4. Let $A_1, A_2, \dots, A_n$ be events where $n \geq 2$, and prove that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n).$$

In each packet of Corn Flakes may be found a plastic bust of one of the last five Vice-Chancellors of Cambridge University, the probability that any given packet contains any specific Vice-Chancellor being $\frac{1}{5}$, independently of all other packets. Show that the probability that each of the last three Vice-Chancellors is obtained in a bulk purchase of six packets is $1 - 3(\frac{4}{5})^6 + 3(\frac{3}{5})^6 - (\frac{2}{5})^6$.

5. Let A_r, $r \geq 1$, be events such that $\mathbb{P}(A_r) = 1$ for all r. Show that $\mathbb{P}\big(\bigcap_{r=1}^{\infty} A_r\big) = 1$.

6. You are given that at least one of the events A_r, $1 \leq r \leq n$, is certain to occur, but certainly no more than two occur. If $\mathbb{P}(A_r) = p$, and $\mathbb{P}(A_r \cap A_s) = q$, $r \neq s$, show that $p \geq 1/n$ and $q \leq 2/n$.

7. You are given that at least one, but no more than three, of the events A_r, $1 \leq r \leq n$, occur, where $n \geq 3$. The probability of at least two occurring is $\frac{1}{2}$. If $\mathbb{P}(A_r) = p$, $\mathbb{P}(A_r \cap A_s) = q$, $r \neq s$, and $\mathbb{P}(A_r \cap A_s \cap A_t) = x$, $r < s < t$, show that $p \geq 3/(2n)$, and $q \leq 4/n$.

1.4 Exercises. Conditional probability

1. Prove that $\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A)\mathbb{P}(A)/\mathbb{P}(B)$ whenever $\mathbb{P}(A)\mathbb{P}(B) \neq 0$. Show that, if $\mathbb{P}(A \mid B) > \mathbb{P}(A)$, then $\mathbb{P}(B \mid A) > \mathbb{P}(B)$.

2. For events $A_1, A_2, \dots, A_n$ satisfying $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$, prove that

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

3. A man possesses five coins, two of which are double-headed, one is double-tailed, and two are normal. He shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head?

He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head? He shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head? He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head?

He discards this coin, picks another at random, and tosses it. What is the probability that it shows heads?

4. What do you think of the following 'proof' by Lewis Carroll that an urn cannot contain two balls of the same colour? Suppose that the urn contains two balls, each of which is either black or white; thus, in the obvious notation, $\mathbb{P}(\mathrm{BB}) = \mathbb{P}(\mathrm{BW}) = \mathbb{P}(\mathrm{WB}) = \mathbb{P}(\mathrm{WW}) = \frac{1}{4}$. We add a black ball, so that $\mathbb{P}(\mathrm{BBB}) = \mathbb{P}(\mathrm{BBW}) = \mathbb{P}(\mathrm{BWB}) = \mathbb{P}(\mathrm{BWW}) = \frac{1}{4}$. Next we pick a ball at random; the chance that the ball is black is (using conditional probabilities) $1 \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{2}{3}$. However, if there is probability $\frac{2}{3}$ that a ball, chosen randomly from three, is black, then there must be two black and one white, which is to say that originally there was one black and one white ball in the urn.

5. The Monty Hall problem: goats and cars. (a) In a game show; you have to choose one of three doors. One conceals a new car, two conceal old goats. You choose, but your chosen door is not opened immediately. Instead the presenter opens another door, which reveals a goat. He offers you the opportunity to change your choice to the third door (unopened and so far unchosen). Let p be the (conditional) probability that the third door conceals the car. The presenter's protocol is:

(i) he is determined to show you a goat; with a choice of two, he picks one at random. Show $p = \frac{2}{3}$.

(ii) he is determined to show you a goat; with a choice of two goats (Bill and Nan, say) he shows you Bill with probability b. Show that, given you see Bill, the probability is $1/(1+b)$.

(iii) he opens a door chosen at random irrespective of what lies behind. Show $p = \frac{1}{2}$.

(b) Show that, for $\alpha \in [\frac{1}{2}, \frac{2}{3}]$, there exists a protocol such that $p = \alpha$. Are you well advised to change your choice to the third door?

(c) In a variant of this question, the presenter is permitted to open the first door chosen, and to reward you with whatever lies behind. If he chooses to open another door, then this door invariably conceals a goat. Let p be the probability that the unopened door conceals the car, conditional on the presenter having chosen to open a second door. Devise protocols to yield the values $p = 0$, $p = 1$, and deduce that, for any $\alpha \in [0, 1]$, there exists a protocol with $p = \alpha$.

6. The prosecutor's fallacy†. Let G be the event that an accused is guilty, and T the event that some testimony is true. Some lawyers have argued on the assumption that $\mathbb{P}(G \mid T) = \mathbb{P}(T \mid G)$. Show that this holds if and only if $\mathbb{P}(G) = \mathbb{P}(T)$.

7. Urns. There are n urns of which the rth contains $r-1$ red balls and $n-r$ magenta balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that:

(a) the second ball is magenta;

(b) the second ball is magenta, given that the first is magenta.

1.5 Exercises. Independence

1. Let A and B be independent events; show that A^c, B are independent, and deduce that A^c, B^c are independent.

2. We roll a die n times. Let A_{ij} be the event that the ith and jth rolls produce the same number. Show that the events $\{A_{ij} : 1 \le i < j \le n\}$ are pairwise independent but not independent.

3. A fair coin is tossed repeatedly. Show that the following two statements are equivalent:

(a) the outcomes of different tosses are independent,

(b) for any given finite sequence of heads and tails, the chance of this sequence occurring in the first m tosses is 2^{-m}, where m is the length of the sequence.

4. Let $\Omega = \{1, 2, \dots, p\}$ where p is prime, $\mathcal{F}$ be the set of all subsets of Ω, and $\mathbb{P}(A) = |A|/p$ for all $A \in \mathcal{F}$. Show that, if A and B are independent events, then at least one of A and B is either $\varnothing$ or Ω.

5. Show that the conditional independence of A and B given C neither implies, nor is implied by, the independence of A and B. For which events C is it the case that, for all A and B, the events A and B are independent if and only if they are conditionally independent given C?

6. Safe or sorry? Some form of prophylaxis is said to be 90 per cent effective at prevention during one year's treatment. If the degrees of effectiveness in different years are independent, show that the treatment is more likely than not to fail within 7 years.

7. Families. Jane has three children, each of which is equally likely to be a boy or a girl independently of the others. Define the events:

$$\begin{aligned} A &= \{\text{all the children are of the same sex}\}, \\ B &= \{\text{there is at most one boy}\}, \\ C &= \{\text{the family includes a boy and a girl}\}. \end{aligned}$$

†The prosecution made this error in the famous Dreyfus case of 1894.