Bistability in Communication Networks

R.J. Gibbens[1], P.J. Hunt[1], and F.P. Kelly

1. Introduction

Advances in the technology of modern telecommunications networks have led to considerable interest in schemes which can dynamically control the routing of calls within a network. The aim of such schemes is to adjust routing patterns within the network in accordance with varying and uncertain offered traffics, to make better use of spare capacity, and to provide extra flexibility and robustness to respond to failures or overloads. However, unless care is taken a dynamic routing strategy that appears to be beneficial may under some circumstances be detrimental.

One of the simplest dynamic routing strategies is Random Alternative Routing, which operates as follows. Every call type that can arrive at the network has a fixed first choice route and a set of possible second choice routes. If possible a call will be carried on its first choice route. If not, then an alternative route is selected at random from the set of possible second choice routes. The call is carried on this route if possible and otherwise it is lost.

At first sight the existence of second choice routes appears beneficial, as it gives each call more ways of being accepted. However if second choice routes require more network resource (hold more circuits or possibly hold circuits for longer) then the network performance, as measured for example by its overall loss probability, may be worse than if a call has access to just its first choice route. Further, Random Alternative Routing (and several other dynamic routing schemes) can lead to instability and hysteresis: several modes of behaviour are possible, the initial conditions of the network determining which is obtained. In this paper we use a combination of analytical, numerical and simulation approaches to investigate these phenomena, in the simplest case of a symmetric fully connected network.

The possibility of bistable behaviour was first noted by Nakagome and Mori (1973), using a simple analytical fixed point approximation for the equilibrium behaviour of a network. Using a development of this approximation Krupp (1982) showed that a network's performance could be improved by using a simple priority technique, known as trunk reservation.

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Akinpelu (1984) and Ackerley (1987) have presented simulation results illustrating hysteresis, and Schwartz (1987) gives a review of this area. In this paper we extend this earlier work by establishing limit theorems for sequences of approximating processes and by obtaining systems of integral equations to describe transient behaviour. We confirm the power of these analytical approaches by comparing numerical solutions to the systems of integral equations with simulated sample paths.

The organisation of this paper is as follows. In Sections 2 and 3 we define formally the network we consider, and obtain a functional law of large numbers for a sequence of approximating processes, using results from Whitt (1985) and Ethier and Kurtz (1986). Fixed point approximations of the form considered by Nakagome and Mori (1973) and, amongst others, Krupp (1982) and Kelly (1986), emerge naturally as fixed points of the integral equations we obtain. Section 4 illustrates the integral equations with a simple example. In Section 5 we deal with systems involving multiple alternative routes and trunk reservation. In Section 6 we introduce a one-dimensional diffusion approximation and use the approximation to elucidate bistable and tunnelling behaviour. Nelson (1986) describes how diffusions can be used to illuminate various types of catastrophic behaviour in performance models of computer systems, and our approximation can be viewed within his framework as an example of a stochastic cusp catastrophe.

Although in this paper we consider just a symmetric fully connected network operating under simple random routing schemes, many of the insights carry over to more general network structures and routing strategies. For example, the insights into trunk reservation obtained from the fixed point approximation of Section 5 were important in the development of Dynamic Alternative Routing (Stacey and Songhurst 1987, Gibbens 1988, Gibbens, Kelly and Key 1988), the dynamic routing strategy currently being implemented by British Telecom in the UK main digital trunk network.

2. A Simple Model

The symmetric fully connected network that we wish to study is as follows. There are a total of N nodes and every pair of nodes is connected by a link of capacity C, giving a total of $K = N(N-1)/2$ links. For all $\alpha \neq \beta$, calls between node α and node β arrive as a Poisson process of rate ν, all arrival streams being independent. If there is free capacity on the direct link between α and β then the call is routed along this path. If not, we try to route the call along two links via a randomly chosen third node $\gamma \neq \alpha, \beta$. If there is free capacity on both these links then the call is routed. Otherwise the call is lost. A call that has been successfully routed holds one circuit from each link on its path for the holding period of the call. The holding period is independent of earlier arrival times and holding periods, and is exponentially distributed with unit mean.

The network described above can be treated as a finite state space Markov process and we can derive equations for the equilibrium distribution. However the full state space is rather complicated, involving the graph structure of the network. It is difficult to analyse the process, even in equilibrium.

So we will consider a simplified model for this network, defined as follows. There are K links, each link comprising C circuits. Calls requesting link k as their first choice arrive as a Poisson process of rate ν. If a call is blocked on its first choice link it tries two other links chosen at random from the $K-1$ remaining links, with each pair of links having equal probability of being chosen. If neither of the links in the chosen pair is full the call is set up along these two links. Otherwise the call is lost. When a circuit is used by a call, the circuit is held for an exponential time, mean 1. All circuit holding times are independent of one another and of earlier arrival times. In particular, a call that requires two links holds each link independently for an exponential length of time, and so these circuits will become free at different times. Thus the simplified model differs in two ways from the original network: circuit holding times are independent, and the graph structure relationship between links has been lost.

The simplified model is an approximation that we would expect to be good for large K. The approximation is much simpler to analyse than the original and can be described by the following Markov process. Let $n_j^K(t)$ be the number of links with j circuits in use at time t, $j = 0, 1, \ldots, C$. Let

$$x_j^K(t) = \frac{n_j^K(t)}{K}, \quad \mathbf{x}^K(t) = (x_j^K(t))_j. \tag{1}$$

So $\sum_{j=0}^{C} x_j^K(t) = 1$ for all t. For $i \neq j$, $0 \leq i, j \leq C$ let T_{ij} be an operator defined on $\mathbf{x}^K$ given by

$$T_{ij}\mathbf{x}^K = \mathbf{x}^K + K^{-1}(\mathbf{e}_j - \mathbf{e}_i)$$

where $\mathbf{e}_i$ is the unit vector in the ith direction. Then $\mathbf{x}^K$ is a Markov process with transition rates

$$\begin{aligned}
&\mathbf{x}^K \to T_{j,j+1}\mathbf{x}^K \qquad \text{at rate} \quad \nu x_j^K K, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 0, 1, \ldots, C-1 \\
&\mathbf{x}^K \to T_{j,j-1}\mathbf{x}^K \qquad \text{at rate} \quad j x_j^K K, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \ldots, C \\
&\mathbf{x}^K \to T_{i,i+1}T_{j,j+1}\mathbf{x}^K \qquad \text{at rate} \quad 2\left(\frac{K}{K-1}\right)\nu K x_C^K x_i^K x_j^K, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad i > j,\ i, j = 0, 1, \ldots, C-1
\end{aligned}$$

$$\mathbf{x}^K \to T^2_{j,j+1}\mathbf{x}^K \qquad \text{at rate} \quad \left(\frac{K}{K-1}\right)\nu K x^K_C x^K_j \left(x^K_j - \frac{1}{K}\right), \qquad j = 0, 1, \ldots, C-1.$$

With this process we can prove results that we also expect to hold for the original fully connected network.

3. Weak Convergence

We now prove a functional law of large numbers for the $\mathbf{x}^K$ process defined in Section 2. Note that $\mathbf{x}^K$ lies in the simplex

$$\Delta = \{\, \mathbf{x}^K \in \mathbb{R}^{C+1}_+ : \sum_{i=0}^{C} x^K_i = 1 \,\}.$$

Let $\Rightarrow$ denote convergence in distribution as $K \to \infty$ of random elements in the state space Δ or the space of all sample paths $D_\Delta[0,\infty)$; for background see Billingsley 1968, Lindvall 1973, Whitt 1980, Ethier and Kurtz 1986.

LEMMA 1. *The sequence $\mathbf{x}^K$ is relatively compact in $D_\Delta[0,\infty)$ and the limit of any convergent subsequence has continuous sample paths.*

PROOF: This result follows from a minor modification of Lemma 1 of Whitt (1985, p. 1843). ■

THEOREM 2. *If $\mathbf{x}^K(0) \Rightarrow \mathbf{x}(0)$ then $\mathbf{x}^K(\cdot) \Rightarrow \mathbf{x}(\cdot)$ where $\mathbf{x}(\cdot)$ is the unique solution to the equations*

$$x_0(t) = x_0(0) + \int_0^t \{x_1(u) - (\nu + \lambda(u))x_0(u)\}\, du \tag{2}$$

$$x_j(t) = x_j(0) + \int_0^t \{(\nu + \lambda(u))x_{j-1}(u) - (\nu + \lambda(u) + j)x_j(u) + (j+1)x_{j+1}(u)\}\, du \qquad j \neq 0, C \tag{3}$$

$$x_C(t) = x_C(0) + \int_0^t \{(\nu + \lambda(u))x_{C-1}(u) - Cx_C(u)\}\, du \tag{4}$$

and

$$\lambda(t) = 2\nu x_C(t)(1 - x_C(t)). \tag{5}$$

PROOF: Let

$$v(\mathbf{x}^K(t)) = \lim_{h \downarrow 0} E\left[\frac{\mathbf{x}^K(t+h) - \mathbf{x}^K(t)}{h}\middle|\mathbf{x}^K(t)\right]$$

$$D^K(t) = \int_0^t v(\mathbf{x}^K(u))\, du$$

$$\mathbf{M}^K(t) = \mathbf{x}^K(t) - \mathbf{x}^K(0) - D^K(t).$$

FIG. 1. Instability of blocking probability:
(i) with one retry (ii) with five retries.

Then $\mathbf{M}^K$ is an $\{\mathcal{F}_t^{\mathbf{x}^K}\}$–martingale. It is now easy to check the conditions in Ethier and Kurtz (1986, Theorem 1.4, p. 339). Hence, since $[M_i^K, M_j^K](t) \to 0$ as $K \to \infty$, we have that $\mathbf{M}^K \Rightarrow \mathbf{0}$.

Now along any convergent subsequence of $\{\mathbf{x}^K\}$ we can use the continuous mapping theorem (see, for example, Whitt 1980) to show that $\mathbf{M}^K \Rightarrow \mathbf{M}$ for some $\mathbf{M}$. But by the above result we know that $\mathbf{M} = \mathbf{0}$ and thus we have (2)–(5) satisfied by the limit of a convergent subsequence. But the result now follows since (2)–(5) have a unique solution. (See Arnold 1973, pp. 50, 57.) ■

From equations (2)–(5)

$$\sum_{i=0}^{j} x_i(t) = \sum_{i=0}^{j} x_i(0) + \int_0^t \big\{(j+1)x_{j+1}(u) - (\nu + \lambda(u))x_j(u)\big\}\, du$$

$$j = 0, 1, \ldots, C-1$$

Thus $\mathbf{x} = (x_0, x_1, \ldots, x_C) \in \Delta$ is a fixed point of the system of equations

(2)–(5) if and only if

$$(j+1)x_{j+1} = (\nu + \lambda)x_j \qquad j = 0, 1, \ldots, C-1$$

where

$$\lambda = 2\nu x_C(1 - x_C).$$

A fixed point $\mathbf{x}$ is thus of the form

$$x_j = \frac{\xi^j}{j!}\left(\sum_{i=0}^{C} \frac{\xi^i}{i!}\right)^{-1} \qquad j = 0, 1, \ldots, C$$

where ξ solves

$$\xi = \nu + 2\nu E(\xi, C)\left(1 - E(\xi, C)\right). \tag{6}$$

Here

$$E(\xi, C) = \frac{\xi^C}{C!}\left(\sum_{i=0}^{C} \frac{\xi^i}{i!}\right)^{-1} \tag{7}$$

is Erlang's formula for the loss probability of a single link offered Poisson traffic at rate ξ. The equation (6) for ξ is equivalent to the equation

$$B = E\big(\nu + 2\nu B(1 - B), C\big) \tag{8}$$

for B, under the transformation $B = E(\xi, C)$. The parameter B corresponds to the *link blocking probability*, x_C. Equation (8) is usually derived from an approximation that links block independently: see, for example, Kelly 1986. Under such an approximation the probability that a call overflows is B, and the probability it can be accepted at the other link of a two-link path is $1 - B$; the arrival rate of overflowing calls at a link is then $2\nu B(1 - B)$. The locus of points satisfying equation (8) is illustrated in Figure 1(i) for $C = 120, 1000$ and infinity. Observe the possibility of multiple solutions for B, for C large enough and for a narrow range of the ratio ν/C. The upper and lower solutions correspond to stable fixed points for the system of equations (2)–(5), while the middle solution corresponds to an unstable fixed point. We discuss the possibility of multiple fixed points further in Section 6.

4. An Illustration

The integral equations of Theorem 2 apply to the limit process obtained from the simplified model. It is natural to ask how well they model the behaviour of the fully connected network. Figure 2 shows that the model is in fact very good. If $\overline{x} = \sum_{i=0}^{C} i x_i$, then Figure 2 shows the projection of the path given by the integral equation for several initial points. Also

FIG. 2. Trajectories for the limit process $\mathbf{x}(\cdot)$.

shown is the sample path for a fully connected network starting with the same initial configuration as one of the points. The parameters used to obtain these simulation results were $\nu = 115$, $C = 120$ and the number of nodes $N = 11$.

Let

$$\Xi = \left\{ \mathbf{x} : x_i = \frac{\xi^i}{i!} \left(\sum_{j=0}^{C} \frac{\xi^j}{j!} \right)^{-1} ; \, \xi \in (0, \infty) \right\},$$

a one-dimensional submanifold of the space Δ. The submanifold Ξ is a natural space to consider: if $\lambda(t)$ is held fixed at a value λ then the solution to the integral equations (2)–(4) will move exponentially quickly to the submanifold Ξ, to the point parametrised by $\xi = \nu + \lambda$. The submanifold Ξ is not closed under the integral equations (2)–(5), but notice the way in which trajectories head rapidly towards the projection of Ξ (shown as a dashed curve), and more slowly towards the fixed point. We exploit this observation later in Section 6.

The two dimensions shown in Figure 2 are natural choices since x_C controls the rates of the process and $\overline{x}$ measures the total network utilization. Calculations of the trajectories for various different initial vectors $\mathbf{x}$ with the same projection have given very similar trajectories supporting the belief that this projection is natural and sufficient to summarise the process $\mathbf{x}(t)$. The overall network loss probability is given by

$$L = x_C[1 - (1 - x_C)^2]$$

and for the example simulated this value is 0.12 . If alternative routing is not allowed, so that a call blocked on its direct link is lost, then the network loss probability is given by Erlang's formula (7) to be 0.05. Observe that allowing a blocked call to attempt a two-link alternative actually *increases* the loss probability of the network.

5. Trunk Reservation and Multiple Alternatives

We have seen that allowing a blocked call to attempt a two-link alternative route may *increase* the loss probability of a network, and we might expect this effect to become even more pronounced if a blocked call can attempt a sequence of alternative routes. Observe that if a link accepts an alternatively routed call it may later have to block a directly routed call which will then attempt to find two circuits elsewhere in the network. A natural response is to allow a link to reject alternatively routed calls if the link occupancy is above a certain level. Suppose then that a call attempting a two-link alternative route is only accepted if on each of the two links the number of circuits occupied is less than $C - s$. This method of giving priority at a link is known as *trunk reservation*, and the constant s is known as the trunk reservation parameter for the link.

The above model for a fully connected network of N nodes is difficult to analyse, and so instead we suppose there are $K = N(N-1)/2$ links, and that a call blocked on its first choice link tries two other links chosen at random from amongst the $K - 1$ remaining links. If the number of circuits occupied on each of the two links is less than $C - s$ then the call is routed via that pair of links. If not the call can try another pair of links chosen at random from amongst the $K - 3$ remaining links. On each link a trunk reservation parameter of s acts against alternatively routed calls, and a call is lost after it has tried r pairs. As in Section 2 we suppose that all circuit holding times are independent, even the holding times of two circuits used by an alternatively routed call. Let $n_j^K(t)$ be the number of links with j circuits in use at time t, and define $\mathbf{x}^K(t)$ by (1). Then the following result can be established by the methods used to prove Theorem 2.

THEOREM 3. *If* $\mathbf{x}^K(0) \Rightarrow \mathbf{x}(0)$ *in* Δ *then* $\mathbf{x}^K(\cdot) \Rightarrow \mathbf{x}(\cdot)$ *in* $D_\Delta[0,\infty)$ *where* $\mathbf{x}(\cdot)$ *is the unique solution to the equations*

$$x_0(t) = x_0(0) + \int_0^t \{x_1(u) - [\nu + \lambda(u)]x_0(u)\}\, du \tag{9}$$

$$x_j(t) = x_j(0) + \int_0^t \{[\nu + \lambda(u)](x_{j-1}(u) - x_j(u)) + (j+1)x_{j+1}(u) - jx_j(u)\}\, du \qquad (10)$$
$$j = 1, 2, \ldots, C-s-1$$

$$x_{C-s}(t) = x_{C-s}(0) + \int_0^t \{[\nu + \lambda(u)](x_{C-s-1}(u) - x_{C-s}(u)) + (C-s-1)x_{C-s-1}(u) - (C-s)x_{C-s}(u)\}\, du \qquad (11)$$

$$x_j(t) = x_j(0) + \int_0^t \{\nu(x_{j-1}(u) - x_j(u)) + (j+1)x_{j+1}(u) - jx_j(u)\}\, du \qquad (12)$$
$$j = C-s+1, \ldots, C-1$$

$$x_C(t) = x_C(0) + \int_0^t \{\nu x_{C-1}(u) - Cx_C(u)\}\, du \qquad (13)$$

and

$$\lambda(t) = 2\nu x_C(t)\left(\sum_{m=0}^{C-s-1} x_m(t)\right)^{-1} \left\{1 - \left[1 - \left(\sum_{m=0}^{C-s-1} x_m(t)\right)^2\right]^r\right\}. \qquad (14)$$

A fixed point $\mathbf{x} = (x_0, x_1, \ldots, x_C) \in \Delta$ of the system of equations (9)–(14) satisfies

$$(j+1)x_{j+1} = (\nu + \lambda)x_j \qquad j = 0, 1, \ldots, C-s-1 \qquad (15)$$
$$(j+1)x_{j+1} = \nu x_j \qquad j = C-s, \ldots, C-1 \qquad (16)$$

where

$$\lambda = 2\nu B_1(1 - B_2)^{-1}\left\{1 - \left[1 - (1 - B_2)^2\right]^r\right\} \qquad (17)$$
$$B_1 = x_C, \quad B_2 = \sum_{i=C-s}^{C} x_i. \qquad (18)$$

The network loss probability corresponding to a solution to the fixed point equations (15)–(18) is

$$L = B_1\left[1 - (1 - B_2)^2\right]^r.$$

We can interpret this form as follows: a call is lost if it is blocked on its first choice route, which happens with probability B_1, and if it is then blocked on each of r alternatives. It is blocked on an alternative route with

FIG. 3. Trajectories for a network with two stable fixed points.

probability $1 - (1 - B_2)^2$, where B_2 is the probability a link is occupied above its trunk reservation parameter.

We illustrate the integral equations of Theorem 3 in Figure 3 for a network in which $N = 11, C = 120, \nu = 100, r = 5$ and $s = 0$. For this network $B_1 = B_2 = B$, say, and Figure 1(ii) shows the locus of points satisfying relations (15)–(18). The chosen values of ν and C lead to two stable fixed points; these correspond to the points marked by crosses in Figure 3.

6. A One-Dimensional Approximation

In Section 3 we developed a functional law of large numbers for the simplified model, and in Section 4 we saw that the resulting integral equations can provide a reasonable approximation for the exact network. In Section 3 we observed that there may be multiple fixed points for the integral equations. Of course a finite network corresponds to an irreducible Markov process, with a unique equilibrium distribution. However the unique distribution may be multi-modal, and the time taken for the process to move from one mode to another may be long. How can we investigate this analytically?

One approach would be to use a diffusion approximation to the simplified Markov process to obtain results about the process of interest. However we are dealing with a C-dimensional process where C may be quite large, and it is difficult to obtain useful analytical results and computationally expensive to obtain numerical results. So instead we will reduce the process to one important dimension, as follows.

Consider a fully connected network with r retries and no trunk reservation. Let $n(t) \in \{0, 1, \dots, CK\}$. We will use $n(t)$ to represent the number of circuits in use at time t. Let $n(\cdot)$ have transition rates

$$\begin{array}{lll} n \to n+1 & \text{at rate} & \nu K(1 - B_n) \\ n \to n+2 & \text{at rate} & \nu K B_n \left\{1 - \left(1 - (1 - B_n)^2\right)^r\right\} \\ n \to n-1 & \text{at rate} & n \end{array}$$

where B_n solves

$$B_n = E(\rho, C), \quad n = \rho(1 - B_n).$$

Thus $n(\cdot)$ has the same transition rates as the total number of circuits in use in the earlier model of Section 2 provided that the number of links full in the earlier model is KB_n. The process $n(\cdot)$ should thus approximate the earlier model close to the one-dimensional submanifold Ξ. (We note that a more refined model would take into account that two circuits may sometimes be freed simultaneously.) Next approximate $n(\cdot)/CK$ by a diffusion $Z(\cdot)$ on the interval $[0, 1]$ with drift $\mu(z)$ and infinitesimal variance $\sigma^2(z)$ given by

$$\begin{aligned} \mu(z) &= \frac{\nu}{C}\left[(1 - B_z) + 2B_z\left\{1 - (1 - (1 - B_z)^2)^r\right\}\right] - z \\ \sigma^2(z) &= \frac{\nu}{C^2 K}\left[(1 - B_z) + 4B_z\left\{1 - (1 - (1 - B_z)^2)^r\right\}\right] + \frac{z}{CK} \end{aligned}$$

where

$$zC = \rho(1 - E(\rho, C))$$

and with reflecting barriers at 0 and 1. Thus $\mu(z)$ and $\sigma^2(z)$ are the natural extensions of the drift and infinitesimal variance of the discrete process $n(\cdot)/CK$ to $[0, 1]$. We can now use the powerful results for one-dimensional diffusions to gain insight into the network's behaviour.

As an example consider the equilibrium density $\psi(z)$ for the diffusion $Z(\cdot)$. This is given by

$$\psi(z) = A\, \frac{\exp\left(\int_0^z 2\mu(y)/\sigma^2(y)\, dy\right)}{\sigma^2(z)}$$

for some constant A (cf. Karlin and Taylor 1981, Kent 1978, Nelson 1986). Figure 4(i)–(iv) shows this equilibrium density for a network with parameters $N = 11$, $C = 120$, $r = 5$ and $s = 0$ as ν varies. Observe that the high

FIG. 4. Equilibrium density for the diffusion $Z(\cdot)$.

blocking state is a lot less stable than the low blocking state for smaller values of ν but becomes more stable as ν increases until finally there is only one stable point. In the region of ν/C for which there are two stable fixed points, illustrated in Figure 1(ii), we expect to see tunnelling. Figure 5 illustrates a sample path for the same network with $\nu = 100.5$ where this tunnelling has occurred. We see that the sample path heads rapidly to the dashed curve; it then moves towards the upper fixed point, about which it wanders for a period before tunnelling to the region of the lower fixed point.

A natural question that arises is how stable are the two fixed points. That is, how long do we expect to wait until we tunnel from one to the other? Again we can use the one-dimensional approximation.

Let $T(x;y)$ be the first time that the diffusion hits y given that it starts at x. Let $f(x;y) = E[T(x;y)]$. Then (Karlin and Taylor 1981, p. 193) $f(x;y)$ satisfies

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2} + \mu(x)\frac{\partial f}{\partial x} = -1$$

FIG. 5. Tunnelling between stable fixed points.

with boundary conditions

$$\begin{aligned} f(y;y) &= 0 \\ \frac{\partial f(0;y)}{\partial x} &= 0 \qquad x > y \\ \frac{\partial f(1;y)}{\partial x} &= 0 \qquad y > x. \end{aligned}$$

So if $x_1 < x_2 < x_3$ are the three fixed points then we can assess stability from $f(x_1;x_2)$ and $f(x_3;x_2)$.

Remark. For $r = 1$ we find that for some A_1, A_2, A_3, A_4

$$\frac{e^{A_1 CK}}{CK} \leq f(x_1;x_2) \leq \frac{e^{A_2 CK}}{CK} \tag{19}$$

$$\frac{e^{A_3 K}}{CK} \leq f(x_3;x_2) \leq \frac{e^{A_4 K}}{CK} \tag{20}$$

as $C, K \to \infty$.

Equations (19) and (20) show that the low blocking state becomes more stable very rapidly as C and K increase. However the high blocking state becomes stable rapidly with K but more *unstable* as C increases. This is as one would expect. In the high blocking state the number of free circuits is $O(1)$ as C becomes large and in the low blocking state it is $O(C - \nu)$. To tunnel from high to low the number of free circuits needs to be unusually large for a time, an $O(1)$ effect, since then more single link calls are routed and the network falls into the low blocking state. To tunnel from low to high blocking the number of free circuits must change by $C - \nu$, an $O(C)$ effect. This accounts for the exponential terms in the expressions. The $1/C$ part comes from the fact that the transition rates increase linearly in C and hence the time taken between events behaves like $1/C$.

The corresponding model with trunk reservation is a two-dimensional process (x_h, x_l), where x_h and x_l are the amounts of, respectively, high and low priority traffic. We do not develop this here: trunk reservation removes the bistability that has been a focus of this paper.

Note Added in Proof: Marbukh (1983), starting from an independent blocking assumption, has derived differential equations corresponding to the integral equations of this paper.

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Statistical Laboratory
University of Cambridge
16 Mill Lane
Cambridge CB2 1SB.