n = 5,000,000$



$d = 2$
$h = 2$
$m = 46,490$

Thank You!



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