Chapter 1

Random-Cluster Measures

Summary. The random-cluster model is introduced, and its relationship to Ising and Potts models is presented via a coupling of probability measures. In the limit as the cluster-weighting factor tends to 0, one arrives at electrical networks and uniform spanning trees and forests.

1.1 Introduction

In 1925 came the Ising model for a ferromagnet, and in 1957 the percolation model for a disordered medium. Each has since been the subject of intense study, and their theories have become elaborate. Each possesses a phase transition marking the onset of long-range order, defined in terms of correlation functions for the Ising model and in terms of the unboundedness of paths for percolation. These two phase transitions have been the scenes of notable exact (and rigorous) calculations which have since inspired many physicists and mathematicians.

It has been known since at least 1847 that electrical networks satisfy so-called 'series/parallel laws'. Piet Kasteleyn noted during the 1960s that the percolation and Ising models also have such properties. This simple observation led in joint work with Cees Fortuin to the formulation of the random-cluster model. This new model has two parameters, an 'edge-weight' p and a 'cluster-weight' q. The (bond) percolation model is retrieved by setting $q = 1$; when $q = 2$, we obtain a representation of the Ising model, and similarly of the Potts model when $q = 2, 3, \ldots$. The discovery of the model is described in Kasteleyn's words in the Appendix of the current work.

The mathematics begins with a finite graph $G = (V, E)$, and the associated Ising model[1] thereon. A random variable σ_x taking values -1 and $+1$ is assigned to each vertex x of G, and the probability of the configuration $\sigma = (\sigma_x : x \in V)$ is taken to be proportional to $e^{-\beta H(\sigma)}$, where $\beta > 0$ and the 'energy' $H(\sigma)$ is the

[1]The so-called Ising model [187] was in fact proposed (to Ising) by Lenz. The Potts model [103, 271] originated in a proposal (to Potts) by Domb.

negative of the sum of $\sigma_x\sigma_y$ over all edges $e = \langle x, y\rangle$ of G. As β increases, greater probability is assigned to configurations having a large number of neighbouring pairs of vertices with equal signs. The Ising model has proved extraordinarily successful in generating beautiful mathematics of relevance to the physics, and it has been useful and provocative in the mathematical theory of phase transitions and cooperative phenomena (see, for example, [115]). The proof of the existence of a phase transition in two dimensions was completed by Peierls, [259], by way of his famous "argument".

There are many possible generalizations of the Ising model in which the σ_x may take a general number q of values, rather than $q = 2$ only. One such extension, the so-called 'Potts model', [271], has attracted especial interest amongst physicists, and has displayed a complex and varied structure. For example, when q is large, it possesses a discontinuous phase transition, in contrast to the continuous transition believed to take place for small q. Ising/Potts models are the first of three principal ingredients in the story of random-cluster models. Note that they are 'vertex-models' in the sense that they involve random variables σ_x indexed by the vertices x of the underlying graph. (There is a related extension of the Ising model due to Ashkin and Teller, [19], see Section 11.3.)

The (bond) percolation model was inspired by problems of physical type, and emerged from the mathematics literature[2] of the 1950s, [68]. In this model for a porous medium, each edge of the graph G is declared 'open' (to the passage of fluid) with probability p, and 'closed' otherwise, different edges having independent states. The problem is to determine the typical large-scale properties of connected components of open edges as the parameter p varies. Percolation theory is now a mature part of probability lying at the core of the study of random media and interacting systems, and it is the second ingredient in the story of random-cluster models. Note that bond percolation is an 'edge-model', in that the random variables are indexed by the set of edges of the underlying graph. (There is a variant termed 'site percolation' in which the vertices are open/closed at random rather than the edges, see [151, Section 1.6].)

The theory of electrical networks on the graph G is of course more ancient than that of Ising and percolation models, dating back at least to the 1847 paper, [212], in which Kirchhoff set down a method for calculating macroscopic properties of an electrical network in terms of its local structure. Kirchhoff's work explains in particular the relevance of counts of certain types of spanning trees of the graph. To import current language, an electrical network on a graph G may be studied via the properties of a 'uniformly random spanning tree' (UST) on G (see [29]).

The three ingredients above seemed fairly distinct until Fortuin and Kasteleyn discovered around 1970, [117, 118, 119, 120, 200], that each features within a certain parametric family of models which they termed 'random-cluster models'. They developed the basic theory of such models — correlation inequalities and the like — in this series of papers. The true power of random-cluster models as

[2] See also the historical curiosity [312].

a mechanism for studying Ising/Potts models has emerged progressively over the intervening three decades.

The configuration space of the random-cluster model is the set of all subsets of the edge-set E, which we represent as the set $\Omega = \{0, 1\}^E$ of 0/1-vectors indexed by E. An edge e is termed *open* in the configuration $\omega \in \Omega$ if $\omega(e) = 1$, and it is termed *closed* if $\omega(e) = 0$. The random-cluster model is thus an *edge*-model, in contrast to the Ising and Potts models which assign spins to the *vertices* of G. The subject of current study is the subgraph of G induced by the set of open edges of a configuration chosen at random from Ω according to a certain probability measure. Of particular importance is the existence (or not) of paths of open edges joining given vertices x and y, and thus the random-cluster model is a model in stochastic geometry.

The model may be viewed as a parametric family of probability measures $\phi_{p,q}$ on Ω, the two parameters being denoted by $p \in [0, 1]$ and $q \in (0, \infty)$. The parameter p amounts to a measure of the density of open edges, and the parameter q is a 'cluster-weighting' factor. When $q = 1$, $\phi_{p,q}$ is a product measure, and the ensuing probability space is usually termed a percolation model or a random graph depending on the context. The integer values $q = 2, 3, \ldots$ correspond in a certain way to the Potts model on G with q local states, and thus $q = 2$ corresponds to the Ising model. The nature of these 'correspondences', as described in Section 1.4, is that 'correlation functions' of the Potts model may be expressed as 'connectivity functions' of the random-cluster model. When extended to infinite graphs, it turns out that long-range order in a Potts model corresponds to the existence of infinite clusters in the corresponding random-cluster model. In this sense the Potts and percolation phase transitions are counterparts of one another.

Therein lies a major strength of the random-cluster model. Geometrical methods of some complexity have been derived in the study of percolation, and some of these may be adapted and extended to more general random-cluster models, thereby obtaining results of significance for Ising and Potts models. Such has been the value of the random-cluster model in studying Ising and Potts models that it is sometimes called simply the 'FK representation' of the latter systems, named after Fortuin and Kasteleyn. We shall see in Chapter 11 that several other spin models of statistical mechanics possess FK-type representations.

The random-cluster and Ising/Potts models on the graph $G = (V, E)$ are defined formally in the next two sections. Their relationship is best studied via a certain coupling on the product $\{0, 1\}^E \times \{1, 2, \ldots, q\}^V$, and this coupling is described in Section 1.4. The 'uniform spanning-tree' (UST) measure on G is a limiting case of the random-cluster measure, and this and related limits are the topic of Section 1.5. This chapter ends with a section devoted to basic notation.

1.2 Random-cluster model

Let $G = (V, E)$ be a finite graph. The graphs considered here will usually possess neither loops nor multiple edges, but we make no such general assumption. An edge e having endvertices x and y is written as $e = \langle x, y \rangle$. A random-cluster measure on G is a member of a certain class of probability measures on the set of subsets of the edge set E. We take as state space the set $\Omega = \{0, 1\}^E$, members of which are 0/1-vectors $\omega = (\omega(e) : e \in E)$. We speak of the edge e as being *open* (in ω) if $\omega(e) = 1$, and as being *closed* if $\omega(e) = 0$. For $\omega \in \Omega$, let $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ denote the set of open edges. There is a one–one correspondence between vectors $\omega \in \Omega$ and subsets $F \subseteq E$, given by $F = \eta(\omega)$. Let $k(\omega)$ be the number of connected components (or 'open clusters') of the graph $(V, \eta(\omega))$, and note that $k(\omega)$ includes a count of isolated vertices, that is, of vertices incident to no open edge. We associate with Ω the σ-field $\mathcal{F}$ of all its subsets.

A *random-cluster measure* on G has two parameters satisfying $p \in [0, 1]$ and $q \in (0, \infty)$, and is defined as the measure $\phi_{p,q}$ on the measurable pair $(\Omega, \mathcal{F})$ given by

$$\phi_{p,q}(\omega) = \frac{1}{Z_{\mathrm{RC}}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \qquad \omega \in \Omega, \tag{1.1}$$

where the 'partition function', or 'normalizing constant', Z_{RC} is given by

$$Z_{\mathrm{RC}} = Z_{\mathrm{RC}}(p, q) = \sum_{\omega \in \Omega} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}. \tag{1.2}$$

This measure differs from product measure through the inclusion of the term $q^{k(\omega)}$. Note the difference between the cases $q \leq 1$ and $q \geq 1$: the former favours fewer clusters, whereas the latter favours a larger number of clusters. When $q = 1$, edges are open/closed independently of one another. This very special case has been studied in detail under the titles 'percolation' and 'random graphs', see [59, 151, 191]. Perhaps the most important values of q are the integers, since the random-cluster model with $q \in \{2, 3, \dots\}$ corresponds, in a way described in the next two sections, to the Potts model with q local states. The bulk of the work presented in this book is devoted to the theory of random-cluster measures when $q \geq 1$. The case $q < 1$ seems to be harder mathematically and less important physically. There is some interest in the limit as $q \downarrow 0$; see Section 1.5.

We shall sometimes write $\phi_{G,p,q}$ for $\phi_{p,q}$ when the choice of graph G is to be stressed. Computer-generated samples from random-cluster measures on $\mathbb{Z}^2$ are presented in Figures 1.1–1.2. When $q = 1$, the measure $\phi_{p,q}$ is a product measure with density p, and we write $\phi_{G,p}$ or ϕ_p for this special case.

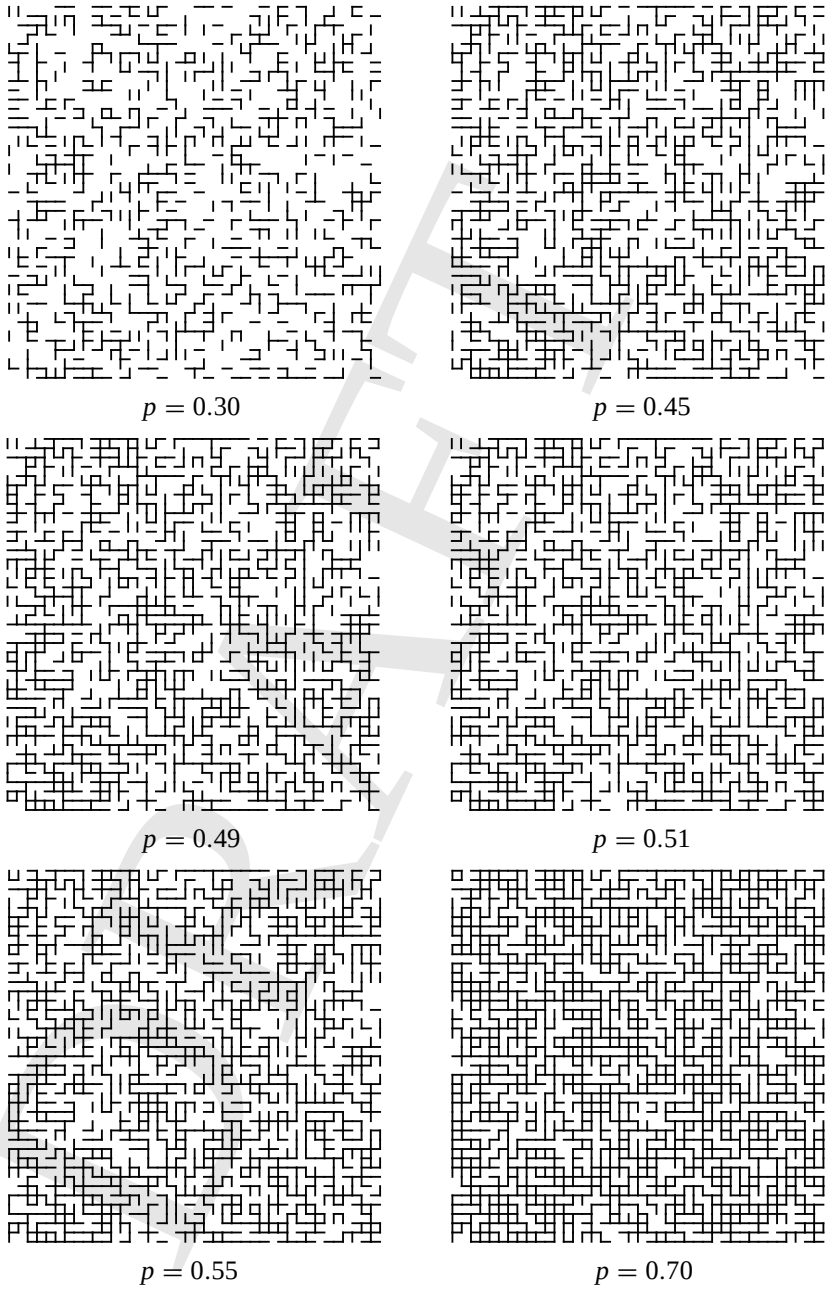

Figure 1.1. Samples from the random-cluster measure with $q = 1$ on a 40×40 box of the square lattice. We have set $q = 1$ for ease of programming, the measure being of product form in this case. The critical value is $p_c(1) = \frac{1}{2}$. Samples with more general values of q may be obtained by the method of 'coupling from the past', as described in Section 8.4.

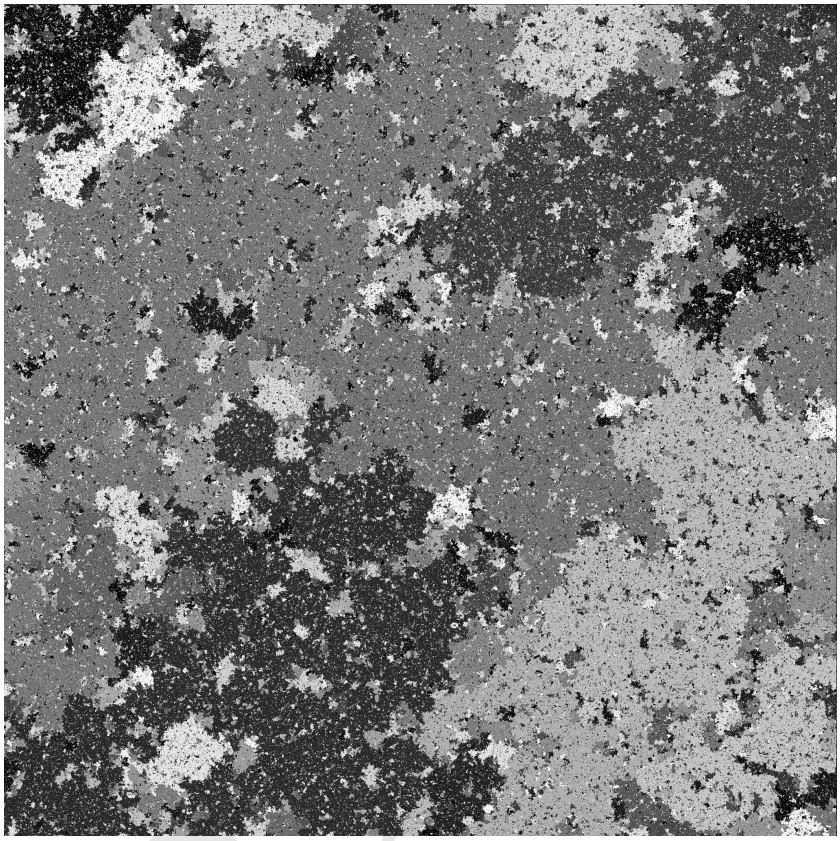

Figure 1.2. A picture of the random-cluster model with free boundary conditions on a 2048×2048 box of $\mathbb{L}^2$, with $p = 0.585816$ and $q = 2$. The critical value of the model with $q = 2$ is $p_c = \sqrt{2}/(1+\sqrt{2}) = 0.585786\ldots$, and therefore the simulation is of a mildly supercritical system. It was obtained by simulating the Ising model using Glauber dynamics (see Section 8.2), and then applying the coupling illustrated in Figure 1.3. Each individual cluster is highlighted with a different tint of gray, and the smaller clusters are not visible in the picture. This and later simulations in Section 5.7 are reproduced by kind permission of Raphaël Cerf.

1.3 Ising and Potts models

In a famous experiment, a piece of iron is exposed to a magnetic field. The field is increased from zero to a maximum, and then diminished to zero. If the temperature is sufficiently low, the iron retains some residual magnetization, otherwise it does not. There is a critical temperature for this phenomenon, often called the *Curie point* after Pierre Curie, who reported this discovery in his 1895 thesis, [96][3]. The

[3] In an example of Stigler's Law, [298], the existence of such a temperature was discovered before 1832 by Pouillet, see [195].

famous (Lenz–)Ising model for such ferromagnetism, [187], may be summarized as follows. One supposes that particles are positioned at the points of some lattice embedded in Euclidean space. Each particle may be in either of two states, representing the physical states of 'spin-up' and 'spin-down'. Spin-values are chosen at random according to a certain probability measure, known as a 'Gibbs state', which is governed by interactions between neighbouring particles. The relevant probability measure is given as follows.

Let $G = (V, E)$ be a finite graph representing part of the lattice. We think of each vertex $x \in V$ as being occupied by a particle having a random spin. Since spins are assumed to come in two basic types, we take as sample space the set $\Sigma = \{-1, +1\}^V$. The appropriate probability mass function $\lambda_{\beta,J,h}$ on Σ has three parameters satisfying $\beta, J \in [0, \infty)$ and $h \in \mathbb{R}$, and is given by

$$\lambda_{\beta,J,h}(\sigma) = \frac{1}{Z_{\mathrm{I}}} e^{-\beta H(\sigma)}, \qquad \sigma \in \Sigma, \tag{1.3}$$

where the partition function Z_{I} and the 'Hamiltonian' $H : \Sigma \to \mathbb{R}$ are given by

$$Z_{\mathrm{I}} = \sum_{\sigma \in \Sigma} e^{-\beta H(\sigma)}, \qquad H(\sigma) = -J \sum_{e=\langle x,y\rangle \in E} \sigma_x \sigma_y - h \sum_{x \in V} \sigma_x. \tag{1.4}$$

The physical interpretation of β is as the reciprocal $1/T$ of temperature, of J as the strength of interaction between neighbours, and of h as the external magnetic field. For reasons of simplicity, we shall consider here only the case of zero external-field, and we assume henceforth that $h = 0$.

Each edge has equal interaction strength J in the above formulation. Since β and J occur only as a product βJ, the measure $\lambda_{\beta,J,0}$ has effectively only a single parameter βJ. In a more complicated measure not studied here, different edges e are permitted to have different interaction strengths J_e, see Chapter 9. In the meantime we shall wrap β and J together by setting $J = 1$, and we write $\lambda_\beta = \lambda_{\beta,1,0}$

As pointed out by Baxter, [24], the Ising model permits an infinity of generalizations. Of these, the extension to so-called 'Potts models' has proved especially fruitful. Whereas the Ising model permits only two possible spin-values at each vertex, the Potts model [271] permits a general number $q \in \{2, 3, \dots\}$, and is governed by a probability measure given as follows.

Let q be an integer satisfying $q \geq 2$, and take as sample space the set of vectors $\Sigma = \{1, 2, \dots, q\}^V$. Thus each vertex of G may be in any of q states. For an edge $e = \langle x, y\rangle$ and a configuration $\sigma = (\sigma_x : x \in V) \in \Sigma$, we write $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ where $\delta_{i,j}$ is the Kronecker delta. The relevant probability measure is given by

$$\pi_{\beta,q}(\sigma) = \frac{1}{Z_{\mathrm{P}}} e^{-\beta H'(\sigma)}, \qquad \sigma \in \Sigma, \tag{1.5}$$

where $Z_{\mathrm{P}} = Z_{\mathrm{P}}(\beta, q)$ is the appropriate normalizing constant and the Hamiltonian H' is given by

$$H'(\sigma) = -\sum_{e=\langle x,y\rangle \in E} \delta_e(\sigma). \tag{1.6}$$

In the special case $q = 2$, the multiplicative formula

$$\delta_{\sigma_x,\sigma_y} = \tfrac{1}{2}(1 + \sigma_x\sigma_y), \quad \sigma_x, \sigma_y \in \{-1, +1\}, \tag{1.7}$$

is valid. It is now easy to see in this case that the ensuing Potts model is simply the Ising model with an adjusted value of β, in that $\pi_{\beta,2}$ is the measure obtained from $\lambda_{\beta/2}$ by re-labelling the local states.

Here is a brief mention of one further generalization of the Ising model, namely the so-called n-vector or O(n) model. Let $n \in \{1, 2, \dots\}$ and let $\mathbb{I}$ be the set of vectors of $\mathbb{R}^n$ with unit length. The n-vector model on $G = (V, E)$ has configuration space $\mathbb{I}^V$ and Hamiltonian

$$H_n(\mathbf{s}) = -\sum_{e=\langle x,y\rangle\in E} \mathbf{s}_x \cdot \mathbf{s}_y, \qquad \mathbf{s} = (\mathbf{s}_v : v \in V) \in \mathbb{I}^V,$$

where $\mathbf{s}_x \cdot \mathbf{s}_y$ denotes the dot product. When $n = 1$, this is the Ising model. It is called the X/Y model when $n = 2$, and the Heisenberg model when $n = 3$.

1.4 Random-cluster and Ising/Potts models coupled

Fortuin and Kasteleyn discovered that Potts models may be re-cast as random-cluster models, and furthermore that the relationship between the two systems facilitates an extended study of phase transitions in Potts models, see [118, 119, 120, 200]. Their methods were elementary in nature. In a more modern approach, we construct the two systems on a common probability space. There may in principle be many ways to do this, but the standard coupling of Edwards and Sokal, [106], is of special value.

Let $q \in \{2, 3, \dots\}$, $p \in [0, 1]$, and let $G = (V, E)$ be a finite graph. We consider the product sample space $\Sigma \times \Omega$ where $\Sigma = \{1, 2, \dots, q\}^V$ and $\Omega = \{0, 1\}^E$ as above. We define a probability mass function μ on $\Sigma \times \Omega$ by

$$\mu(\sigma, \omega) \propto \prod_{e\in E} \big\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\big\}, \qquad (\sigma, \omega) \in \Sigma \times \Omega, \tag{1.8}$$

where, as before, $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ for $e = \langle x, y\rangle \in E$. The constant of proportionality is exactly that which ensures the normalization

$$\sum_{(\sigma,\omega)\in\Sigma\times\Omega} \mu(\sigma, \omega) = 1.$$

By an expansion of (1.8),

$$\mu(\sigma, \omega) \propto \psi(\sigma)\phi_p(\omega)1_F(\sigma, \omega), \qquad (\sigma, \omega) \in \Sigma \times \Omega,$$

where ψ is the uniform probability measure on Σ, ϕ_p is product measure on Ω with density p, and 1_F is the indicator function of the event

$$F = \big\{(\sigma, \omega) : \delta_e(\sigma) = 1 \text{ for any } e \text{ satisfying } \omega(e) = 1\big\} \subseteq \Sigma \times \Omega. \tag{1.9}$$

Therefore, μ may be viewed as the product measure $\psi \times \phi_p$ conditioned on F.

Elementary calculations reveal the following facts.

(1.10) Theorem (Marginal measures of μ) [106]. *Let $q \in \{2, 3, \dots\}$, $p \in [0, 1)$, and suppose that $p = 1 - e^{-\beta}$.*

(a) Marginal on Σ. *The marginal measure $\mu_1(\sigma) = \sum_{\omega \in \Omega} \mu(\sigma, \omega)$ on Σ is the Potts measure*

$$\mu_1(\sigma) = \frac{1}{Z_{\mathrm{P}}} \exp\left\{\beta \sum_{e \in E} \delta_e(\sigma)\right\}, \qquad \sigma \in \Sigma.$$

(b) Marginal on Ω. *The marginal measure $\mu_2(\omega) = \sum_{\sigma \in \Sigma} \mu(\sigma, \omega)$ on Ω is the random-cluster measure*

$$\mu_2(\omega) = \frac{1}{Z_{\mathrm{RC}}} \left\{\prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega)}, \qquad \omega \in \Omega.$$

(c) Partition functions. *We have that*

$$\sum_{\omega \in \Omega} \left\{\prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega)} = \sum_{\sigma \in \Sigma} \prod_{e \in E} \exp[\beta(\delta_e(\sigma) - 1)], \quad (1.11)$$

which is to say that

$$Z_{\mathrm{RC}}(p, q) = e^{-\beta|E|} Z_{\mathrm{P}}(\beta, q). \quad (1.12)$$

The conditional measures of μ are given in the following theorem[4], and illustrated in Figure 1.3.

(1.13) Theorem (Conditional measures of μ) [106]. *Let $q \in \{2, 3, \dots\}$, $p \in [0, 1)$, and suppose that $p = 1 - e^{-\beta}$.*

(a) *For $\omega \in \Omega$, the conditional measure $\mu(\cdot \mid \omega)$ on Σ is obtained by putting random spins on entire clusters of ω (of which there are $k(\omega)$). These spins are constant on given clusters, are independent between clusters, and each is uniformly distributed on the set $\{1, 2, \dots, q\}$.*

(b) *For $\sigma \in \Sigma$, the conditional measure $\mu(\cdot \mid \sigma)$ on Ω is obtained as follows. If $e = \langle x, y \rangle$ is such that $\sigma_x \neq \sigma_y$, we set $\omega(e) = 0$. If $\sigma_x = \sigma_y$, we set*

$$\omega(e) = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{otherwise}, \end{cases}$$

the values of different $\omega(e)$ being (conditionally) independent random variables.

[4] The corresponding facts for the infinite lattice are given in Theorem 4.91.

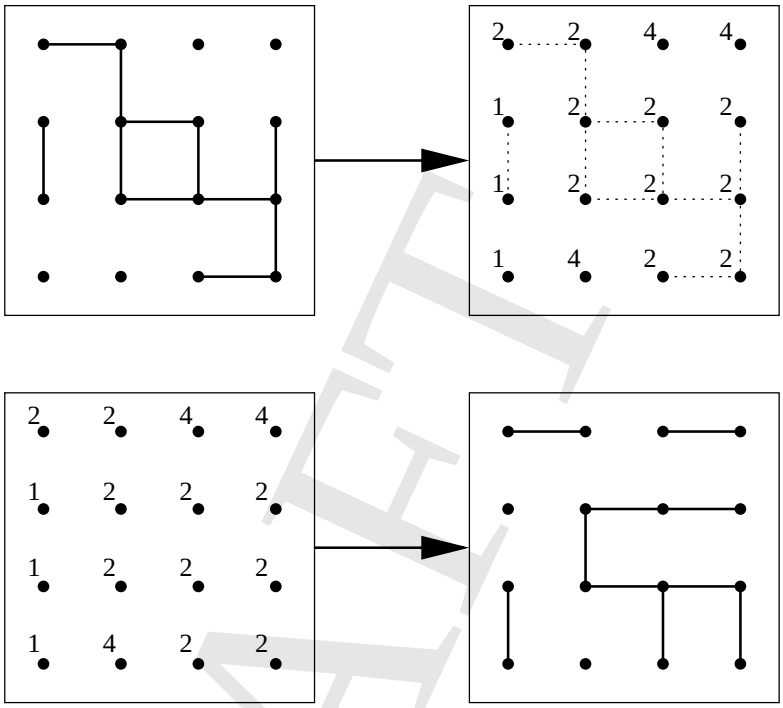

Figure 1.3. The upper diagram is an illustration of the conditional measure of μ on Σ given ω, with $q = 4$. To each open cluster of ω is allocated a spin-value chosen uniformly from $\{1, 2, 3, 4\}$. Different clusters are allocated independent values. In the lower diagram, we begin with a configuration σ. An edge is placed between vertices x, y with probability p (respectively, 0) if $\sigma_x = \sigma_y$ (respectively, $\sigma_x \neq \sigma_y$), and the outcome has as law the conditional measure of μ on Ω given σ.

In conclusion, the measure μ is a coupling of a Potts measure $\pi_{\beta,q}$ on V, together with the random-cluster measure $\phi_{p,q}$ on Ω. The parameters of these measures are related by the equation $p = 1 - e^{-\beta}$. Since $0 \leq p < 1$, we have that $0 \leq \beta < \infty$.

This special coupling may be used in a particularly simple way to show that correlations in Potts models correspond to open connections in random-cluster models. When extended to infinite graphs, this will imply that the phase transition of a Potts model corresponds to the creation of an infinite open cluster in the random-cluster model. Thus, arguments of stochastic geometry, and particularly those developed for the percolation model, may be harnessed directly in order to understand the correlation structure of the Potts system. The basic step is as follows.

Let $\{x \leftrightarrow y\}$ denote the set of all $\omega \in \Omega$ for which there exists an open path joining vertex x to vertex y. The complement of the event $\{x \leftrightarrow y\}$ is denoted by $\{x \not\leftrightarrow y\}$.

The 'two-point correlation function' of the Potts measure $\pi_{\beta,q}$ on the finite graph $G = (V, E)$ is defined to be the function $\tau_{\beta,q}$ given by

$$\tau_{\beta,q}(x, y) = \pi_{\beta,q}(\sigma_x = \sigma_y) - \frac{1}{q}, \qquad x, y \in V. \tag{1.14}$$

The term q^{-1} is the probability that two independent and uniformly distributed spins are equal. Thus[5],

$$\tau_{\beta,q}(x, y) = \frac{1}{q}\pi_{\beta,q}(q\delta_{\sigma_x,\sigma_y} - 1). \tag{1.15}$$

The 'two-point connectivity function' of the random-cluster measure $\phi_{p,q}$ is defined as the function $\phi_{p,q}(x \leftrightarrow y)$ for $x, y \in V$, that is, the probability that x and y are joined by a path of open edges. It turns out that these 'two-point functions' are (except for a constant factor) the same.

(1.16) Theorem (Correlation/connection) [200]. *Let $q \in \{2, 3, \dots\}$, $p \in [0, 1)$, and suppose that $p = 1 - e^{-\beta}$. Then*

$$\tau_{\beta,q}(x, y) = (1 - q^{-1})\phi_{p,q}(x \leftrightarrow y), \qquad x, y \in V.$$

The theorem may be generalized as follows. Suppose we are studying the Potts model, and are interested in some 'observable' $f : \Sigma \to \mathbb{R}$. The mean value of $f(\sigma)$ satisfies

$$\begin{aligned}\pi_{\beta,q}(f) &= \sum_{\sigma} f(\sigma)\pi_{\beta,q}(\sigma) = \sum_{\sigma,\omega} f(\sigma)\mu(\sigma, \omega)\\ &= \sum_{\omega} F(\omega)\phi_{p,q}(\omega) = \phi_{p,q}(F)\end{aligned}$$

where $F : \Omega \to \mathbb{R}$ is given by

$$F(\omega) = \mu(f \mid \omega) = \sum_{\sigma} f(\sigma)\mu(\sigma \mid \omega).$$

Theorem 1.16 is obtained by setting $f(\sigma) = \delta_{\sigma_x,\sigma_y} - q^{-1}$.

The Potts models considered above have zero external-field. Some complications arise when an external field is added; see the discussions in [13, 42].

Proof of Theorem 1.10. (a) Let $\sigma \in \Sigma$ be given. Then

$$\begin{aligned}\sum_{\omega\in\Omega} \mu(\sigma, \omega) &\propto \sum_{\omega\in\Omega} \prod_{e\in E} \left\{(1 - p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\right\}\\ &= \prod_{e\in E} [1 - p + p\delta_e(\sigma)].\end{aligned}$$

[5] If μ is a probability measure and X a random variable, the expectation of X with respect to μ is written $\mu(X)$.

Now $p = 1 - e^{-\beta}$ and

$$1 - p + p\delta = e^{\beta(\delta-1)}, \qquad \delta \in \{0, 1\},$$

whence

$$\sum_{\omega\in\Omega} \prod_{e\in E} \bigl\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\bigr\} = \prod_{e\in E} \exp[\beta(\delta_e(\sigma) - 1)]. \tag{1.17}$$

Viewed as a set of weights on Σ, the latter expression generates the Potts measure.
(b) Let $\omega \in \Omega$ be given. We have that

$$\prod_{e\in E} \bigl\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\bigr\} = p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}1_F(\sigma,\omega), \tag{1.18}$$

where $1_F(\sigma, \omega)$ is the indicator function that $\delta_e(\sigma) = 1$ whenever $\omega(e) = 1$, see (1.9). Now, $1_F(\sigma, \omega) = 1$ if and only if σ is constant on every open cluster of ω. There are $k(\omega)$ such clusters, and therefore $q^{k(\omega)}$ qualifying spin-vectors σ. Thus,

$$\sum_{\sigma\in\Sigma} \prod_{e\in E} \bigl\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\bigr\} = p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}q^{k(\omega)}. \tag{1.19}$$

This set of weights on Ω generates the random-cluster measure.
(c) We obtain the same answer if we sum (1.17) over all σ, or we sum (1.19) over all ω. □

Proof of Theorem 1.13. (a) Let $\omega \in \Omega$ be given. From (1.18)–(1.19),

$$\mu(\sigma \mid \omega) = \frac{1_F(\sigma,\omega)}{q^{k(\omega)}}, \qquad \sigma \in \Sigma,$$

whence the conditional measure is uniform on those σ with $1_F(\sigma, \omega) = 1$.
(b) Let $\sigma \in \Sigma$ be given. By (1.8),

$$\mu(\omega \mid \sigma) = K_\sigma \prod_{e\in E:\,\delta_e(\sigma)=0} \delta_{\omega(e),0} \prod_{e\in E:\,\delta_e(\sigma)=1} \bigl\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\bigr\},$$

where $K_\sigma = K_\sigma(p, q)$. Therefore, $\mu(\omega \mid \sigma)$ is a product measure on Ω with

$$\omega(e) = 1 \quad \text{with probability} \begin{cases} 0 & \text{if } \delta_e(\sigma) = 0, \\ p & \text{if } \delta_e(\sigma) = 1. \end{cases}$$

□

Proof of Theorem 1.16. By Theorem 1.13(a),

$$\begin{aligned}
\tau_{\beta,q}(x, y) &= \sum_{\sigma,\omega} \bigl\{1_{\{\sigma_x=\sigma_y\}}(\sigma) - q^{-1}\bigr\}\mu(\sigma,\omega) \\
&= \sum_{\omega} \phi_{p,q}(\omega) \sum_{\sigma} \mu(\sigma \mid \omega)\bigl\{1_{\{\sigma_x=\sigma_y\}}(\sigma) - q^{-1}\bigr\} \\
&= \sum_{\omega} \phi_{p,q}(\omega)\bigl\{(1-q^{-1})1_{\{x\leftrightarrow y\}}(\omega) + 0\cdot 1_{\{x\not\leftrightarrow y\}}(\omega)\bigr\} \\
&= (1-q^{-1})\phi_{p,q}(x \leftrightarrow y),
\end{aligned}$$

where μ is the above coupling of the Potts and random-cluster measures. □

Here is a final note. The random-cluster measure $\phi_{p,q}$ has two parameters p, q. In a more general version, we replace p by a vector $\mathbf{p} = (p_e : e \in E)$ of reals each of which satisfies $p_e \in [0, 1]$. The corresponding random-cluster measure $\phi_{\mathbf{p},q}$ on $(\Omega, \mathcal{F})$ is given by

$$\phi_{\mathbf{p},q}(\omega) = \frac{1}{Z}\left\{\prod_{e\in E} p_e^{\omega(e)}(1-p_e)^{1-\omega(e)}\right\} q^{k(\omega)}, \qquad \omega \in \Omega, \tag{1.20}$$

where Z is the appropriate normalizing factor. The measure $\phi_{p,q}$ is retrieved by setting $p_e = p$ for all $e \in E$.

1.5 The limit as $q \downarrow 0$

Let $G = (V, E)$ be a finite connected graph, and let $\phi_{p,q}$ be the random-cluster measure on G with parameters $p \in (0, 1)$, $q \in (0, \infty)$. We consider in this section the set of weak limits which may arise as $q \downarrow 0$. In preparation, we introduce three graph-theoretic terms.

A subset F of the edge-set E is called:

- a *forest* of G if the graph (V, F) contains no circuit,
- a *spanning tree* of G if (V, F) is connected and contains no circuit,
- a *connected subgraph* of G if (V, F) is connected.

In each case we consider the graph (V, F) containing every vertex of V; in this regard, sets F of edges satisfying one of the above conditions are sometimes termed *spanning*. Note that F is a spanning tree if and only if it is both a forest and a connected subgraph. For $\Omega = \{0, 1\}^E$ and $\omega \in \Omega$, we call ω a forest (respectively, spanning tree, connected subgraph) if $\eta(\omega)$ is a forest (respectively, spanning tree, connected subgraph). Write Ω_{for}, Ω_{st}, Ω_{cs} for the subsets of Ω containing all forests, spanning trees, and connected subgraphs, respectively, and write USF, UST, UCS for the uniform probability measures[6] on the respective sets Ω_{for}, Ω_{st}, Ω_{cs}.

We consider first the weak limit of $\phi_{p,q}$ as $q \downarrow 0$ for fixed $p \in (0, 1)$. This limit may be ascertained by observing that the dominant terms in the partition function

$$Z_{\text{RC}}(p, q) = \sum_{\omega\in\Omega} p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|} q^{k(\omega)}$$

are those for which $k(\omega)$ is a minimum, that is, those with $k(\omega) = 1$. It follows that $\lim_{q\downarrow 0} \phi_{p,q}$ is precisely the product measure $\phi_p = \phi_{p,1}$ (that is, percolation

[6] This usage of the term 'uniform spanning forest' differs from that of [29].

with intensity p) *conditioned on the resulting graph* $(V, \eta(\omega))$ *being connected*. That is, $\phi_{p,q} \Rightarrow \phi_r^{\mathrm{cs}}$ as $q \downarrow 0$, where $r = p/(1-p)$,

$$\phi_r^{\mathrm{cs}}(\omega) = \begin{cases} \dfrac{1}{Z_{\mathrm{cs}}} r^{|\eta(\omega)|} & \text{if } \omega \in \Omega_{\mathrm{cs}}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.21}$$

and $Z_{\mathrm{cs}} = Z_{\mathrm{cs}}(r)$ is the appropriate normalizing constant. In the special case $p = \frac{1}{2}$, we have that $\phi_{p,q} \Rightarrow \mathrm{UCS}$ as $q \downarrow 0$.

Further limits arise if we allow both p and q to converge to 0. Suppose $p = p_q$ is related to q in such a way that $p \to 0$ and $q/p \to 0$ as $q \downarrow 0$; thus, p approaches zero slower than does q. We may write Z_{RC} in the form

$$Z_{\mathrm{RC}}(p,q) = (1-p)^{|E|} \sum_{\omega \in \Omega} \left(\frac{p}{1-p} \right)^{|\eta(\omega)|+k(\omega)} \left(\frac{q(1-p)}{p} \right)^{k(\omega)}.$$

Note that $p/(1-p) \to 0$ and $q(1-p)/p \to 0$ as $q \downarrow 0$. Now, $k(\omega) \geq 1$ and $|\eta(\omega)| + k(\omega) \geq |V|$ for $\omega \in \Omega$; these two inequalities are satisfied simultaneously with equality if and only if $\omega \in \Omega_{\mathrm{st}}$. Therefore, in the limit as $q \downarrow 0$, the 'mass' is concentrated on spanning trees, and it is easily seen that the limit mass is uniformly distributed. That is, $\phi_{p,q} \Rightarrow \mathrm{UST}$.

Another limit emerges if p approaches 0 at the same rate as does q. Take $p = \alpha q$ where $\alpha \in (0, \infty)$ is constant, and consider the limit as $q \downarrow 0$. This time we write

$$Z_{\mathrm{RC}}(p,q) = (1-\alpha q)^{|E|} \sum_{\omega \in \Omega} \left(\frac{\alpha}{1-\alpha q} \right)^{|\eta(\omega)|} q^{|\eta(\omega)|+k(\omega)}.$$

We have that $|\eta(\omega)| + k(\omega) \geq |V|$ with equality if and only if $\omega \in \Omega_{\mathrm{for}}$, and it follows that $\phi_{p,q} \Rightarrow \phi_\alpha^{\mathrm{for}}$, where

$$\phi_\alpha^{\mathrm{for}}(\omega) = \begin{cases} \dfrac{1}{Z_{\mathrm{for}}} \alpha^{|\eta(\omega)|} & \text{if } \omega \in \Omega_{\mathrm{for}}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.22}$$

and $Z_{\mathrm{for}} = Z_{\mathrm{for}}(\alpha)$ is the appropriate normalizing constant. In the special case $\alpha = 1$, we find that $\phi_{p,q} \Rightarrow \mathrm{USF}$.

Finally, if p approaches 0 faster than does q, in that $p/q \to 0$ as $p, q \to 0$, it is easily seen that the limit measure is concentrated on the empty set of edges. We summarize the three special cases above in a theorem.

(1.23) Theorem. *We have in the limit as* $q \downarrow 0$ *that*:

$$\phi_{p,q} \Rightarrow \begin{cases} \mathrm{UCS} & \text{if } p = \frac{1}{2}, \\ \mathrm{UST} & \text{if } p \to 0 \text{ and } q/p \to 0, \\ \mathrm{USF} & \text{if } p = q. \end{cases}$$

The spanning-tree limit is especially interesting for historical and mathematical reasons. As explained in the Appendix, the random-cluster model originated in a systematic study by Fortuin and Kasteleyn of systems of a certain type which satisfy certain parallel and series laws (see Section 3.8). Electrical networks are the best known such systems: two resistors of resistances r_1 and r_2 in parallel (respectively, in series) may be replaced by a single resistor with resistance $(r_1^{-1} + r_2^{-1})^{-1}$ (respectively, $r_1 + r_2$). Fortuin and Kasteleyn [120] realized that the electrical-network theory of a graph G is related to the limit as $q \downarrow 0$ of the random-cluster model on G, where p is given[7] by $p = \sqrt{q}/(1+\sqrt{q})$. It has been known since Kirchhoff's theorem, [212], that the electrical currents which flow in a network may be expressed in terms of counts of spanning trees. We return to this discussion of UST in Section 3.9.

The theory of the uniform-spanning-tree measure UST is beautiful in its own right (see [29]), and is linked in an important way to the emerging field of stochastic growth processes of 'stochastic Löwner evolution' (SLE) type (see [228, 277]), to which we return in Section 6.7. Further discussions of USF and UCS may be found in [162, 261].

1.6 Basic notation

We present some of the basic notation necessary for a study of random-cluster measures. Let $G = (V, E)$ be a graph, with finite or countably infinite vertex-set V and edge-set E. If two vertices x and y are joined by an edge e, we write $x \sim y$, and $e = \langle x, y \rangle$, and we say that x is *adjacent* to y. The (graph-theoretic) distance $\delta(x, y)$ from x to y is defined to be the number of edges in a shortest path of G from x to y.

The configuration space of the random-cluster model on G is the set $\Omega = \{0, 1\}^E$, points of which are represented as vectors $\omega = (\omega(e) : e \in E)$ and called *configurations*. For $\omega \in \Omega$, we call an edge e *open* (or ω-open, when the role of ω is to be emphasized) if $\omega(e) = 1$, and *closed* (or ω-closed) if $\omega(e) = 0$. We speak of a set F of edges as being 'open' (respectively, 'closed') in the configuration ω if $\omega(f) = 1$ (respectively, $\omega(f) = 0$) for all $f \in F$.

The indicator function of a subset A of Ω is the function $1_A : \Omega \to \{0, 1\}$ given by

$$1_A(\omega) = \begin{cases} 0 & \text{if } \omega \notin A, \\ 1 & \text{if } \omega \in A. \end{cases}$$

For $e \in E$, we write $J_e = \{\omega \in \Omega : \omega(e) = 1\}$, the event that the edge e is open. We use J_e to denote also the indicator function of this event, so that $J_e(\omega) = \omega(e)$. A function $X : \Omega \to \mathbb{R}$ is called a *cylinder* function if there exists a finite subset F of E such that $X(\omega) = X(\omega')$ whenever $\omega(e) = \omega'(e)$ for $e \in F$. A subset A of Ω is called a *cylinder event* if its indicator function is a cylinder function. We

[7] This choice of p is convenient, but actually one requires only that $q/p \to 0$, see [163].

take $\mathcal{F}$ to be the σ-field of subsets of Ω generated by the cylinder events, and we shall consider certain probability measures on the measurable pair $(\Omega, \mathcal{F})$. If G is finite, then $\mathcal{F}$ is the set of all subsets of Ω; all events are cylinder events, and all functions are cylinder functions. The complement of an event A is written A^{c} or $\overline{A}$.

For $W \subseteq V$, let E_W denote the set of edges of G having both endvertices in W. We write $\mathcal{F}_W$ (respectively, $\mathcal{T}_W$) for the smallest σ-field of $\mathcal{F}$ with respect to which each of the random variables $\omega(e)$, $e \in E_W$ (respectively, $e \notin E_W$), is measurable. The notation $\mathcal{F}_F$, $\mathcal{T}_F$ is to be interpreted similarly for $F \subseteq E$. The intersection of the $\mathcal{T}_F$ over all finite sets F is called the *tail* σ-field and is denoted by $\mathcal{T}$. Sets in $\mathcal{T}$ are called *tail events*.

There is a natural partial order on the set Ω of configurations given by: $\omega_1 \le \omega_2$ if and only if $\omega_1(e) \le \omega_2(e)$ for all $e \in E$. Rather than working always with the vector $\omega \in \Omega$, we shall sometimes work with its set of open edges, given by

$$\eta(\omega) = \{e \in E : \omega(e) = 1\}. \tag{1.24}$$

Clearly,

$$\omega_1 \le \omega_2 \qquad \text{if and only if} \qquad \eta(\omega_1) \subseteq \eta(\omega_2).$$

The smallest (respectively, largest) configuration is that with $\omega(e) = 0$ (respectively, $\omega(e) = 1$) for all e, and this is denoted by 0 (respectively, 1). A function $X : \Omega \to \mathbb{R}$ is called *increasing* if $X(\omega_1) \le X(\omega_2)$ whenever $\omega_1 \le \omega_2$. Similarly, X is *decreasing* if $-X$ is increasing. Note that every increasing function $X : \Omega \to \mathbb{R}$ is necessarily bounded since $X(0) \le X(\omega) \le X(1)$ for all $\omega \in \Omega$. A subset A of Ω is called *increasing* (respectively, *decreasing*) if it has increasing (respectively, decreasing) indicator function.

For $\omega \in \Omega$ and $e \in E$, let ω^e and ω_e be the configurations obtained from ω by 'switching on' and 'switching off' the edge e, respectively. That is,

$$\begin{aligned} \omega^e(f) &= \begin{cases} \omega(f) & \text{if } f \ne e, \\ 1 & \text{if } f = e, \end{cases} \qquad \text{for } f \in E, \\ \omega_e(f) &= \begin{cases} \omega(f) & \text{if } f \ne e, \\ 0 & \text{if } f = e, \end{cases} \qquad \text{for } f \in E. \end{aligned} \tag{1.25}$$

More generally, for $J \subseteq E$ and $K \subseteq E \setminus J$, we denote by ω_K^J the configuration that equals 1 on J, equals 0 on K, and agrees with ω on $E \setminus (J \cup K)$. When J and/or K contain only one or two edges, we may omit the necessary parentheses. The *Hamming distance* between two configurations is given by

$$H(\omega_1, \omega_2) = \sum_{e \in E} |\omega_1(e) - \omega_2(e)|, \qquad \omega_1, \omega_2 \in \Omega. \tag{1.26}$$

A *path* of G is defined as an alternating sequence $x_0, e_0, x_1, e_1, \ldots, e_{n-1}, x_n$ of distinct vertices x_i and edges $e_i = \langle x_i, x_{i+1} \rangle$. Such a path has *length* n and

is said to connect x_0 to x_n. A *circuit* or *cycle* of G is an alternating sequence $x_0, e_0, x_1, \dots, e_{n-1}, x_n, e_n, x_0$ of vertices and edges such that $x_0, e_0, \dots, e_{n-1}, x_n$ is a path and $e_n = \langle x_n, x_0 \rangle$; such a circuit has length $n+1$. For $\omega \in \Omega$, we call a path or circuit *open* if all its edges are open, and *closed* if all its edges are closed. Two subgraphs of G are called *edge-disjoint* if they have no edges in common, and *disjoint* if they have neither edges nor vertices in common.

Let $\omega \in \Omega$. Consider the random subgraph of G containing the vertex set V and the open edges only, that is, the edges in $\eta(\omega)$. The connected components of this graph are called *open clusters*. We write $C_x = C_x(\omega)$ for the open cluster containing the vertex x, and we call C_x the *open cluster at* x. The vertex-set of C_x is the set of all vertices of G that are connected to x by open paths, and the edges of C_x are those edges of $\eta(\omega)$ that join pairs of such vertices. We shall occasionally use the term C_x to represent the *set of vertices* joined to x by open paths, rather than the graph of this open cluster. We shall be interested in the size of C_x, and we denote by $|C_x|$ the number of vertices in C_x. Note that $C_x = \{x\}$ whenever x is an isolated vertex, which is to say that x is incident to no open edge. We denote by $k(\omega)$ the number of open clusters in the configuration ω, that is, $k(\omega)$ is the number of components of the graph $(V, \eta(\omega))$. The random variable k plays an important role in the definition of a random-cluster measure, and the reader is warned of the importance of including in k a count of the number of isolated vertices of the graph.

Let $\omega \in \Omega$. If A and B are sets of vertices of G, we write '$A \leftrightarrow B$' if there exists an open path joining some vertex in A to some vertex in B; if $A \cap B \neq \varnothing$ then $A \leftrightarrow B$ trivially. Thus, for example, $C_x = \{y \in V : x \leftrightarrow y\}$. We write '$A \not\leftrightarrow B$' if there exists no open path from any vertex of A to any vertex of B, and '$A \leftrightarrow B$ off D' if there exists an open path joining some vertex in A to some vertex in B that uses no vertex in the set D.

If W is a set of vertices of the graph, we write ∂W for the *boundary* of A, being the set of vertices in A that are adjacent to some vertex not in A,

$$\partial W = \{x \in W : \text{there exists } y \notin W \text{ such that } x \sim y\}.$$

We write $\Delta_{\mathrm{e}} W$ for the set of edges of G having exactly one endvertex in W, and we call $\Delta_{\mathrm{e}} W$ the *edge-boundary* of W.

We shall be mostly interested in the case when G is a subgraph of a d-dimensional lattice with $d \geq 2$. Rather than embarking on a debate of just what constitutes a 'lattice-graph', we shall, almost without exception, consider only the case of the (hyper)cubic lattice. This restriction enables a clear exposition of the theory and open problems without suffering the complications which arise through allowing greater generality.

Let d be a positive integer. We write $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ for the set of all integers, and $\mathbb{Z}^d$ for the set of all d-vectors $x = (x_1, x_2, \dots, x_d)$ with integral coordinates. For $x \in \mathbb{Z}^d$, we generally write x_i for the ith coordinate of x, and we

define

$$\delta(x, y) = \sum_{i=1}^{d} |x_i - y_i|.$$

The *origin* of $\mathbb{Z}^d$ is denoted by 0. The set $\{1, 2, \dots\}$ of natural numbers is denoted by $\mathbb{N}$, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The real line is denoted by $\mathbb{R}$.

We turn $\mathbb{Z}^d$ into a graph, called the *d-dimensional cubic lattice*, by adding edges between all pairs x, y of points of $\mathbb{Z}^d$ with $\delta(x, y) = 1$. We denote this lattice by $\mathbb{L}^d$, and we write $\mathbb{Z}^d$ for the set of vertices of $\mathbb{L}^d$, and $\mathbb{E}^d$ for the set of its edges. Thus, $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$. We shall often think of $\mathbb{L}^d$ as a graph embedded in $\mathbb{R}^d$, the edges being straight line-segments between their endvertices. The *edge-set* $\mathbb{E}_V$ of $V \subseteq \mathbb{Z}^d$ is the set of all edges of $\mathbb{L}^d$ both of whose endvertices lie in V.

Let x, y be vertices of $\mathbb{L}^d$. The (graph-theoretic) distance from x to y is simply $\delta(x, y)$, and we write $|x|$ for the distance $\delta(0, x)$ from the origin to x. We shall make occasional use of another distance function on $\mathbb{Z}^d$, namely

$$\|x\| = \max\bigl\{|x_i| : i = 1, 2, \dots, d\bigr\}, \qquad x \in \mathbb{Z}^d,$$

and we note that

$$\|x\| \le |x| \le d\|x\|, \qquad x \in \mathbb{Z}^d.$$

For $\omega \in \Omega = \{0, 1\}^{\mathbb{E}^d}$, we abbreviate to C the open cluster C_0 at the origin.

A *box* of $\mathbb{L}^d$ is a subset of $\mathbb{Z}^d$ of the form

$$\Lambda_{a,b} = \bigl\{x \in \mathbb{Z}^d : a_i \le x_i \le b_i \text{ for } i = 1, 2, \dots, d\bigr\}, \qquad a, b \in \mathbb{Z}^d,$$

and we sometimes write

$$\Lambda_{a,b} = \prod_{i=1}^{d} [a_i, b_i]$$

as a convenient shorthand. The expression $\Lambda_{a,b}$ is used also to denote the graph with vertex-set $\Lambda_{a,b}$ together with those edges of $\mathbb{L}^d$ joining two vertices in $\Lambda_{a,b}$. For $x \in \mathbb{Z}^d$, we write $x + \Lambda_{a,b}$ for the translate by x of the box $\Lambda_{a,b}$. The expression Λ_n denotes the box with side-length $2n$ and centre at the origin,

$$\Lambda_n = [-n, n]^d = \{x \in \mathbb{Z}^d : \|x\| \le n\}. \tag{1.27}$$

Note that $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$.

In taking what is called a 'thermodynamic limit', one works often on a finite box Λ of $\mathbb{Z}^d$, and then takes the limit as $\Lambda \uparrow \mathbb{Z}^d$. Such a limit is to be interpreted along a sequence $\mathbf{\Lambda} = (\Lambda_n : n = 1, 2, \dots)$ of boxes such that: Λ_n is non-decreasing in n and, for all m, $\Lambda_n \supseteq [-m, m]^d$ for all large n.

For any random variable X and appropriate probability measure μ, we write $\mu(X)$ for the expectation of X,

$$\mu(X) = \int X \, d\mu.$$

Let $\lfloor a \rfloor$ and $\lceil a \rceil$ denote the integer part of the real number a, and the least integer not less than a, respectively. Finally, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.