Mitosis in an evolving voter model

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A result of the 2009-2010 SAMSI program on complex networks

Holme and Newman (2006)

They begin with a network of N nodes and M edges, where each node i has an opinion g_i from a set of G possible opinions and the number of people per opinion $\gamma = N/G$ stays bounded as N gets large.

On each step a vertex i is picked at random. If its degree $k_i = 0$, nothing happens. For $k_i > 0$,

(i) with probability ϕ an edge attached to vertex i is selected and the other end of that edge is moved to a vertex chosen at random from those with opinion g_i.

(ii) otherwise (i.e., with probability $1 - \phi$) a random neighbor j of i is selected and the opinion of i is set to $g_i = g_j$.

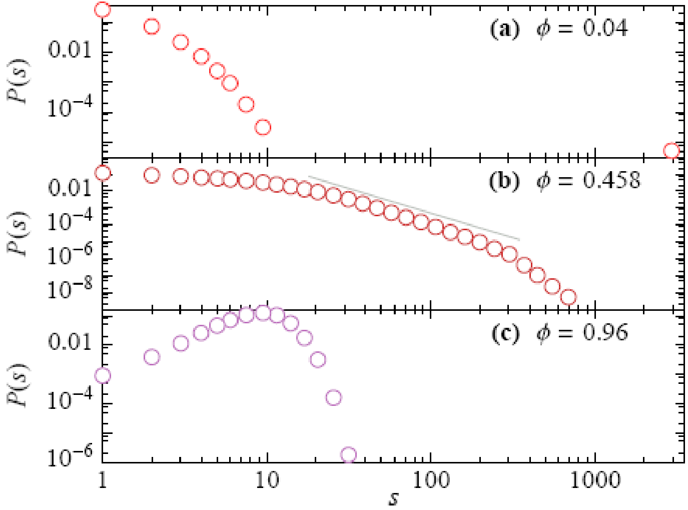
Eventually there are no edges that connect different opinions and the system freezes at time τ_N.

Extreme Cases

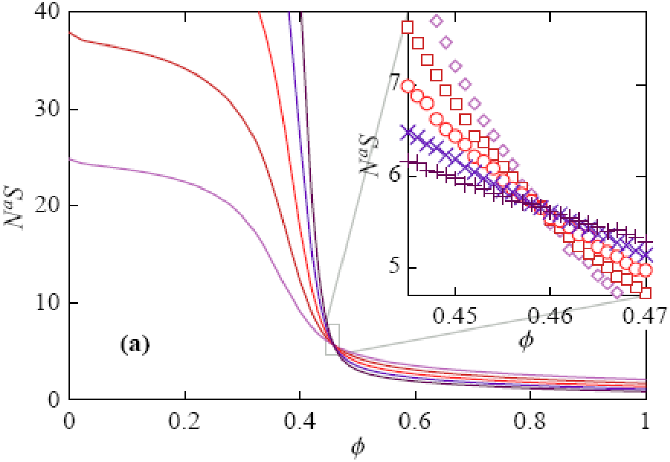
When $\phi = 1$ only rewiring steps occur, so once all of the M edges have been touched the graph has been disconnected into G components, each of which is small. By results for the coupon collector's problem, this $\tau_N \sim M \log M$ updates.

When $\phi = 0$ this is a voter model on a static graph. If we use an Erdös-Renyi random graph in which each vertex has average degree $\lambda > 1$ then there is a giant component with a positive fraction of the vertices and a large number of small components. The giant component will reach consensus in $\tau_N \sim N^2$ steps, so the end result is one opinion with a large number of followers while all of the other populations are small.

Community sizes $N = 3200$, $M = 6400$, $\gamma = 10$.



Finite size scaling



Our model

On each step an edge is picked at random and assigned a random orientation (x, y). (Isothermal voter model.)

If the voters at the two ends of the edge agree then we do nothing.

If they disagree, then with probability $1 - \alpha$ the voter at x adopts the opinion of the voter at y.

With probability α, x breaks its connection to y and makes a new connection to a voter chosen at random:

(i) from all of the vertices in the graph "rewire to random",

(ii) from those that share its opinion "rewire to same."

Opinions $\{0, 1\}$. Initial state product measure with density u.

Rewire to random with $u = 1/2$ (Alun Lloyd)

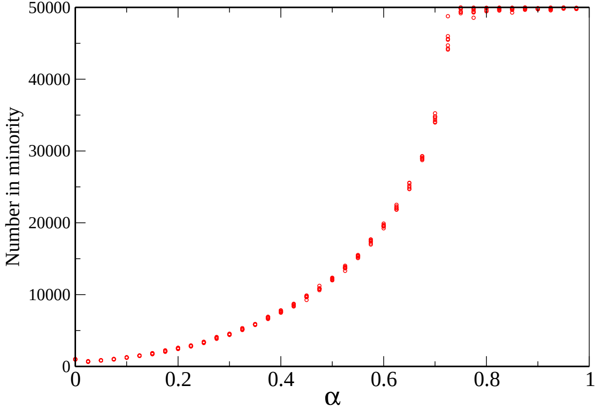
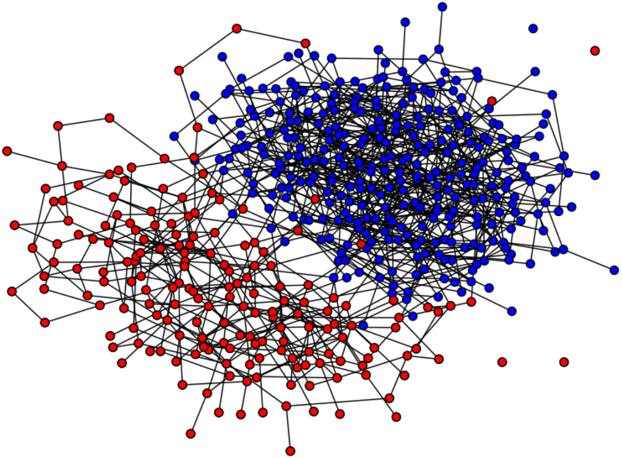
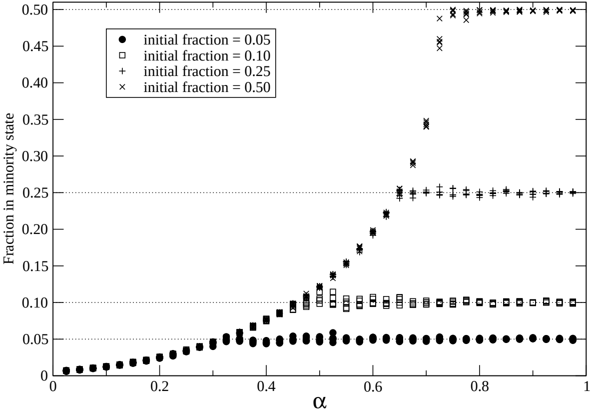


Figure: Erdos-Renyi, $\lambda = 4$, $N = 100,000$

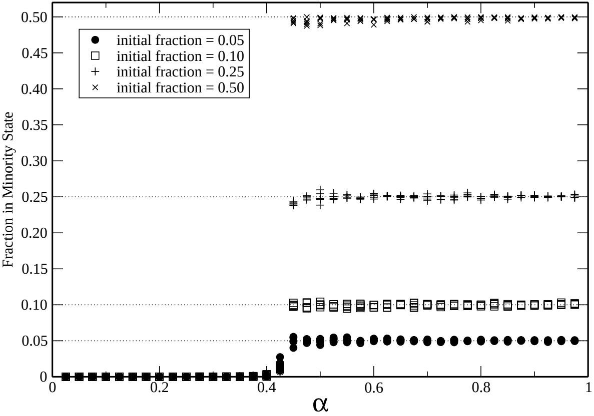
Mitosis for $\alpha = 0.65$ (David Sivakoff)



To random: A universal curve? (Alun Lloyd)



Rewire to same (Alun Lloyd)



The simulation that showed us the answer

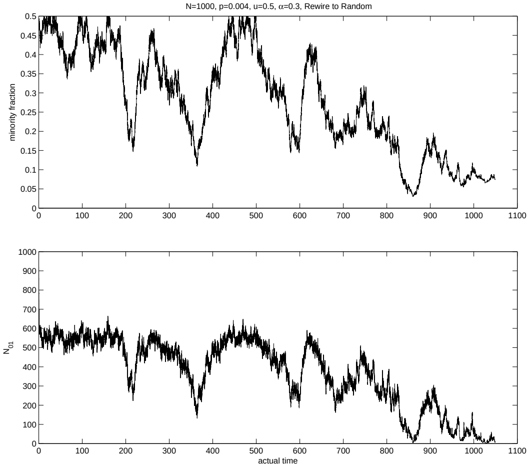


Figure: $N = 1000$, $\lambda = 4$, $u = 1/2$, Initial $N_{10} = 1000$. (Chris Varghese)

Holley and Liggett (1975)

Consider the voter model on the d-dimensional integer lattice $\mathbb{Z}^d$ in which each vertex decides to change its opinion at rate 1, and when it does, it adopts the opinion of one of its $2d$ nearest neighbors chosen at random.

In $d \leq 2$, the system approaches complete consensus. That is if $x \neq y$ then $P(\xi_t(x) \neq \xi_t(y)) \to 0$.

In $d \geq 3$ if we start from ξ_0^p product measure with density p, i.e., $\xi_0^p(x)$ are independent and equal to 1 with probability then ξ_t^p converges in distribution to a limit ν_p, which is a stationary distribution for the voter model.

Cox (1989)

Let τ be the time to consensus τ for the voter model on the d-dimensional torus T_d. If we let $N = L^d$ be the number of points and start from product measure with density $p \in (0,1)$ then

$$E\tau \sim \begin{cases} C_p N^2 & d = 1 \\ C_p N \log N & d = 2 \\ C_p N & d \geq 3 \end{cases}$$

Time $N = N^2$ simulation steps.

Cox and Greven (1990)

The voter model on the torus in $d \geq 3$ at time Nt then it locally looks like $\nu_{\theta(t)}$ where the density changes according to the Wright-Fisher diffusion:

$$d\theta_t = \sqrt{\beta_d \cdot 2\theta_t(1-\theta_t)}dB_t$$

Here β_d is the probability that two random walks starting from neighboring sites never hit.

To explain the diffusion constant, note that the number of 1's changes at rate $2/2d$ times the number of 1-0 edges and use duality.

There is a one parameter family of quasi-stationary distributions, and the parameter changes according to a diffusion.

N_{01} versus N_1, $\alpha = 0.5$ (Bill Shi)

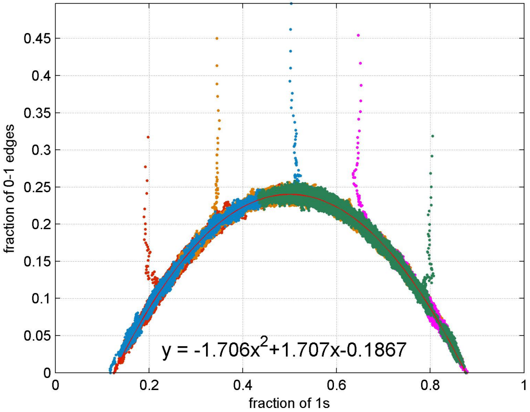
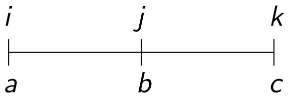


Figure: Process comes quickly to the arch then diffuses along it, splitting into two when it reaches the end.

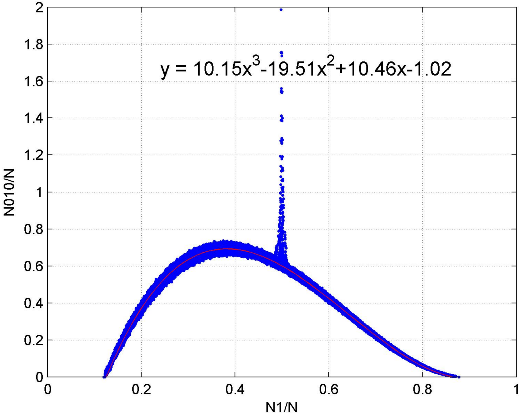
Finite dim. distr. on a random graph

A definition from the theory of graph limits of Lovasz et al. N_{ijk} is the number of homomorphisms from the labeled graph



into our labeled graph (G, ξ). When $i = 0$, $j = 1$, $k = 0$ every triple is counted twice but this seems like the natural definition.

N_{010} versus N_1, $\alpha = 0.5$ (Bill Shi)



N_{ijk} are polynomials?

Bill Shi's simulations for $\lambda = 4$, $\alpha = 0.5$

$$\begin{aligned}
N_{01} &= -3.42x^2 + 3.42x - 0.38 \\
N_{110} &= -13.53x^3 + 10.87x^2 + 1.19x - 0.30 \\
N_{100} &= 13.54x^3 - 29.74x^2 + 17.67x - 1.77 \\
N_{101} &= -10.14x^3 + 10.93x^2 - 1.89x + 0.08 \\
N_{010} &= 10.15x^3 - 19.51x^2 + 10.46x - 1.02
\end{aligned}$$

I have multiplied Bill's N_{01} by 2 since he divides by the number of edges and I divide by N.

Evolution Equations

$$\frac{dN_{10}}{dt} = -(2-\alpha)N_{10} + (1-\alpha)[N_{100} - N_{010} + N_{110} - N_{101}]$$
$$\frac{1}{2}\frac{dN_{11}}{dt} = (1-\alpha(1-u))N_{10} + (1-\alpha)[N_{101} - N_{011}]$$
$$\frac{1}{2}\frac{dN_{00}}{dt} = (1-\alpha u)N_{10} + (1-\alpha)[N_{010} - N_{100}]$$

Of course $N_{11} + 2N_{10} + N_{00} = M$, the number of edges.

$$\sum_{ijk} N_{ijk} = \sum_y d(y)(d(y)-1)$$
$$\frac{d}{dt}\sum_{ijk} N_{ijk} = -2\alpha[N_{101} + N_{010} + N_{100} + N_{110}] + 4\alpha N_{10} \cdot \frac{M}{N}$$

One equation short

When $\lambda = 4$, $\alpha = 0.5$

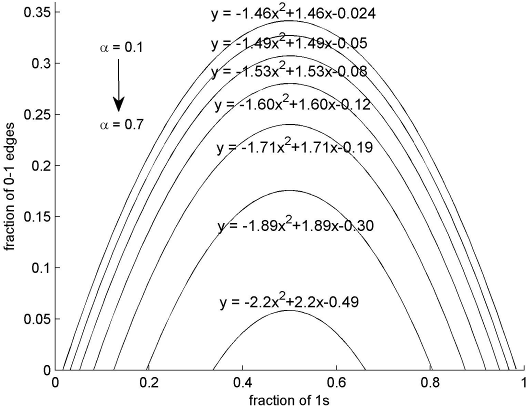
	from equations	from simulation
N_{101}/N_{01}	$(2a_3 + 2b_3) - 2b_3 u$	$-0.23 + 2.96u$
N_{010}/N_{01}	$2a_3 + 2b_3 u$	$2.73 - 2.96u$
N_{110}/N_{01}	$(2a_3 + 2b_3 + 1) + (2b_3 - 1)u$	$0.77 + 3.96u$
N_{100}/N_{01}	$(2a_3 + 2) + (2b_3 - 1)u$	$4.73 - 3.96u$

From $(d/dt)\sum_{ijk} N_{ijk} = 0$ we get

$$2\lambda(1-\alpha) = 4a_3 + 2 + 2b_3 - \alpha$$

Equations and simulation agree if $2a_3 = 2.73$ and $2b_3 = -2.96$.

Arches for rewire to random



Arches for rewire to same

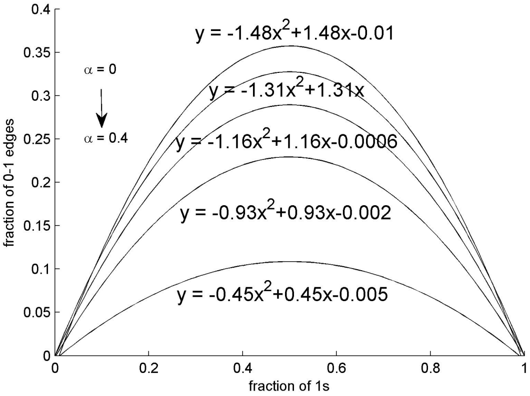


Figure: Note that constant term ≈ 0.

What makes the two models different?

Rewire to same

$$\frac{1}{2}\frac{dN_{11}}{dt} = N_{10} + (1-\alpha)[N_{101} - N_{011}]$$
$$\frac{1}{2}\frac{dN_{00}}{dt} = N_{10} + (1-\alpha)[N_{010} - N_{100}]$$

Using the pair approximation $N_{101} = N_{10}N_{01}/N_0$ and algebra gives

$$N_{11} + N_{00} - \left(\frac{u}{1-u} + \frac{1-u}{u}\right) N_{01} = \frac{N}{1-\alpha}$$

When $N_{01} = 0$ we have $N_{11} + N_{00} = \lambda N$ which means

$$\alpha_c = \frac{\lambda - 1}{\lambda} \quad \text{which does not depend on } u$$

When $\lambda = 4$, $\alpha_c = 3/4$ (versus 0.42 from simulation).

Rewire to random

Using the pair approximation as before:

$$N_{11} + N_{00} - \left(\frac{u}{1-u} + \frac{1-u}{u} \right) N_{01} = \left[1 + \frac{(u^2 + (1-u)^2)\alpha}{1-\alpha} \right] N$$

When $N_{01} = 0$ we have $N_{11} + N_{00} = \lambda N$ which means

$$\alpha_c(u) = \frac{\lambda - 1}{\lambda - 1 + u^2 + (1-u)^2}$$

When $u = 1/2$ and $\lambda = 4$, we get $\alpha_c = 6/7 = 0.85$ (versus 0.72).
As $u \to 0$, $\alpha_c(u) \to (\lambda - 1)/\lambda = 3/4$.

$$N_{01} = u(1-u) \left(\lambda - 1 - \frac{(u^2 + (1-u)^2)\alpha}{1-\alpha} \right) N$$

Pair approximation and approx. master eq.

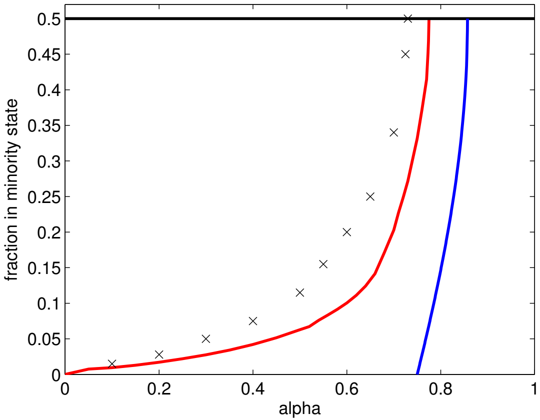


Figure: × are values from simulation

Approx. Master Eq. (James Gleeson)

$\bar{S}_{k,m}$ = vertices in state 0 (susceptible) with degree k and m neighbors in state 1. $\bar{I}_{k,m}$ = vertices in state 1 (infected).

$$\begin{aligned}
\frac{d}{dt}\bar{S}_{k,m} = \alpha\{&-(2-u)m\bar{S}_{k,m} \\
&+ (1-u)(m+1)\bar{S}_{k,m+1} + (m+1)\bar{S}_{k+1,m+1})\} \\
&+ \alpha N_{01}[-2\bar{S}_{k,m} + \bar{S}_{k-1,m-1} + \bar{S}_{k-1,m}]/N \\
&+ (1-\alpha)[-k\bar{S}_{k,m} + (m-k)\bar{I}_{k,m}] \\
&+ (1-\alpha)[-\beta^S(k-m)\bar{S}_{k,m} + \beta^S(k-m+1)\bar{S}_{k,m-1} \\
&\qquad - \gamma^S m\bar{S}_{k,m} + \gamma^S(m+1)\bar{S}_{k,m+1}]
\end{aligned}$$

where $\beta^S = N_{001}/N_{00}$ and $\gamma^S = 1 + (N_{010}/N_{01})$.
When $\lambda = 4$ truncate at $k = 15$. Solve 250 DE using Mathematica.

Two remarks

AME is better than PA because we compute

$$N_{001} = \sum_k \bar{S}_{k,m} m(k-m) \quad \text{etc}$$

rather than use the bad approximation $N_{001}/N_{00} = N_{01}$. LHS linear, RHS quadratic.

The AME equations for dN_{ij}/dt are exact. Those for dN_{ijk}/dt approximate the terms

$$N_{ijk\ell} = \frac{N_{ijk} N_{jk\ell}}{N_{jk}}$$

but leave the $N^Y_{i;jk\ell}$ for 3-star homomorphisms alone.

Aprox. Master Eq. versus random arches

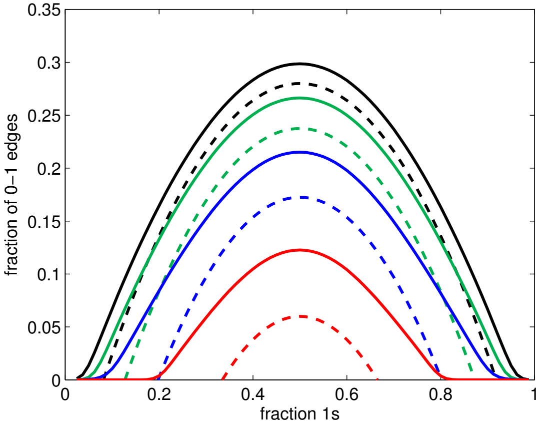
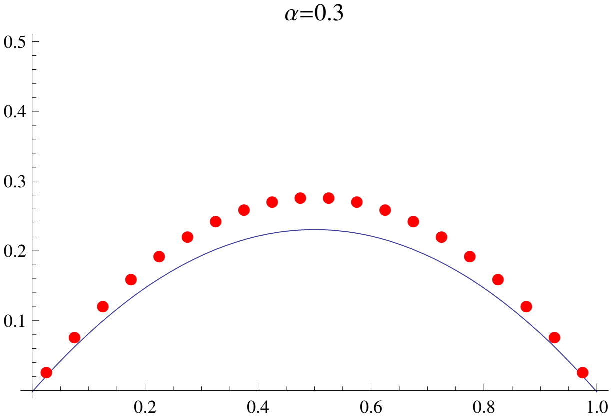


Figure: $\alpha = 0.4,\ 0.5,\ 0.6,\ 0.7$

AME. versus " rewire to same" arch $\alpha = 0.3$



Extensions

We get the same result if we start with a random 4-regular graph

OR

if we designate uN vertices as 1 and $(1-u)N$ as 0 and connect an i node to a j node with probability p_{ij}/N.

In the second case by choosing the p_{ij} correctly we can achieve any possible value of N_1/N and N_{10}/M where $M = \lambda N/2$.

For these initial conditions we quickly move to the arch of quasistationary distributions.

Degree distribution Poisson?

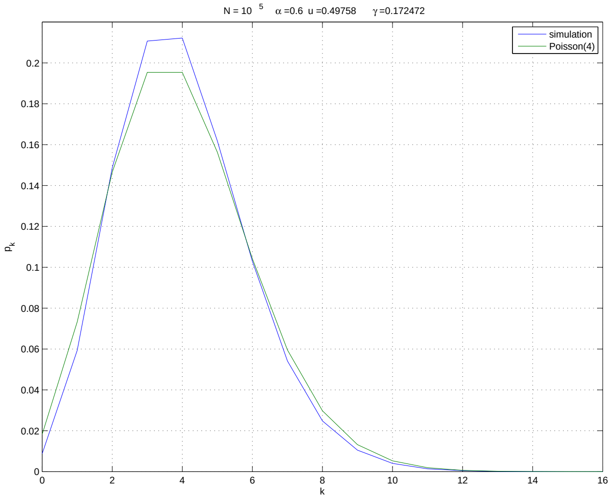
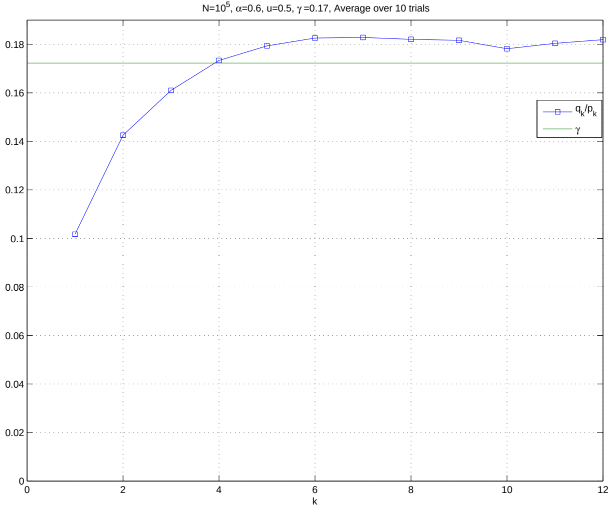
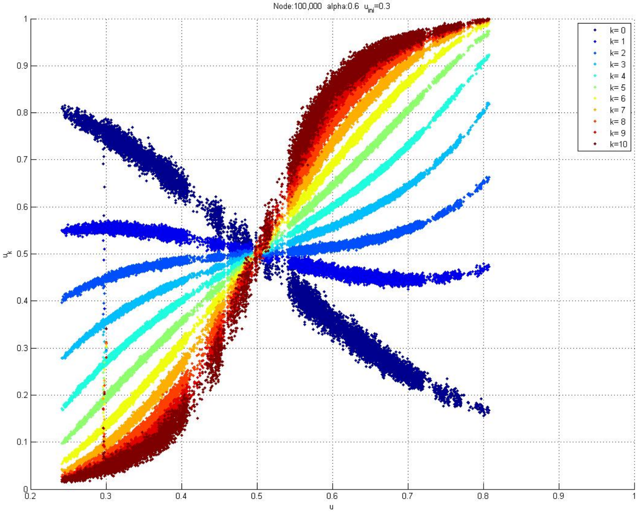


Figure: This and the next simulation by Chris Varghese

Poisson iff fraction of $\neq$ neighbors is constant



Fraction of degree k nodes = 1 (Bill Shi)



Open Problems

Prove $\alpha_c < 1$.

Prove quasi-stationary distributions exist when α is small.

When $\alpha = 0$ we know the quasi-stationary distribution for the voter model, so it is natural to try a perturbation argument. However when we consider (G, ξ) for the voter model the G does not change. For $\alpha > 0$ the G converges to some G_α, which should be $\approx$ Erdos-Renyi(λ) when α is small.