TWO OBSERVATIONS

1. Analysis of equation (12.68)

It is helpful to observe that the discriminant of (12.68) is strictly positive for $|s| < 1$. Since both roots are continuous on $(-1, 1)$ (except possibly at $s = 0$), we have to choose a root and stick to it for all $s \in (0, 1)$, and for all $s \in (-1, 0)$. Now $|G(s)| \leq 1$ on $(-1, 1)$, and the positive square root diverges at $s = 0$. Hence the negative square root holds on the entire interval $(-1, 1)$.

2. Proof of Theorem 12.75(a)

The following argument is preferred.

Since $i \leftrightarrow j$, there exist $m, n \geq 1$ such that

$$\alpha := p_{i,j}(m)p_{j,i}(n) > 0.$$

By the Chapman–Kolmogorov equations, Theorem 12.13,

$$p_{i,i}(m + r + n) \geq p_{i,j}(m)p_{j,j}(r)p_{j,i}(n) = \alpha p_{j,j}(r) \qquad \text{for } r \geq 0. \tag{2.1}$$

In particular, on setting $r = 0$ we obtain $p_{i,i}(m + n) \geq \alpha > 0$, so that $d_i \mid m + n$.

Let $R = \{r \geq 1 : p_{j,j}(r) > 0\}$. By (2.1), $d_i \mid m + r + n$ for all $r \in R$. Since $d_i \mid m + n$, we have that $d_i \mid r$ for all $r \in R$. Therefore, $d_i \mid d_j$. By the reverse argument, $d_j \mid d_i$, and hence $d_i = d_j$.

Date: 29 October 2018.