# Chapter 2

# Monotonic Measures

*Summary.* The property of monotonicity of measures leads naturally to positive association and the FKG inequality. A monotonic measure may be used as the seed for a parametric family of measures satisfying probabilistic inequalities including influence, sharp-threshold, and exponential-steepness inequalities.

## 2.1 Stochastic ordering of measures

The stochastic ordering of probability measures provides a technique which is fundamental to the study of random-cluster measures. Let $E$ be a finite or countably infinite set, let $\Omega = \{0, 1\}^E$, and let $\mathcal{F}$ be the $\sigma$-field generated by the cylinder events of $\Omega$. In applications of the arguments of this section, $E$ will be the edge-set of a graph, and thus we refer to members of $E$ as 'edges', although the graphical structure is not itself relevant at this stage.

The configuration space $\Omega$ is a partially ordered set with partial order given by: $\omega_1 \le \omega_2$ if $\omega_1(e) \le \omega_2(e)$ for all $e \in E$. A random variable $X : \Omega \to \mathbb{R}$ is called *increasing* if $X(\omega_1) \le X(\omega_2)$ whenever $\omega_1 \le \omega_2$. An event $A \in \mathcal{F}$ is called *increasing* (respectively, *decreasing*) if its indicator function $1_A$ is increasing (respectively, decreasing). The set $\Omega$, equipped with the topology of open sets generated by the cylinder events, is a metric space, and we speak of a random variable $X : \Omega \to \mathbb{R}$ as being 'continuous' if it is a continuous function on this metric space. Since $\Omega$ is compact, any continuous function on $\Omega$ is necessarily bounded. In addition, any increasing function $X : \Omega \to \mathbb{R}$ is bounded since $X(0) \le X(\omega) \le X(1)$ for $\omega \in \Omega$.

Given two probability measures $\mu_1$, $\mu_2$ on $(\Omega, \mathcal{F})$, we write $\mu_1 \le_{\text{st}} \mu_2$ (or $\mu_2 \ge_{\text{st}} \mu_1$), and we say that $\mu_1$ is stochastically smaller than $\mu_2$, if[1] $\mu_1(X) \le \mu_2(X)$ for all increasing continuous random variables $X$ on $\Omega$.

[1]Recall that $\mu(X)$ denotes the expectation of $X$ under $\mu$, that is, $\mu(X) = \int X \, d\mu$.

For two probability measures $\phi_1$, $\phi_2$ on $(\Omega, \mathcal{F})$, a *coupling* of $\phi_1$ and $\phi_1$ is a probability measure $\kappa$ on $(\Omega, \mathcal{F}) \times (\Omega, \mathcal{F})$ whose first (respectively, second) marginal is $\phi_1$ (respectively, $\phi_2$). There exist many couplings of any given pair $\phi_1$, $\phi_2$, and the art of coupling lies in finding one that is useful. Let $\mu_1$, $\mu_2$ be probability measures on $(\Omega, \mathcal{F})$. The theorem known sometimes as 'Strassen's theorem' states that $\mu_1 \leq_{\text{st}} \mu_2$ if and only if there exists a coupling $\kappa$ satisfying $\kappa(S) = 1$, where $S = \{(\omega_1, \omega_2) \in \Omega^2 : \omega_1 \leq \omega_2\}$ is the 'sub-diagonal' of the product space $\Omega^2$. A useful account of coupling and its applications may be found in [237].

We call a probability measure $\mu$ on $(\Omega, \mathcal{F})$ *strictly positive* if $\mu(\omega) > 0$ for all $\omega \in \Omega$. For $\omega_1, \omega_2 \in \Omega$, we denote by $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ the 'maximum' and 'minimum' configurations given by

$$\begin{aligned} \omega_1 \vee \omega_2(e) &= \max\{\omega_1(e), \omega_2(e)\}, \qquad e \in E, \\ \omega_1 \wedge \omega_2(e) &= \min\{\omega_1(e), \omega_2(e)\}, \qquad e \in E. \end{aligned}$$

We suppose for the remainder of this section that $E$ is finite. There is a useful sufficient condition for the stochastic inequality $\mu_1 \leq_{\text{st}} \mu_2$, as follows.

**(2.1) Theorem (Holley inequality) [185].** *Let $\mu_1$ and $\mu_2$ be strictly positive probability measures on the finite space $(\Omega, \mathcal{F})$ such that*

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2), \qquad \omega_1, \omega_2 \in \Omega. \tag{2.2}$$

*Then*

$$\mu_1(X) \leq \mu_2(X) \qquad \text{for increasing functions } X : \Omega \to \mathbb{R},$$

*which is to say that $\mu_1 \leq_{\text{st}} \mu_2$.*

This may be extended in (at least) two ways. Firstly, a similar claim[2] is valid in the more general setting where $\Omega = T^E$ and $T$ is a finite subset of $\mathbb{R}$. Secondly, one may relax the condition that the measures be *strictly* positive. See, for example, [136, Section 4].

Let $S \subseteq \Omega^2$ $(= \Omega \times \Omega)$ be the set of all ordered pairs $(\pi, \omega)$ of configurations satisfying $\pi \leq \omega$, as above. In the proof of Theorem 2.1, we shall construct a coupling $\kappa$ of $\mu_1$ and $\mu_2$ such that $\kappa(S) = 1$. It is an immediate consequence that $\mu_1 \leq_{\text{st}} \mu_2$. There is a variety of couplings of measures which play roles in the theory of random-cluster measures. Another may be found in the proof of Theorem 3.45.

Condition (2.2) in key to Theorem 2.1, and it is equivalent to a condition of monotonicity on the one-point conditional distributions.

[2]An application of such a claim may be found in the analysis of the Ashkin–Teller model at Theorem 11.12.

**(2.3) Theorem.** *Let $\mu_1$, $\mu_2$ be strictly positive probability measures on $(\Omega, \mathcal{F})$. The following are equivalent.*

(a) *The pair $\mu_1$, $\mu_2$ satisfies* (2.2).

(b) *The one-point conditional probabilities are monotonic in that*

$$\mu_2\big(\omega(e) = 1 \,\big|\, \omega(f) = \zeta(f) \text{ for all } f \in E \setminus \{e\}\big) \\ \geq \mu_1\big(\omega(e) = 1 \,\big|\, \omega(f) = \xi(f) \text{ for all } f \in E \setminus \{e\}\big) \qquad (2.4)$$

*for all $e \in E$, and all pairs $\xi, \zeta \in \Omega$ satisfying $\xi \leq \zeta$.*

(c) *It is the case that*

$$\frac{\mu_2(\zeta^e)}{\mu_2(\zeta_e)} \geq \frac{\mu_1(\xi^e)}{\mu_1(\xi_e)}, \qquad \xi \leq \zeta,\ e \in E. \qquad (2.5)$$

The following is sufficient for (2.2).

**(2.6) Theorem.** *Let $\mu_1$, $\mu_2$ be strictly positive probability measures on $(\Omega, \mathcal{F})$ such that*

$$\text{(2.7)} \qquad \mu_2(\omega^e)\mu_1(\omega_e) \geq \mu_1(\omega^e)\mu_2(\omega_e), \qquad \omega \in \Omega,\ e \in E.$$

*If either $\mu_1$ or $\mu_2$ satisfies*

$$\text{(2.8)} \qquad \mu(\omega^{ef})\mu(\omega_{ef}) \geq \mu(\omega^e_f)\mu(\omega^f_e), \qquad \omega \in \Omega,\ e, f \in E,$$

*then* (2.2) *holds.*

*Proof of Theorem 2.1.* The theorem amounts to a 'mere' numerical inequality involving a finite number of positive reals. It may in principle be proved in a totally elementary manner, using essentially no general mechanism. The proof given here proceeds by constructing certain reversible Markov chains. There is some extra mechanism required, but the method is beautiful, and in addition yields a structure which finds applications elsewhere.

The main step of the proof is designed to show that, under condition (2.2), $\mu_1$ and $\mu_2$ may be 'coupled' in such a way that the sub-diagonal $S$ has full measure. This is achieved by constructing a certain Markov chain with the coupled measure as invariant measure.

Here is a preliminary calculation. Let $\mu$ be a strictly positive probability measure on $(\Omega, \mathcal{F})$. We may construct a reversible Markov chain with state space $\Omega$ and unique invariant measure $\mu$ by choosing a suitable generator (or '$Q$-matrix') satisfying the detailed balance equations. The dynamics of the chain involve the 'switching on or off' of components of the current state. Let $G : \Omega^2 \to \mathbb{R}$ be given by

$$\text{(2.9)} \qquad G(\omega_e, \omega^e) = 1, \quad G(\omega^e, \omega_e) = \frac{\mu(\omega_e)}{\mu(\omega^e)}, \qquad \omega \in \Omega,\ e \in E.$$

We let $G(\omega, \omega') = 0$ for all other pairs $\omega, \omega'$ with $\omega \neq \omega'$. The diagonal elements $G(\omega, \omega)$ are chosen in such a way that

$$\sum_{\omega' \in \Omega} G(\omega, \omega') = 0, \qquad \omega \in \Omega.$$

It is elementary that

$$\mu(\omega)G(\omega, \omega') = \mu(\omega')G(\omega', \omega), \qquad \omega, \omega' \in \Omega,$$

and therefore $G$ generates a Markov chain on the state space $\Omega$ which is reversible with respect to $\mu$. The chain is irreducible, for the following reason. For $\omega, \omega' \in \Omega$, one may add edges one by one to $\eta(\omega)$ thus arriving at the unit vector 1, and then one may remove edges one by one thus arriving at $\omega'$. By (2.9), each such transition has a strictly positive intensity, whence the chain is irreducible. It follows that the chain has unique invariant measure $\mu$. Similar constructions are explored in Chapter 8. An account of the general theory of reversible Markov chains may be found in [164, Section 6.5].

We follow next a similar route for *pairs* of configurations. Let $\mu_1$ and $\mu_2$ satisfy the hypotheses of the theorem, and let $S$ be the set of all ordered pairs $(\pi, \omega)$ of configurations in $\Omega$ satisfying $\pi \leq \omega$. We define $H : S \times S \to \mathbb{R}$ by

$$H(\pi_e, \omega; \pi^e, \omega^e) = 1, \tag{2.10}$$

$$H(\pi, \omega^e; \pi_e, \omega_e) = \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)}, \tag{2.11}$$

$$H(\pi^e, \omega^e; \pi_e, \omega^e) = \frac{\mu_1(\pi_e)}{\mu_1(\pi^e)} - \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)}, \tag{2.12}$$

for all $(\pi, \omega) \in S$ and $e \in E$; all other off-diagonal values of $H$ are set to 0. The diagonal terms $H(\pi, \omega; \pi, \omega)$ are chosen in such a way that

$$\sum_{(\pi', \omega') \in S} H(\pi, \omega; \pi', \omega') = 0, \qquad (\pi, \omega) \in S.$$

Equation (2.10) specifies that, for $\pi \in \Omega$ and $e \in E$, the edge $e$ is acquired by $\pi$ (if it does not already contain it) at rate 1; any edge so acquired is added also to $\omega$ if it does not already contain it. (Here, we speak of a configuration $\psi$ 'containing the edge $e$' if $\psi(e) = 1$.) Equation (2.11) specifies that, for $\omega \in \Omega$ and $e \in E$ with $\omega(e) = 1$, the edge $e$ is removed from $\omega$ (and also from $\pi$ if $\pi(e) = 1$) at the rate given in (2.11). For $e$ with $\pi(e) = 1$, there is an additional rate given in (2.12) at which $e$ is removed from $\pi$ but not from $\omega$. This additional rate is indeed non-negative, since the required inequality

$$\mu_2(\omega^e)\mu_1(\pi_e) \geq \mu_1(\pi^e)\mu_2(\omega_e) \qquad \text{whenever } \pi \leq \omega, \tag{2.13}$$

follows from (2.2) with $\omega_1 = \pi^e$ and $\omega_2 = \omega_e$.

Let $(Y_t, Z_t)_{t\ge 0}$ be a Markov chain on $S$ with generator $H$, and set $(Y_0, Z_0) = (0, 1)$, where 0 (respectively, 1) is the state of all zeros (respectively, ones). We write $\mathbb{P}$ for the appropriate probability measure. Since all transitions retain the ordering of the two components of the state, we may assume that the chain satisfies $\mathbb{P}(Y_t \le Z_t \text{ for all } t) = 1$. By examination of (2.10)–(2.12) we see that $Y = (Y_t : t \ge 0)$ is a Markov chain with generator given by (2.9) with $\mu = \mu_1$, and that $Z = (Z_t : t \ge 0)$ arises similarly with $\mu = \mu_2$. Here is a brief explanation of this elementary step in the case of $Y$, a similar argument holds for $Z$. For $\pi \in \Omega$ and $e \in E$,

$$\begin{aligned}
&\mathbb{P}(Y_{t+h} = \pi^e \mid Y_t = \pi_e)\\
&\qquad = \sum_{\omega\in\Omega} \mathbb{P}\big(Y_{t+h} = \pi^e \,\big|\, (Y_t, Z_t) = (\pi_e, \omega)\big)\mathbb{P}(Z_t = \omega \mid Y_t = \pi_e)\\
&\qquad = \sum_{\omega\in\Omega} [h + \mathrm{o}(h)]\mathbb{P}(Z_t = \omega \mid Y_t = \pi_e) \qquad \text{by (2.10)}\\
&\qquad = h + \mathrm{o}(h).
\end{aligned}$$

Similarly, with $J_e$ the event that $e$ is open,

$$\begin{aligned}
&\mathbb{P}(Y_{t+h} = \pi_e \mid Y_t = \pi^e)\\
&\quad = \sum_{\omega\in J_e,\ \omega'\in\Omega} \mathbb{P}\big((Y_{t+h}, Z_{t+h}) = (\pi_e, \omega') \,\big|\, (Y_t, Z_t) = (\pi^e, \omega^e)\big)\\
&\qquad\qquad\qquad\qquad \times\mathbb{P}(Z_t = \omega^e \mid Y_t = \pi^e)\\
&\quad = \sum_{\omega\in J_e} \big[\big\{H(\pi^e, \omega^e; \pi_e, \omega_e) + H(\pi^e, \omega^e; \pi_e, \omega^e)\big\}h + \mathrm{o}(h)\big]\\
&\qquad\qquad\qquad\qquad \times\mathbb{P}(Z_t = \omega^e \mid Y_t = \pi^e)\\
&\quad = \sum_{\omega\in J_e} \left[\frac{\mu_1(\pi_e)}{\mu_1(\pi^e)}h + \mathrm{o}(h)\right]\mathbb{P}(Z_t = \omega^e \mid Y_t = \pi^e) \qquad \text{by (2.11) and (2.12)}\\
&\quad = \frac{\mu_1(\pi_e)}{\mu_1(\pi^e)}h + \mathrm{o}(h).
\end{aligned}$$

Let $\kappa$ be an invariant measure for the paired chain $(Y_t, Z_t)_{t\ge 0}$. Since $Y$ and $Z$ have (respective) unique invariant measures $\mu_1$ and $\mu_2$, the marginals of $\kappa$ are $\mu_1$ and $\mu_2$. Since $\mathbb{P}(Y_t \le Z_t \text{ for all } t) = 1$,

$$\kappa(S) = \kappa\big(\{(\pi, \omega) : \pi \le \omega\}\big) = 1,$$

and $\kappa$ is the required 'coupling' of $\mu_1$ and $\mu_2$.

Let $(\pi, \omega) \in S$ be chosen according to the measure $\kappa$. Then

$$\mu_1(X) = \kappa(X(\pi)) \le \kappa(X(\omega)) = \mu_2(X),$$

for any increasing function $X$. Therefore $\mu_1 \le_{\mathrm{st}} \mu_2$. □

*Proof of Theorem 2.3.* Inequality (2.4) is equivalent to

$$\mu_2(\zeta^e)[\mu_1(\xi^e) + \mu_1(\xi_e)] \geq \mu_1(\xi^e)[\mu_2(\zeta^e) + \mu_2(\zeta_e)],$$

which is the same as (2.5). Therefore, (b) and (c) are equivalent.

It is clear that (a) implies (c). Suppose conversely that (c) holds. We identify a configuration $\omega \in \Omega$ with the set of indices $\eta(\omega)$ at which $\omega$ takes the value 1. Let $\omega_1, \omega_2 \in \Omega$, and write $A_k = \eta(\omega_k)$. Let $B = A_1 \setminus A_2 = \{b_1, b_2, \dots, b_r\}$, and write $B_s = \{b_1, b_2, \dots, b_s\}$ for $s \geq 1$. Assume $\omega_1 \neq \omega_2$, and without loss of generality that $r \geq 1$. By (2.5),

$$\begin{aligned}\frac{\mu_2(\omega_1 \vee \omega_2)}{\mu_2(\omega_2)} &= \frac{\mu_2(A_2 \cup B)}{\mu_2(A_2 \cup B_{r-1})} \cdot \frac{\mu_2(A_2 \cup B_{r-1})}{\mu_2(A_2 \cup B_{r-2})} \cdots \frac{\mu_2(A_2 \cup B_1)}{\mu_2(A_2)} \\ &\geq \frac{\mu_1((A_1 \cap A_2) \cup B)}{\mu_1((A_1 \cap A_2) \cup B_{r-1})} \cdot \frac{\mu_1((A_1 \cap A_2) \cup B_{r-1})}{\mu_1((A_1 \cap A_2) \cup B_{r-2})} \\ &\qquad \cdots \frac{\mu_1((A_1 \cap A_2) \cup B_1)}{\mu_1(A_1 \cap A_2)} \\ &= \frac{\mu_1(\omega_1)}{\mu_1(\omega_1 \wedge \omega_2)}\end{aligned}$$

as required for (a). The above may be called a 'telescoping' argument. □

*Proof of Theorem 2.6.* We prove first by a telescoping argument that (2.7) is equivalent to

$$\frac{\mu_2(\zeta)}{\mu_2(\xi)} \geq \frac{\mu_1(\zeta)}{\mu_1(\xi)}, \qquad \xi, \zeta \in \Omega,\ \xi \leq \zeta. \tag{2.14}$$

As above, we identify a configuration $\omega \in \Omega$ with the set of indices $\eta(\omega)$ at which $\omega$ takes the value 1. That (2.14) implies (2.7) is immediate on setting $\zeta = \omega^e, \xi = \omega_e$. Conversely, let $\xi, \zeta \in \Omega$ satisfy $\xi \leq \zeta$. Let $B = \eta(\zeta) \setminus \eta(\xi) = \{b_1, b_2, \dots, b_r\}$, and write $B_s = \{b_1, b_2, \dots, b_s\}$ for $s \geq 0$. We may assume $\xi \neq \zeta$ so that $r \geq 1$. By (2.7),

$$\begin{aligned}\frac{\mu_2(\zeta)}{\mu_2(\xi)} &= \prod_{s=1}^{r} \frac{\mu_2(\eta(\xi) \cup B_s)}{\mu_2(\eta(\xi) \cup B_{s-1})} \\ &\geq \prod_{s=1}^{r} \frac{\mu_1(\eta(\xi) \cup B_s)}{\mu_1(\eta(\xi) \cup B_{s-1})} = \frac{\mu_1(\zeta)}{\mu_1(\xi)}.\end{aligned}$$

Inequality (2.8) may be written as

$$\frac{\mu(\omega^{ef})}{\mu(\omega_e^f)} \geq \frac{\mu(\omega_f^e)}{\mu(\omega_{ef})}, \qquad \omega \in \Omega,\ e,\ f \in E. \tag{2.15}$$

The edge $f$ is 'switched on' in both numerator and denominator of the left side, and 'switched off' on the right side. Let $\xi, \zeta \in \Omega$ and $\xi \le \zeta$. By a sequential application of (2.15) to all edges (other than possibly $e$) in $\eta(\zeta)\setminus\eta(\xi)$, (2.8) implies

$$\frac{\mu(\zeta^e)}{\mu(\zeta_e)} \ge \frac{\mu(\xi^e)}{\mu(\xi_e)}, \qquad \xi \le \zeta,\ e \in E. \tag{2.16}$$

It follows by Theorem 2.3 that

$$\frac{\mu(\omega_1 \vee \omega_2)}{\mu(\omega_2)} \ge \frac{\mu(\omega_1)}{\mu(\omega_1 \wedge \omega_2)}, \qquad \omega_1, \omega_2 \in \Omega. \tag{2.17}$$

Assume that (2.7) holds, and let $\omega_1, \omega_2 \in \Omega$. If $\mu_1$ satisfies (2.8), then it satisfies (2.17), and (2.2) follows from (2.14) with $\zeta = \omega_1 \vee \omega_2$, $\xi = \omega_2$. Similarly, if $\mu_2$ satisfies (2.8), it satisfies (2.17), and (2.2) follows from (2.14) with $\zeta = \omega_1$, $\xi = \omega_1 \wedge \omega_2$. □

## 2.2 Positive association

Let $E$ be a finite set as in the last section, and let $\Omega = \{0,1\}^E$. A probability measure $\mu$ on $\Omega$ is said to have the *FKG lattice property* if it satisfies the so-called *FKG lattice condition*:

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \ge \mu(\omega_1)\mu(\omega_2), \qquad \omega_1, \omega_2 \in \Omega. \tag{2.18}$$

It is a consequence of the Holley inequality (Theorem 2.1), as follows, that any strictly positive probability measure with the FKG lattice property satisfies the so-called FKG inequality. A stronger result will appear at Theorem 2.27.

**(2.19) Theorem (FKG inequality) [124, 185].** *Let $\mu$ be a strictly positive probability measure on $\Omega$ satisfying the FKG lattice condition. Then*

$$\mu(XY) \ge \mu(X)\mu(Y) \qquad \text{for increasing functions } X, Y : \Omega \to \mathbb{R}. \tag{2.20}$$

There is an extensive literature on the FKG inequality[3] and its extensions. See, for example, [2, 25, 184]. One may extend the inequality to probability measures on sample spaces of the form $T^E$ with $T$ a finite subset of $\mathbb{R}$. In addition, some of the results of this section are valid for measures that are not strictly positive. Any probability measure $\mu$ satisfying (2.20) is said to have the property of 'positive association' or, more concisely, to be 'positively associated'. We consider in Section 4.1 the positive association of measures on $\Omega = \{0,1\}^E$ when $E$ is countably infinite.

[3] The history and origins of the FKG inequality are described in the Appendix.

Correlation-type inequalities play an important role in mathematical physics. For example, the FKG inequality is a fundamental tool in the study of the Ising and random-cluster models, see Chapter 3. There are many other correlation inequalities in statistical physics (see [118]), but these do not generally have a random-cluster equivalent and are omitted from the current work.

*Proof.* Since inequality (2.20) involves a finite set of real numbers only, it may in principle be proved in a totally elementary manner, [280]. We follow here the more interesting route via the Holley inequality, Theorem 2.1. Assume that $\mu$ satisfies the FKG lattice condition (2.18), and let $X$ and $Y$ be increasing functions. Let $a > 0$ and $Y' = Y + a$. Since

$$\mu(XY') - \mu(X)\mu(Y') = \mu(XY) - \mu(X)\mu(Y),$$

it suffices to prove (2.20) with $Y$ replaced by $Y'$. We may pick $a$ sufficiently large that $Y(\omega) > 0$ for all $\omega \in \Omega$. Thus, it suffices to prove (2.20) under the additional hypothesis that $Y$ is strictly positive, and we assume henceforth that this holds. Define the strictly positive probability measures $\mu_1$ and $\mu_2$ on $(\Omega, \mathcal{F})$ by $\mu_1 = \mu$ and

$$\mu_2(\omega) = \frac{Y(\omega)\mu(\omega)}{\sum_{\omega' \in \Omega} Y(\omega')\mu(\omega')}, \qquad \omega \in \Omega.$$

Since $Y$ is increasing, inequality (2.2) follows from (2.18). By the Holley inequality, $\mu_2(X) \ge \mu_1(X)$, which is to say that

$$\frac{\sum_{\omega \in \Omega} X(\omega)Y(\omega)\mu(\omega)}{\sum_{\omega' \in \Omega} Y(\omega')\mu(\omega')} \ge \sum_{\omega \in \Omega} X(\omega)\mu(\omega). \qquad \square$$

If $X$ is increasing and $Y$ is decreasing, we may apply (2.20) to $X$ and $-Y$ to find, under the conditions of the theorem, that $\mu(XY) \le \mu(X)\mu(Y)$. In the special case when $X = 1_A$, $Y = 1_B$, the indicator functions of events $A$ and $B$, we obtain similarly that

$$\mu(A \cap B) \ge \mu(A)\mu(B) \qquad \text{for increasing events } A, B. \tag{2.21}$$

Let $X = (X_1, X_2, \dots, X_r)$ be a vector of random variables taking values in $\{0,1\}^r$. We speak of $X$ as being positively associated if its law on $\{0,1\}^r$ is itself positively associated. Let $Y = h(X)$ where $h : \{0,1\}^r \to \{0,1\}^s$ is a non-decreasing function. It is standard that the vector $Y$ is positively associated whenever $X$ is positively associated. The proof is straightforward, as follows. Let $A$, $B$ be increasing subsets of $\{0,1\}^s$. Then

$$\begin{aligned}\mathbb{P}(Y \in A \cap B) &= \mathbb{P}\big(X \in \{h^{-1}A\} \cap \{h^{-1}B\}\big)\\ &\ge \mathbb{P}(X \in h^{-1}A)\mathbb{P}(X \in h^{-1}B)\\ &= \mathbb{P}(Y \in A)\mathbb{P}(Y \in B),\end{aligned}$$

since $h^{-1}A$ and $h^{-1}B$ are increasing subsets of $\{0,1\}^r$.

We turn now to a consideration of the FKG lattice condition. Recall the Hamming distance between configurations defined in (1.26). A pair $\omega_1, \omega_2 \in \Omega$ is called *comparable* if either $\omega_1 \le \omega_2$ or $\omega_1 \ge \omega_2$, and *incomparable* otherwise.

**(2.22) Theorem.** *A strictly positive probability measure $\mu$ on $(\Omega, \mathcal{F})$ satisfies the FKG lattice condition if and only the inequality of* (2.18) *holds for all incomparable pairs $\omega_1, \omega_2 \in \Omega$ with $H(\omega_1, \omega_2) = 2$.*

For pairs $\omega_1$, $\omega_2$ that differ on exactly two edges $e$ and $f$, the inequality of (2.18) is equivalent to the statement that, conditional on the states of all other edges, the states of $e$ and $f$ are positively associated.

*Proof.* The inequality of (2.18) is a triviality when $H(\omega_1, \omega_2) = 1$, and the claim now follows by Theorem 2.6 with $\mu_1 = \mu_2 = \mu$. See also [257, Lemma 6.5]. □

The FKG lattice condition is sufficient but not necessary for positive association. It is equivalent for strictly positive measures to a stronger property termed 'strong positive-association' (or, sometimes, 'strong FKG'). For $F \subseteq E$ and $\xi \in \Omega$, we write $\Omega_F = \{0, 1\}^F$ and

$$\Omega_F^\xi = \{\omega \in \Omega : \omega(e) = \xi(e) \text{ for all } e \in E \setminus F\}, \tag{2.23}$$

the set of configurations that agree with $\xi$ on the complement of $F$. Let $\mu$ be a probability measure on $(\Omega, \mathcal{F})$, and let $F, \xi$ be such that $\mu(\Omega_F^\xi) > 0$. We define the conditional probability measure $\mu_F^\xi$ on $\Omega_F$ by

$$\mu_F^\xi(\omega_F) = \mu(\omega_F \mid \Omega_F^\xi) = \frac{\mu(\omega_F \times \xi)}{\mu(\Omega_F^\xi)}, \qquad \omega_F \in \Omega_F, \tag{2.24}$$

where $\omega_F \times \xi$ denotes the configuration that agrees with $\omega_F$ on $F$ and with $\xi$ on its complement. We say that $\mu$ is *strongly positively-associated* if: for all $F \subseteq E$ and all $\xi \in \Omega$ such that $\mu(\Omega_F^\xi) > 0$, the measure $\mu_F^\xi$ is positively associated.

We call $\mu$ *monotonic* if: for all $F \subseteq E$, all increasing subsets $A$ of $\Omega_F$, and all $\xi, \zeta \in \Omega$ such that $\mu(\Omega_F^\xi), \mu(\Omega_F^\zeta) > 0$,

$$\mu_F^\xi(A) \le \mu_F^\zeta(A) \qquad \text{whenever } \xi \le \zeta. \tag{2.25}$$

That is, $\mu$ is monotonic if, for all $F \subseteq E$,

$$\mu_F^\xi \le_{\text{st}} \mu_F^\zeta \qquad \text{whenever } \xi \le \zeta. \tag{2.26}$$

We call $\mu$ 1-*monotonic* if (2.26) holds for all singleton sets $F$. That is, $\mu$ is 1-monotonic if and only if, for all $f \in E$, $\mu(J_f \mid \Omega_f^\xi)$ is a non-decreasing function of $\xi$. Here, $J_f$ denotes the event that $f$ is open.

**(2.27) Theorem**[4]**.** *Let $\mu$ be a strictly positive probability measure on $(\Omega, \mathcal{F})$. The following are equivalent.*

(a) *$\mu$ is strongly positively-associated.*

(b) *$\mu$ satisfies the FKG lattice condition.*

(c) *$\mu$ is monotonic.*

(d) *$\mu$ is* 1*-monotonic.*

It is a near triviality to check that any product measure on $\Omega$ satisfies the FKG lattice condition, and thus product measures are strongly positively-associated. This is the $q = 1$ case of Theorem 3.8, and is usually referred to as Harris's inequality, [181]. We give two examples of probability measures that are positively associated but do not satisfy the statements of the above theorem.

**(2.28) Example**[5]**.** Let $\epsilon, \delta \in (0, 1)$, and let $\mu_0$, $\mu_1$ be the probability measures on $\{0, 1\}^3$ given by

$$\mu_0(010) = \mu_0(001) = \delta, \quad \mu_0(000) = 1 - 2\delta,$$
$$\mu_1(111) = \mu_1(100) = \tfrac{1}{2}.$$

Let $\epsilon \in [0, 1]$ and set $\mu = \epsilon\mu_0 + (1 - \epsilon)\mu_1$. Note that

$$\mu(011) = \mu(101) = \mu(110) = 0.$$

It may be checked that $\mu$ does not satisfy the FKG lattice condition whereas, for sufficiently small positive values of the constants $\epsilon$, $\delta$, the measure $\mu$ is positively associated. Note from the above that $\mu$ is not strictly positive. However, a strictly positive example may be arranged by replacing $\mu$ by the probability measure $\mu' = (1 - \eta)\mu + \eta\mu_2$ where

$$\mu_2(011) = \mu_2(101) = \mu_2(110) = \tfrac{1}{3}$$

and $\eta$ is small and positive.

**(2.29) Example**[6]**.** Let $X$ and $Y$ be independent Bernoulli random variables with parameter $\frac{1}{2}$, so that

$$\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = \tfrac{1}{2},$$

and similarly for $Y$. Let $Z = \max\{X, Y\}$. It is clear that

$$\mathbb{P}(X = 1 \mid Z = 1) > \mathbb{P}(X = 1), \quad \mathbb{P}(X = 1 \mid Y = Z = 1) = \mathbb{P}(X = 1).$$

[4]Closely related material is discussed in [204]. The equivalence of (a) and (b) is attributed in [8] to J. van den Berg and R. M. Burton (1987). See [136] for a further discussion of monotonic measures.

[5]Proposed by J. Steif.

[6]Proposed by J. van den Berg.

It is easy to deduce that the law $\mu$ of the triple $(X, Y, Z)$ is not monotonic. It is however positively associated since the triple $(X, Y, Z)$ is an increasing function of the independent pair $X, Y$.

As in the previous example, $\mu$ is not strictly positive, a weakness which we remedy differently than before. Let $X'$, $Y'$, $Z'$ (respectively, $X''$, $Y''$, $Z''$) be Bernoulli random variables with parameter $\delta$ (respectively, $1-\delta$), and assume the maximal amount of independence. The triple

$$(A, B, C) = \big((X \vee X') \wedge X'', (Y \vee Y') \wedge Y'', (Z \vee Z') \wedge Z''\big)$$

is an increasing function of positively associated random variables, and is therefore positively associated. However, for small positive $\delta$, it is only a small (stochastic) perturbation of the original triple $(X, Y, Z)$, and one may check that $(A, B, C)$ is not monotonic. It is easily verified that $\mathbb{P}((A, B, C) = \omega) > 0$ for all $\omega \in \{0, 1\}^3$.

*Proof of Theorem 2.27.* Throughout, $\mu$ is assumed strictly positive.

(a) $\iff$ (b). We prove first that (a) implies (b). By Theorem 2.22, it suffices to prove (2.18) for two incomparable configurations $\omega_1, \omega_2$ that disagree on exactly two distinct edges $e, f \in E$. We order $E = (e_1, e_2, \dots, e_m)$ with $e_1 = e$ and $e_2 = f$, and we express a configuration $\omega$ as a 'word' $\omega(e_1) \cdot \omega(e_2) \cdot \dots \cdot \omega(e_m)$ in the alphabet with two letters. Thus $\omega_1 = 0 \cdot 1 \cdot w$ and $\omega_2 = 1 \cdot 0 \cdot w$ for some word $w$ of length $|E| - 2$. By strong positive-association, $\alpha(xy) = \mu(x \cdot y \cdot w)$ satisfies

$$\alpha(11)\big[\alpha(00) + \alpha(01) + \alpha(10) + \alpha(11)\big] \geq \big[\alpha(01) + \alpha(11)\big]\big[\alpha(10) + \alpha(11)\big],$$

which may be simplified to obtain as required that

$$\alpha(11)\alpha(00) \geq \alpha(01)\alpha(10).$$

We prove next that (b) implies (a). Suppose (b) holds, and let $F \subseteq E$ and $\xi \in \Omega$. It is immediate from (2.24) that

$$\mu_F^\xi(\omega_1 \vee \omega_2)\mu_F^\xi(\omega_1 \wedge \omega_2) \geq \mu_F^\xi(\omega_1)\mu_F^\xi(\omega_2), \qquad \omega_1, \omega_2 \in \Omega_F.$$

By Theorem 2.19, $\mu_F^\xi$ is positively associated.

(b) $\implies$ (c). By the Holley inequality, Theorem 2.1, it suffices to prove for $\omega_F, \rho_F \in \Omega_F$ that

$$\mu_F^\zeta(\omega_F \vee \rho_F)\mu_F^\xi(\omega_F \wedge \rho_F) \geq \mu_F^\zeta(\omega_F)\mu_F^\xi(\rho_F) \qquad \text{whenever } \xi \leq \zeta.$$

This is, by (2.24), an immediate consequence of the FKG lattice property applied to the pair $\omega_F \times \zeta$, $\rho_F \times \xi$.

(c) $\implies$ (d). This is trivial.

(d) $\implies$ (b). Let $\mu$ be 1-monotonic. By Theorem 2.3, the pair $\mu$, $\mu$ satisfies (2.2), which is to say that $\mu$ satisfies the FKG lattice condition. □

## 2.3 Influence for monotonic measures

Let $N \geq 1$, and let $E$ be an arbitary finite set with $|E| = N$. We write $\Omega = \{0, 1\}^E$ as usual, and $\mathcal{F}$ for the set of all subsets of $\Omega$. Let $\mu$ be a probability measure on $(\Omega, \mathcal{F})$, and $A$ an increasing event. The (*conditional*) *influence* on $A$ of the edge $e \in E$ is defined by

$$I_A(e) = \mu(A \mid J_e = 1) - \mu(A \mid J_e = 0), \tag{2.30}$$

where $J = (J_e : e \in E)$ denotes[7] the identity function on $\Omega$. There has been an extensive study of the largest influence, $\max_e I_A(e)$, when $\mu$ is a product measure, and this has been used to obtain concentration theorems for $\phi_p(A)$ viewed as a function of $p$, where $\phi_p$ denotes product measure with density $p$ on $\Omega$. Such results have applications to several topics including random graphs, random walks, and percolation. Theorems concerning influence were first proved for product measures, but they may be extended in a natural way to monotonic measures.

**(2.31) Theorem (Influence) [141].** *There exists a constant $c$ satisfying $c \in (0, \infty)$ such that the following holds. Let $N \geq 1$, let $E$ be a finite set with $|E| = N$, and let $A$ be an increasing subset of $\Omega = \{0, 1\}^E$. Let $\mu$ be a strictly positive probability measure on $(\Omega, \mathcal{F})$ that is monotonic. There exists $e \in E$ such that*

$$I_A(e) \geq c \min\big\{\mu(A), 1 - \mu(A)\big\} \frac{\log N}{N}.$$

There are several useful references concerning influence for product measures, see [125, 126, 200, 201, 329] and their bibliographies. The order of magnitude $N^{-1} \log N$ is the best possible, see [34].

*Proof.* Let $\mu$ be strictly positive and monotonic. The idea is to encode $\mu$ in terms of Lebesgue measure $\lambda$ on the Euclidean cube $[0, 1]^E$, and then to apply the influence theorem[8] of [67]. This will be done via a certain function $f : [0, 1]^E \to \{0, 1\}^E$ constructed next. A similar argument will be used to prove Theorem 3.45.

We may suppose without loss of generality that $E = \{1, 2, \ldots, N\}$. Let $\mathbf{x} = (x_i : i = 1, 2, \ldots, N) \in [0, 1]^E$, and let $f(\mathbf{x}) = (f_i(\mathbf{x}) : i = 1, 2, \ldots, N) \in \mathbb{R}^E$ be given recursively as follows. The first coordinate $f_1(\mathbf{x})$ is defined by:

$$\text{with} \quad a_1 = \mu(J_1 = 1), \quad \text{let} \quad f_1(\mathbf{x}) = \begin{cases} 1 & \text{if } x_1 > 1 - a_1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}$$

Suppose we know the values $f_i(\mathbf{x})$ for $i = 1, 2, \ldots, k - 1$. Let

$$a_k = \mu\big(J_k = 1 \,\big|\, J_i = f_i(\mathbf{x}) \text{ for } i = 1, 2, \ldots, k - 1\big), \tag{2.33}$$

[7] Thus, $J_e$ denotes both the event $\{\omega \in \Omega : \omega(e) = 1\}$ and its indicator function.

[8] An interesting aspect of the proof of this theorem is the use of discrete Fourier transforms and hypercontractivity.

and define

$$f_k(\mathbf{x}) = \begin{cases} 1 & \text{if } x_k > 1 - a_k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.34}$$

It may be shown as follows that the function $f : [0,1]^E \to \{0,1\}^E$ is non-decreasing. Let $\mathbf{x} \le \mathbf{x}'$, and write $a_k = a_k(\mathbf{x})$ and $a'_k = a_k(\mathbf{x}')$ for the values in (2.32)–(2.33) corresponding to the vectors $\mathbf{x}$ and $\mathbf{x}'$. Clearly $a_1 = a'_1$, so that $f_1(\mathbf{x}) \le f_1(\mathbf{x}')$. Since $\mu$ is monotonic, $a_2 \le a'_2$, implying that $f_2(\mathbf{x}) \le f_2(\mathbf{x}')$. Continuing inductively, we find that $f_k(\mathbf{x}) \le f_k(\mathbf{x}')$ for all $k$, which is to say that $f(\mathbf{x}) \le f(\mathbf{x}')$.

Let $A \in \mathcal{F}$ be an increasing event, and let $B$ be the increasing subset of $[0,1]^E$ given by $B = f^{-1}(A)$. We make four notes concerning the definition of $f$.

(a) For given $\mathbf{x}$, each $a_k$ depends only on $x_1, x_2, \dots, x_{k-1}$.

(b) Since $\mu$ is strictly positive, the $a_k$ satisfy $0 < a_k < 1$ for all $\mathbf{x} \in [0,1]^N$ and $k \in E$.

(c) For any $\mathbf{x} \in [0,1]^N$ and $k \in E$, the values $f_k(\mathbf{x}), f_{k+1}(\mathbf{x}), \dots, f_N(\mathbf{x})$ depend on $x_1, x_2, \dots, x_{k-1}$ only through the values $f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_{k-1}(\mathbf{x})$.

(d) The function $f$ and the event $B$ depend on the ordering of the set $E$.

Let $U = (U_i : i = 1, 2, \dots, N)$ be the identity function on $[0,1]^E$, so that $U$ has law $\lambda$. By the definition of $f$, $f(U)$ has law $\mu$. Hence,

$$\mu(A) = \lambda(f(U) \in A) = \lambda(U \in f^{-1}(A)) = \lambda(B). \tag{2.35}$$

Let

$$K_B(i) = \lambda(B \mid U_i = 1) - \lambda(B \mid U_i = 0),$$

where the conditional probabilities are interpreted as

$$\lambda(B \mid U_i = u) = \lim_{\epsilon \downarrow 0} \lambda\big(B \,\big|\, U_i \in (u - \epsilon, u + \epsilon)\big).$$

By [67, Thm 1], there exists a constant $c \in (0, \infty)$, independent of the choice of $N$ and $A$, such that: there exists $e \in E$ with

$$K_B(e) \ge c \min\big\{\lambda(B), 1 - \lambda(B)\big\} \frac{\log N}{N}. \tag{2.36}$$

We choose $e$ accordingly. We claim that

$$I_A(j) \ge K_B(j) \qquad \text{for } j \in E. \tag{2.37}$$

By (2.35) and (2.36), it suffices to prove (2.37). We prove first that

$$I_A(1) \ge K_B(1), \tag{2.38}$$

which is stronger than (2.37) with $j = 1$. By (b) and (c) above,

$$I_A(1) = \mu(A \mid J_1 = 1) - \mu(A \mid J_1 = 0)$$
$$= \lambda(B \mid f_1(U) = 1) - \lambda(B \mid f_1(U) = 0)$$
$$= \lambda(B \mid U_1 > 1 - a_1) - \lambda(B \mid U_1 \le 1 - a_1)$$
$$= \lambda(B \mid U_1 = 1) - \lambda(B \mid U_1 = 0)$$
$$= K_B(1). \tag{2.39}$$

We turn to (2.37) with $j \ge 2$. We re-order the set $E$ to bring the index $j$ to the front. That is, we let $F$ be the re-ordered index set $F = (k_1, k_2, \dots, k_N) = (j, 1, 2, \dots, j-1, j+1, \dots, N)$. Let $g = (g_{k_r} : r = 1, 2, \dots, N)$ denote the associated function given by (2.32)–(2.34) subject to the new ordering, and let $C = g^{-1}(A)$. We claim that

$$K_C(k_1) \ge K_B(j). \tag{2.40}$$

By (2.39) with $E$ replaced by $F$, $K_C(k_1) = I_A(j)$, and (2.37) follows. It remains to prove (2.40), and we use monotonicity again for this. It suffices to prove that

$$\lambda(C \mid U_j = 1) \ge \lambda(B \mid U_j = 1), \tag{2.41}$$

together with the reversed inequality given $U_j = 0$. Let

$$\overline{U} = (U_1, U_2, \dots, U_{j-1}, 1, U_{j+1}, \dots, U_N). \tag{2.42}$$

The 0/1-vector $f(\overline{U}) = (f_i(\overline{U}) : i = 1, 2, \dots, N)$ is constructed sequentially (as above) by considering the indices $1, 2, \dots, N$ in turn. At stage $k$, we declare $f_k(\overline{U})$ equal to 1 if $U_k$ exceeds a certain function $a_k$ of the variables $f_i(\overline{U})$, $1 \le i < k$. By the monotonicity of $\mu$, this function is non-increasing in these variables. The index $j$ plays a special role in that: (i) $f_j(\overline{U}) = 1$, and (ii) given this fact, it is more likely than before that the variables $f_k(\overline{U})$, $j < k \le N$, will take the value 1. The values $f_k(\overline{U})$, $1 \le k < j$ are unaffected by the value of $U_j$.

Consider now the 0/1-vector $g(\overline{U}) = (g_{k_r}(\overline{U}) : r = 1, 2, \dots, N)$, constructed in the same manner as above but with the new ordering $F$ of the index set $E$. First we examine index $k_1$ $(= j)$, and we automatically declare $g_{k_1}(\overline{U}) = 1$ (since $U_j = 1$). We then construct $g_{k_r}(\overline{U})$, $r = 2, 3, \dots, N$, in sequence. Since the $a_k$ are non-decreasing in the variables constructed so far,

$$g_{k_r}(\overline{U}) \ge f_{k_r}(\overline{U}), \qquad r = 2, 3, \dots, N. \tag{2.43}$$

Therefore, $g(\overline{U}) \ge f(\overline{U})$, and hence

$$\lambda(C \mid U_j = 1) = \lambda(g(\overline{U}) \in A) \ge \lambda(f(\overline{U}) \in A) = \lambda(B \mid U_j = 1). \tag{2.44}$$

Inequality (2.41) has been proved. The same argument implies the reversed inequality obtained from (2.41) by changing the conditioning to $U_j = 0$. Inequality (2.40) follows, and the proof is complete. □

## 2.4 Sharp thresholds for increasing events

We consider next certain families of probability measures $\mu_p$ indexed by a parameter $p \in (0,1)$, and we prove a sharp-threshold theorem subject to a hypothesis of monotonicity. The idea is as follows. Let $A$ be a non-empty increasing event in $\Omega = \{0,1\}^N$. Subject to a certain hypothesis on the $\mu_p$, the function $f(p) = \mu_p(A)$ is non-decreasing with $f(0) = 0$ and $f(1) = 1$. If $A$ has a certain property of symmetry, the sharp-threshold theorem asserts that $f(p)$ increases steeply from 0 to 1 over a short interval of $p$-values with length of order $1/\log N$.

We use the notation of the previous section. Let $\mu$ be a probability measure on $(\Omega, \mathcal{F})$. For $p \in (0,1)$, let $\mu_p$ be the probability measure given by

$$\mu_p(\omega) = \frac{1}{Z_p}\mu(\omega)\left\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\}, \qquad \omega \in \Omega, \tag{2.45}$$

where $Z_p$ is the normalizing constant

$$Z_p = \sum_{\omega\in\Omega}\mu(\omega)\left\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\}.$$

It is elementary that $\mu = \mu_{\frac{1}{2}}$, and that (each) $\mu_p$ is strictly positive if and only if $\mu$ is strictly positive. It is easy to check that (each) $\mu_p$ satisfies the FKG lattice condition (2.18) if and only if $\mu$ satisfies this condition, and it follows by Theorem 2.27 that, for strictly positive $\mu$, $\mu$ is monotonic if and only if (each) $\mu_p$ is monotonic. In order to prove a sharp-threshold theorem for the family $\mu_p$, we present first a differential formula of the type referred to as Russo's formula, [154, Section 2.4].

**(2.46) Theorem [39].** *For a random variable $X : \Omega \to \mathbb{R}$,*

$$\frac{d}{dp}\mu_p(X) = \frac{1}{p(1-p)}\operatorname{cov}_p(|\eta|, X), \qquad p \in (0,1), \tag{2.47}$$

*where $\operatorname{cov}_p$ denotes covariance with respect to the probability measure $\mu_p$, and $\eta(\omega)$ is the set of $\omega$-open edges.*

We note for later use that

$$\operatorname{cov}_p(|\eta|, X) = \sum_{e\in E}\operatorname{cov}_p(J_e, X). \tag{2.48}$$

*Proof.* We follow [39, Prop. 4] and [156, Section 2.4]. Write

$$\nu_p(\omega) = p^{|\eta(\omega)|}(1-p)^{N-|\eta(\omega)|}\mu(\omega), \qquad \omega \in \Omega,$$

so that

$$\mu_p(X) = \frac{1}{Z_p} \sum_{\omega \in \Omega} X(\omega) \nu_p(\omega). \tag{2.49}$$

It is elementary that

$$\frac{d}{dp}\mu_p(X) = \frac{1}{Z_p} \sum_{\omega \in \Omega} \left( \frac{|\eta(\omega)|}{p} - \frac{N - |\eta(\omega)|}{1-p} \right) X(\omega)\nu_p(\omega) - \frac{Z_p'}{Z_p}\mu_p(X), \tag{2.50}$$

where $Z_p' = dZ_p/dp$. Setting $X = 1$, we find that

$$0 = \frac{1}{p(1-p)}\mu_p(|\eta| - pN) - \frac{Z_p'}{Z_p},$$

whence

$$\begin{aligned} p(1-p)\frac{d}{dp}\mu_p(X) &= \mu_p\big([|\eta| - pN]X\big) - \mu_p(|\eta| - pN)\mu_p(X) \\ &= \mu_p(|\eta|X) - \mu_p(|\eta|)\mu_p(X) \\ &= \operatorname{cov}_p(|\eta|, X). \end{aligned}$$

□

Let $\Pi$ be the group of permutations of $E$. Any $\pi \in \Pi$ acts[9] on $\Omega$ by $\pi\omega = (\omega(\pi_e) : e \in E)$. We say that a subgroup $\mathcal{A}$ of $\Pi$ *acts transitively* on $E$ if, for all pairs $j, k \in E$, there exists $\alpha \in \mathcal{A}$ with $\alpha_j = k$.

Let $\mathcal{A}$ be a subgroup of $\Pi$. A probability measure $\phi$ on $(\Omega, \mathcal{F})$ is called $\mathcal{A}$-*invariant* if $\phi(\omega) = \phi(\alpha\omega)$ for all $\alpha \in \mathcal{A}$. An event $A \in \mathcal{F}$ is called $\mathcal{A}$-*invariant* if $A = \alpha A$ for all $\alpha \in \mathcal{A}$. It is easily seen that, for any subgroup $\mathcal{A}$, $\mu$ is $\mathcal{A}$-invariant if and only if (each) $\mu_p$ is $\mathcal{A}$-invariant.

**(2.51) Theorem (Sharp threshold) [141].** *There exists a constant $c$ satisfying $c \in (0, \infty)$ such that the following holds. Let $N = |E| \geq 1$ and let $A \in \mathcal{F}$ be an increasing event. Let $\mu$ be a strictly positive probability measure on $(\Omega, \mathcal{F})$ that is monotonic. Suppose there exists a subgroup $\mathcal{A}$ of $\Pi$ acting transitively on $E$ such that $\mu$ and $A$ are $\mathcal{A}$-invariant. Then*

$$\frac{d}{dp}\mu_p(A) \geq \frac{cm_p}{p(1-p)} \min\big\{\mu_p(A), 1 - \mu_p(A)\big\} \log N, \qquad p \in (0,1), \tag{2.52}$$

*where $m_p = \mu_p(J_e)(1 - \mu_p(J_e))$.*

Let $\epsilon \in (0, \frac{1}{2})$ and let $A$ be non-empty and increasing. Under the conditions of the theorem, $\mu_p(A)$ increases from $\epsilon$ to $1 - \epsilon$ over an interval of values of $p$ having length of order $1/\log N$. This amounts to a quantification of the so-called S-shape results described and cited in [154, Section 2.5]. Note that $m_p$ does not depend on the choice of edge $e$.

The proof is preceded by an easy lemma. Let

$$I_{p,A}(e) = \mu_p(A \mid J_e = 1) - \mu_p(A \mid J_e = 0), \qquad e \in E.$$

[9] This differs slightly from the definition of Section 4.3, for reasons of local convenience.

**(2.53) Lemma.** *Let $A \in \mathcal{F}$. Suppose there exists a subgroup $\mathcal{A}$ of $\Pi$ acting transitively on $E$ such that $\mu$ and $A$ are $\mathcal{A}$-invariant. Then $I_{p,A}(e) = I_{p,A}(f)$ for all $e, f \in E$ and all $p \in (0,1)$.*

*Proof.* Since $\mu$ is $\mathcal{A}$-invariant, so is $\mu_p$ for every $p$. Let $e, f \in E$, and find $\alpha \in \mathcal{A}$ such that $\alpha_e = f$. Under the given conditions,

$$\begin{aligned}\mu_p(A,\ J_f = 1) &= \sum_{\omega \in A} \mu_p(\omega) J_f(\omega) = \sum_{\omega \in A} \mu_p(\alpha\omega) J_e(\alpha\omega) \\ &= \sum_{\omega' \in A} \mu_p(\omega') J_e(\omega') = \mu_p(A,\ J_e = 1).\end{aligned}$$

We deduce with $A = \Omega$ that $\mu_p(J_f = 1) = \mu_p(J_e = 1)$. On dividing, we obtain that $\mu_p(A \mid J_f = 1) = \mu_p(A \mid J_e = 1)$. A similar equality holds with 1 replaced by 0, and the claim of the lemma follows. □

*Proof of Theorem 2.51.* By Lemma 2.53, $I_{p,A}(e) = I_{p,A}(f)$ for all $e, f \in E$. Since $A$ is increasing and $\mu_p$ is monotonic, each $I_{p,A}(e)$ is non-negative, and therefore

$$\begin{aligned}\mathrm{cov}_p(J_e, 1_A) &= \mu_p(J_e 1_A) - \mu_p(J_e)\mu_p(A) \\ &= \mu_p(J_e)(1 - \mu_p(J_e)) I_{p,A}(e) \\ &\geq m_p I_{p,A}(e), \qquad e \in E.\end{aligned}$$

Summing over the index set $E$ as in (2.47)–(2.48), we deduce (2.52) by Theorem 2.31 applied to the monotonic measure $\mu_p$. □

## 2.5 Exponential steepness

This chapter closes with a further differential inequality for the probability of a monotonic event. Let $A \in \mathcal{F}$ and $\omega \in \Omega$. We define $H_A(\omega)$ to be the Hamming distance from $\omega$ to $A$, that is,

$$H_A(\omega) = \inf\{H(\omega', \omega) : \omega' \in A\}, \tag{2.54}$$

where $H(\omega', \omega)$ is given in (1.26). Note that

(2.55)
$$H_A(\omega) = \begin{cases} \inf\left\{\sum_e [\omega'(e) - \omega(e)] : \omega' \geq \omega,\ \omega' \in A\right\} & \text{if } A \text{ is increasing}, \\ \inf\left\{\sum_e [\omega(e) - \omega'(e)] : \omega' \leq \omega,\ \omega' \in A\right\} & \text{if } A \text{ is decreasing}. \end{cases}$$

Suppose now that $A$ is increasing (respectively, decreasing). Here are three useful facts concerning $H_A$.

(i) $H_A$ is a decreasing (respectively, increasing) random variable.

(ii) The function $|\eta| + H_A$ (respectively, $|\eta| - H_A$) is increasing, since the addition of a single open edge to a configuration $\omega$ causes $|\eta(\omega)|$ to increase by 1, and $H_A(\omega)$ to decrease (respectively, increase) by at most 1.

(iii) We have that $H_A(\omega)1_A(\omega) = 0$ for $\omega \in \Omega$.

Given a probability measure $\mu$ on $(\Omega, \mathcal{F})$, the associated measures $\mu_p$, $p \in (0, 1)$, are given by (2.45).

**(2.56) Theorem [153, 163].** *Let $\mu$ be a strictly positive probability measure on $(\Omega, \mathcal{F})$ that is monotonic. For a non-empty event $A \in \mathcal{F}$, and $p \in (0, 1)$,*

$$\frac{d}{dp} \log \mu_p(A) \geq \frac{\mu_p(H_A)}{p(1-p)}, \qquad \text{if } A \text{ is increasing}, \tag{2.57}$$

$$\frac{d}{dp} \log \mu_p(A) \leq -\frac{\mu_p(H_A)}{p(1-p)}, \qquad \text{if } A \text{ is decreasing}. \tag{2.58}$$

Inequality (2.57) bears a resemblance to a formula valid for percolation that may be written as

$$\frac{d}{dp} \log \phi_p(A) = \frac{1}{p} \phi_p(N_A \mid A),$$

where $N_A$ is the number of pivotal edges for the increasing event $A$, and $\phi_p$ denotes product measure with density $p$ on $(\Omega, \mathcal{F})$. See [154, p. 44] for further details.

*Proof.* Since $\mu$ is assumed strictly positive and monotonic, it satisfies the FKG lattice property. Therefore, every $\mu_p$ satisfies the FKG lattice property, and hence is positively associated. Let $A \in \mathcal{F}$ be non-empty and increasing. By (2.47), (ii)–(iii) above, and positive association,

$$\begin{aligned}
\frac{d}{dp}\mu_p(A) &= \frac{1}{p(1-p)} \mathrm{cov}_p(|\eta|, 1_A) \\
&= \frac{1}{p(1-p)} \big[\mathrm{cov}_p(|\eta| + H_A, 1_A) - \mathrm{cov}_p(H_A, 1_A)\big] \\
&\geq -\frac{1}{p(1-p)} \mathrm{cov}_p(H_A, 1_A) \\
&= \frac{\mu_p(H_A)\mu_p(A)}{p(1-p)},
\end{aligned}$$

and (2.57) follows. The argument is easily adapted for decreasing $A$. □

Let $A \in \mathcal{F}$ be non-empty and increasing. Inequality (2.57) is usually used in integrated form. Integrating over the interval $[r, s]$, and using the facts that $p(1-p) \leq \frac{1}{4}$ and that $H_A$ is decreasing, we obtain that

$$\begin{aligned}
\mu_r(A) &\leq \mu_s(A) \exp\left\{-4 \int_r^s \mu_p(H_A)\, dp\right\} \\
&\leq \mu_s(A) \exp\big\{-4(s-r)\mu_s(H_A)\big\}, \qquad 0 < r \leq s < 1.
\end{aligned} \tag{2.59}$$

This may sometimes be combined with a complementary inequality derived by a consideration of 'finite energy', see Theorem 3.45.