Theorem 5.47 of The Random-Cluster Model corrected

I am grateful to Remco van der Hofstad for pointing out a problem with Theorem 5.47, which has lost two important conditions. Here is a corrected but weaker version.

A probability measure $\mu$ on $\Omega$, $\mathcal{F}$ is said to satisfy the ‘uniform insertion-tolerance condition’ if, for some $\alpha, \beta \in (0, 1)$,

\[(1') \quad \alpha \leq \mu(J_e \mid \mathcal{T}_e) \leq \beta, \quad \mu\text{-almost-surely, for } e \in \mathbb{E}^d,
\]

where $J_e$ is the event that $e$ is open. Let $E$ be a finite set of edges, and let $K_1, K_2, \ldots, K_I$ be the components of the graph $(\mathbb{Z}^d, \mathbb{E}^d \setminus E)$. We say that $\mu$ has the ‘empty-boundary Markov property’ if: for all such sets $E$, given that every edge in $E$ is closed, the configurations on the $K_i, i = 1, 2, \ldots, I$, are independent. [We shall need a slightly weaker form of this property in the proof of Theorem 5.47', following.]

**Theorem 5.47'.** Let $\mu$ be a translation-invariant, positively associated probability measure on $(\Omega, \mathcal{F})$ satisfying $(1')$ for $\alpha, \beta \in (0, 1)$, and with the empty-boundary Markov property. The limit

\[(5.48) \quad \zeta(\mu) = \lim_{n \to \infty} \left\{ -\frac{1}{n} \log \mu(|C| = n) \right\}
\]

exists and satisfies

\[(5.49) \quad \mu(|C| = n) \leq \frac{(1 - \alpha)^2}{\alpha} ne^{-n\zeta(\mu)}, \quad n \geq 1.
\]

Furthermore, $0 \leq \zeta(\mu) \leq -\log[\alpha(1 - \beta)^{2(d - 1)}]$.

The quoted proof remains valid. One needs one further step in addition to those in the proof of Lemma 6.102 of reference [154]. In proving (6.105) of [154] in the current context, one needs to condition on the event that all edges in the external boundary of $\sigma \ast \tau$ are closed, to break this probability into a product of two terms, using positive association, and then to use the empty-boundary Markov property.

A small note concerning the second display of Theorem 5.51, which should read

\[
\phi_{p,q}^0(|C| = n) \leq \frac{q^2(1 - p)^2}{p(p + q(1 - p))} ne^{-n\zeta}.
\]

**Proof.** In the terminology of (6.105) of [154], it suffices to prove that

\[(2') \quad \mu(C = \sigma \ast \tau) \geq \frac{\alpha}{(1 - \alpha)^2} \mu(C = \sigma) \mu(C = \tau).
\]

http://www.statslab.cam.ac.uk/~ggr/books/rcm.html
Let $E_\sigma$ be the set of edges of $\mathbb{L}^d$ that do not belong to $\sigma$ but have one or more endpoints in $\sigma$, and let $F_\sigma$ be the event that every edge in $E_\sigma$ is closed. Let $e = (x, y)$ denote the edge with endpoints $x = \text{tr}(\sigma)$ and $y = \text{bl}(\sigma)'$. Then

$$\mu(C = \sigma) = \mu(F_\sigma)\mu(C = \sigma \mid F_\sigma),$$

and, by the empty-boundary Markov property and positive association,

$$\mu(C = \sigma, C_y = \tau + y) = \mu(F_{\sigma \cap e} \cap F_e)\mu(C = \sigma, C_y = \tau + y \mid F_{\sigma \cap e} \cap F_e)$$

$$= \mu(F_{\sigma \cap e} \cap F_e)\mu(C = \sigma \mid F_\sigma)\mu(C_y = \tau + y \mid F_{\tau + y})$$

$$\geq \frac{\mu(E_{\tau + y} \setminus \{e\} \text{ closed})}{\mu(F_{\tau + y})}\mu(C = \sigma)\mu(C_y = \tau + y)$$

where $F_e = \{e \text{ is closed}\}$. By the uniform insertion-tolerance property (1'),

$$\frac{\mu(E_{\tau + y} \setminus \{e\} \text{ closed})}{\mu(F_{\tau + y})} \geq \frac{1}{1 - \alpha},$$

and therefore, by the translation-invariance of $\mu$,

$$\mu(C = \sigma, C_y = \tau + y) \geq \frac{1}{1 - \alpha}\mu(C = \sigma)\mu(C = \tau).$$

Inequality (2') follows by a further application of (1').