Resource sharing in networks

Frank Kelly
University of Cambridge
www.statslab.cam.ac.uk/~frank

IMA MATHEMATICS 2010
London, April 2010

Outline

- The processor sharing queue
- Sharing in networks – proportional fairness
- Multipath routing
- Markov chain description, and heavy traffic
- Ramp metering
- Resource pooling

Processor sharing discipline

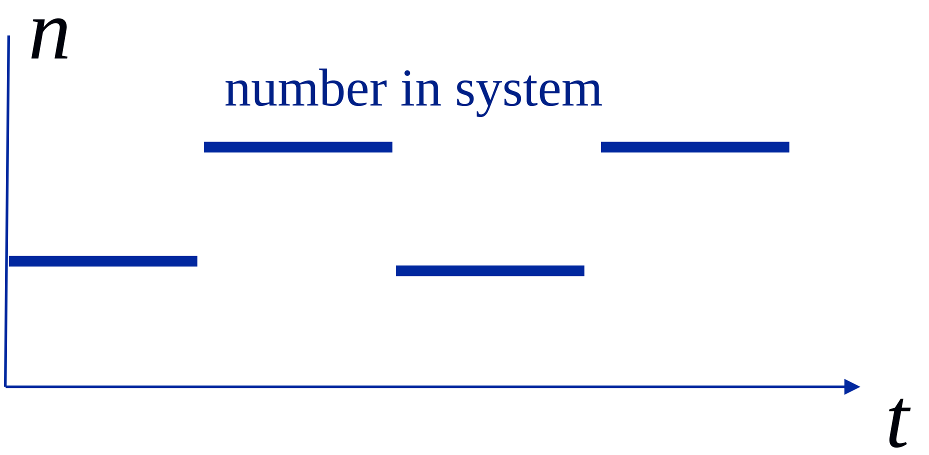
Kleinrock, 1967, 1976; Boxma tutorial, informs 2005

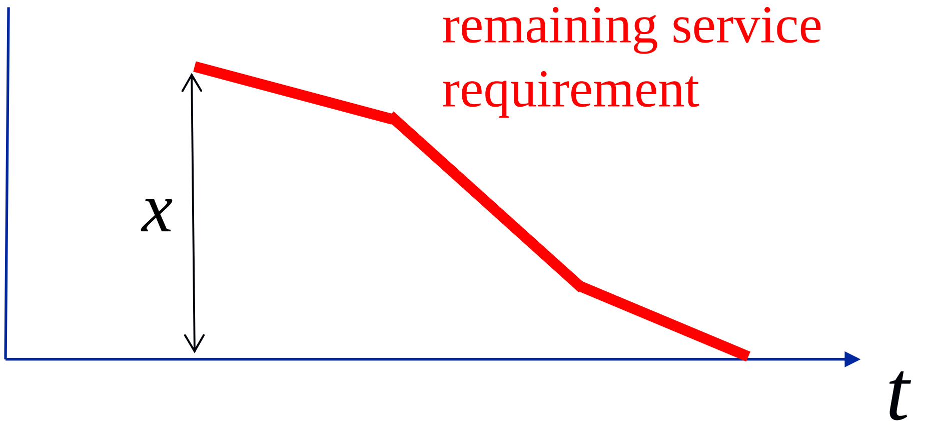
- Often attractive in practice, since gives
  - rapid service for short jobs
  - the appearance of a processor continuously available (albeit of varying capacity)
- Tractable analytically – a symmetric discipline. E.g. for M/G/1 PS

$$E\left[\text{sojourn time, } S \mid \text{ job size, } x\right] = \frac{x}{C-\rho}$$

(similar tractability for LCFS, Erlang loss system, networks of symmetric queues)

The M/G/1 processor sharing queue





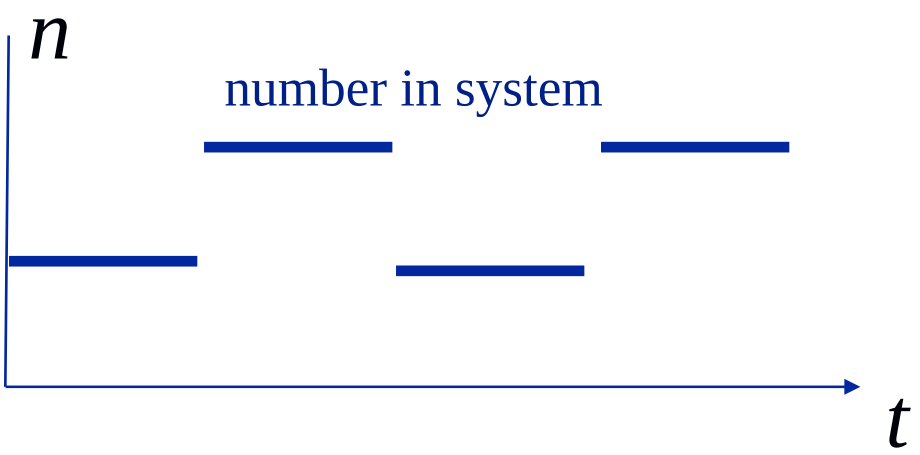
The M/G/1 processor sharing queue

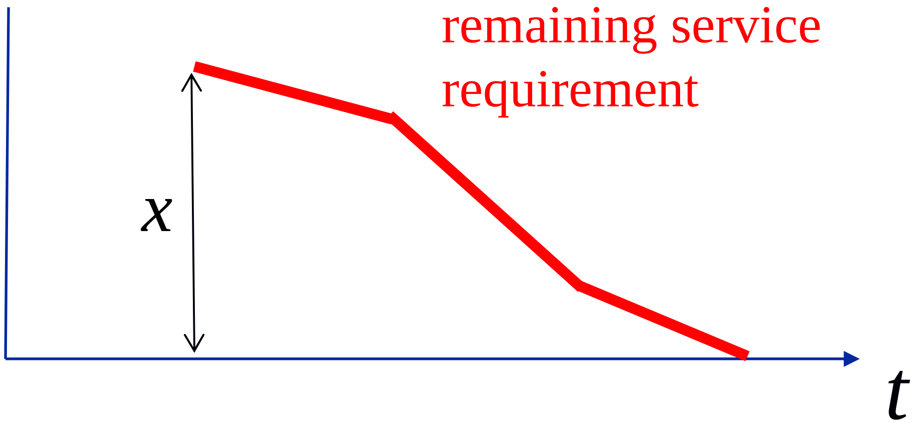
$$\left[S \mid x\right] \cong \frac{x}{C-\rho} + o(1/x)$$

if x is large;

$$\left[S \mid x\right] \cong x.\frac{n+1}{C} + o(x)$$

if x is small, where n is a geometric random variable.





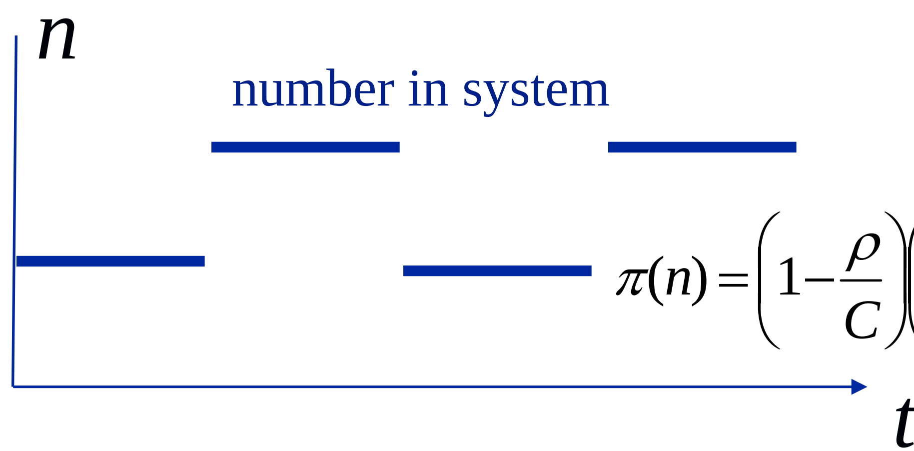
The M/G/1 processor sharing queue

$$\left[S \mid x\right] \cong \frac{x}{C-\rho} + o(1/x)$$

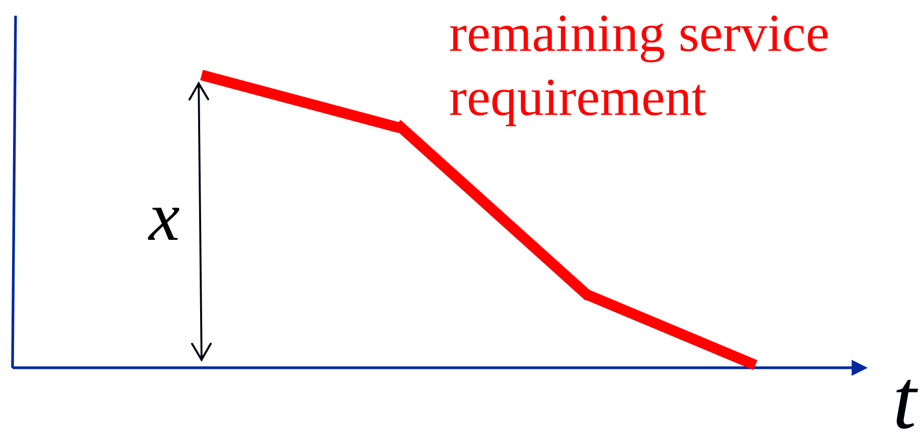
if x is large;

$$\left[S \mid x\right] \cong x.\frac{n+1}{C} + o(x)$$

if x is small, where n is a geometric random variable.



$$\pi(n) = \left(1-\frac{\rho}{C}\right)\left(\frac{\rho}{C}\right)^n$$



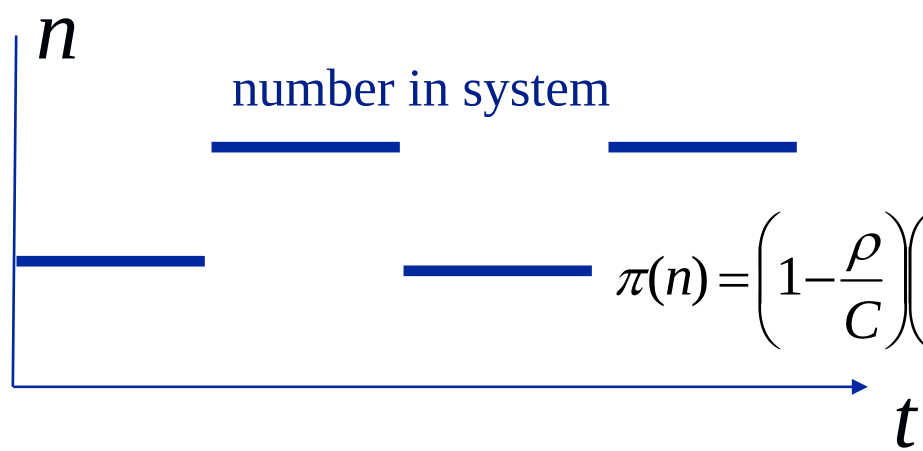
The M/G/1 processor sharing queue

$$\left[S \mid x\right] \cong \frac{x}{C-\rho} + o(1/x)$$

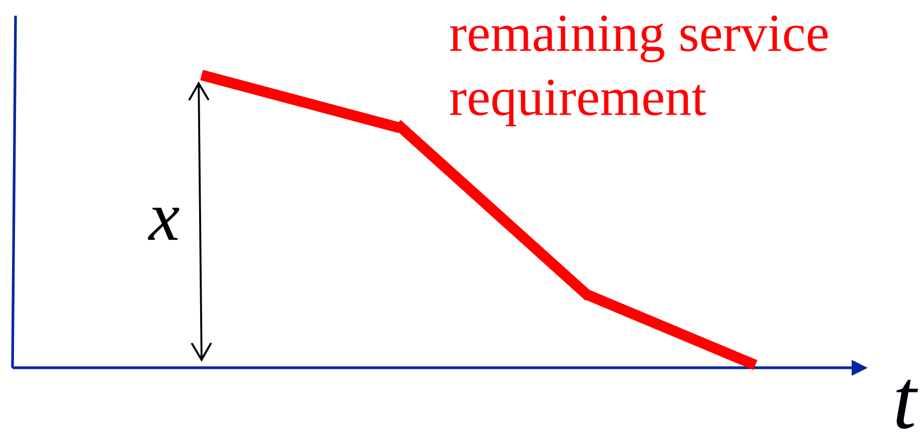
if x is large;

$$\left[S \mid x\right] \cong x.\frac{n+1}{C} + o(x)$$

if x is small, where n is a geometric random variable.



$$\pi(n) = \left(1-\frac{\rho}{C}\right)\left(\frac{\rho}{C}\right)^n$$



$$E\left[S \mid x\right] = \frac{x}{C-\rho}$$ in both cases, of course!

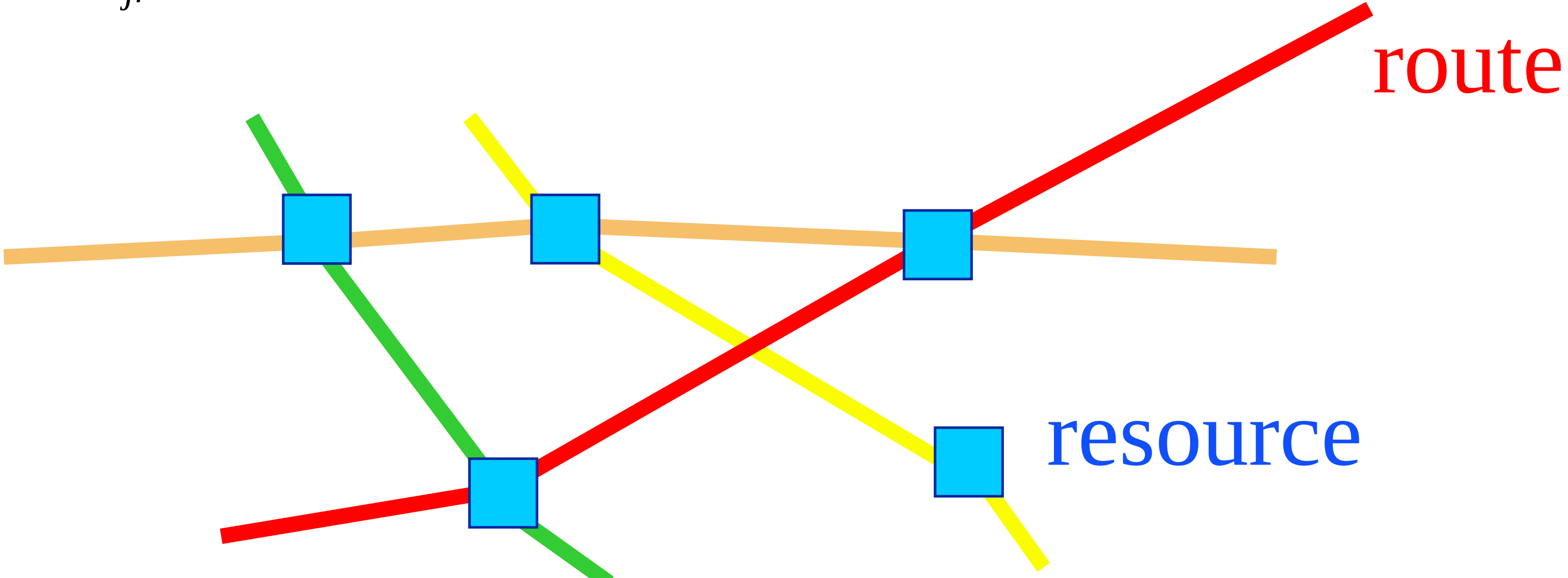
What is the network equivalent?

J - set of resources

R - set of routes

$A_{jr} = 1$ - if resource j is on route r

$A_{jr} = 0$ - otherwise



Rate allocation

n_r - number of flows on route r

x_r - rate of each flow on route r

Given the vector how are the rates chosen ?

$$n = (n_r, r \in R)$$

$$x = (x_r, r \in R)$$

Optimization formulation

Suppose $x = x(n)$ is chosen to

maximize $$\sum_r w_r n_r \frac{x_r^{1-\alpha}}{1-\alpha}$$

subject to $$\sum_r A_{jr} n_r x_r \le C_j \quad j \in J$$

$$x_r \ge 0 \quad r \in R$$

(weighted α-fair allocations, Mo and Walrand 2000)

$0 < \alpha < \infty$ (replace $\frac{x_r^{1-\alpha}}{1-\alpha}$ by $\log(x_r)$ if $\alpha = 1$)

Solution

$$x_r = \left(\frac{w_r}{\sum_j A_{jr} p_j(n)} \right)^{1/\alpha} \qquad r \in R$$

where

$$\sum_r A_{jr} n_r x_r \le \quad C_j \quad j \in J; \quad x_r \ge 0 \quad r \in R$$

$$p_j(n) \ge 0 \quad j \in J$$

$$p_j(n) \left(C_j - \sum_r A_{jr} n_r x_r \right) \ge 0 \quad j \in J$$

KKT conditions

$p_j(n)$ - *shadow price* (Lagrange multiplier) for the resource j capacity constraint

Examples of α-fair allocations

maximize $\sum_r w_r n_r \frac{x_r^{1-\alpha}}{1-\alpha}$

subject to $\sum_r A_{jr} n_r x_r \le C_j \quad j \in J$

$$x_r \ge 0 \quad r \in R$$

$$x_r = \left(\frac{w_r}{\sum_j A_{jr} p_j(n)} \right)^{1/\alpha} \quad r \in R$$

$\alpha \to 0 \quad (w = 1)$ - maximum flow

$\alpha \to 1 \quad (w = 1)$ - proportionally fair

$\alpha = 2 \quad (w_r = 1/T_r^2)$ - TCP fair

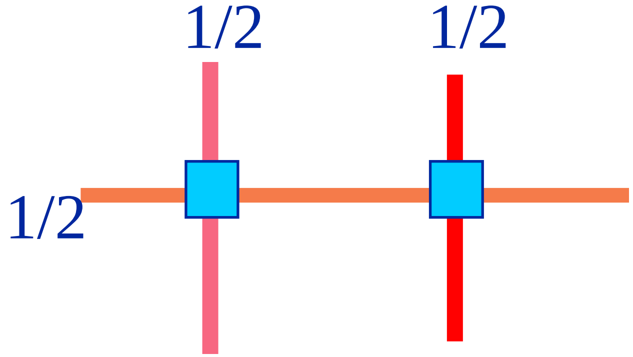
$\alpha \to \infty \quad (w = 1)$ - max-min fair

Example

$$n_r = 1, \ \ w_r = 1 \ \ r \in R,$$
$$C_j = 1 \ \ j \in J$$

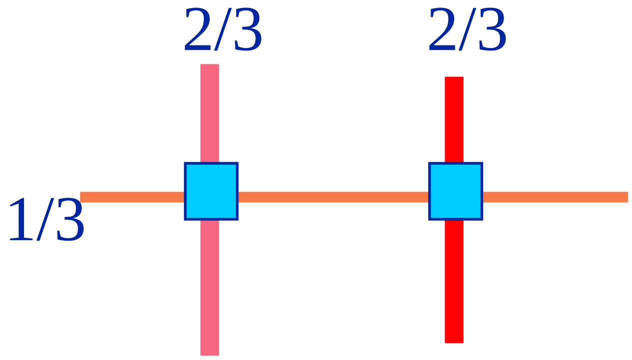
max-min fairness:

$$\alpha \to \infty$$



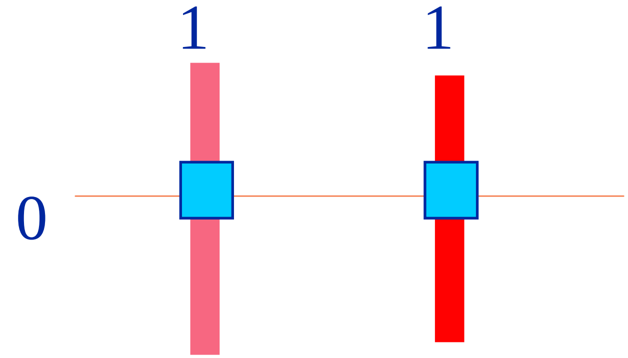
proportional fairness:

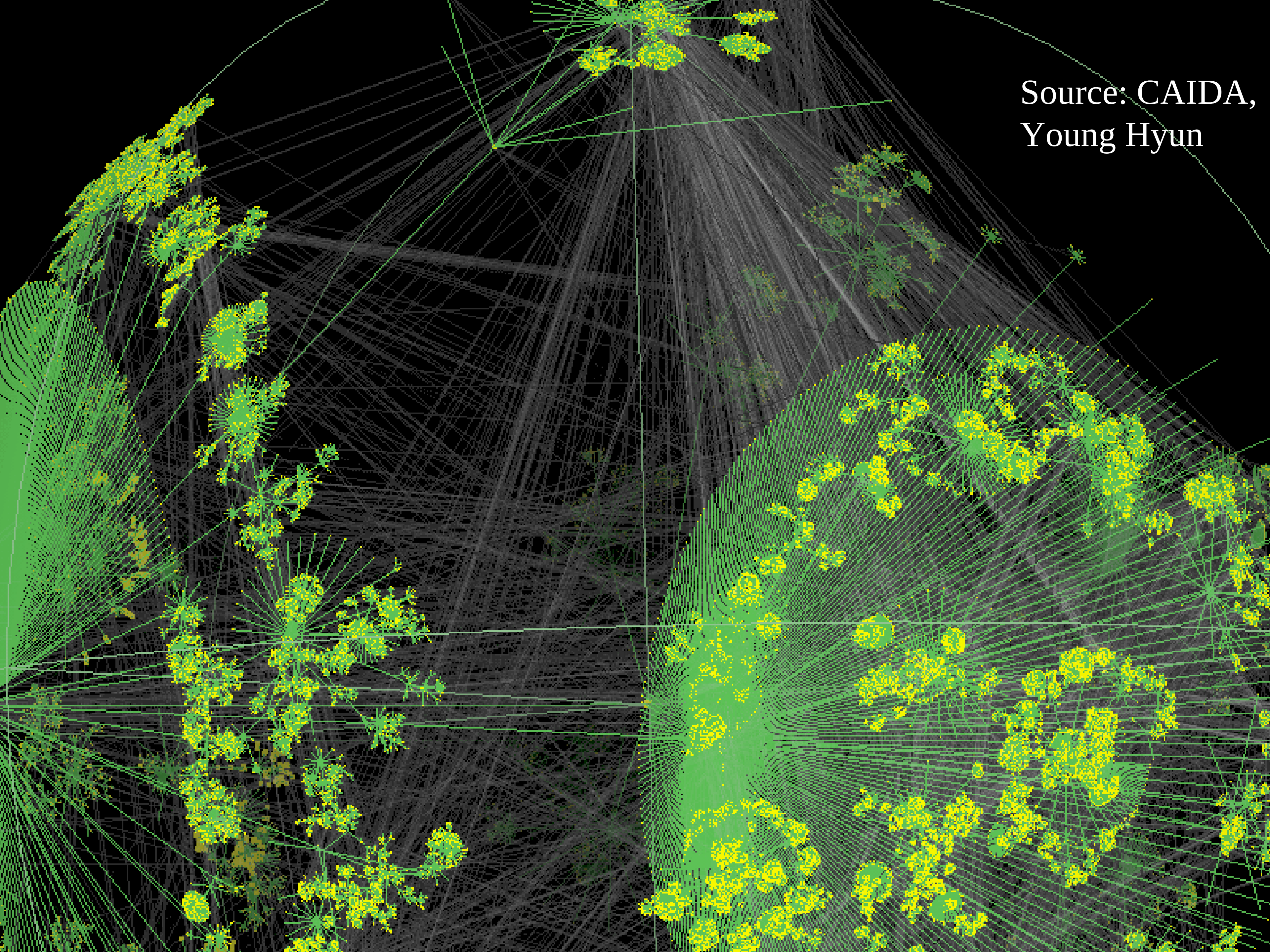
$$\alpha = 1$$



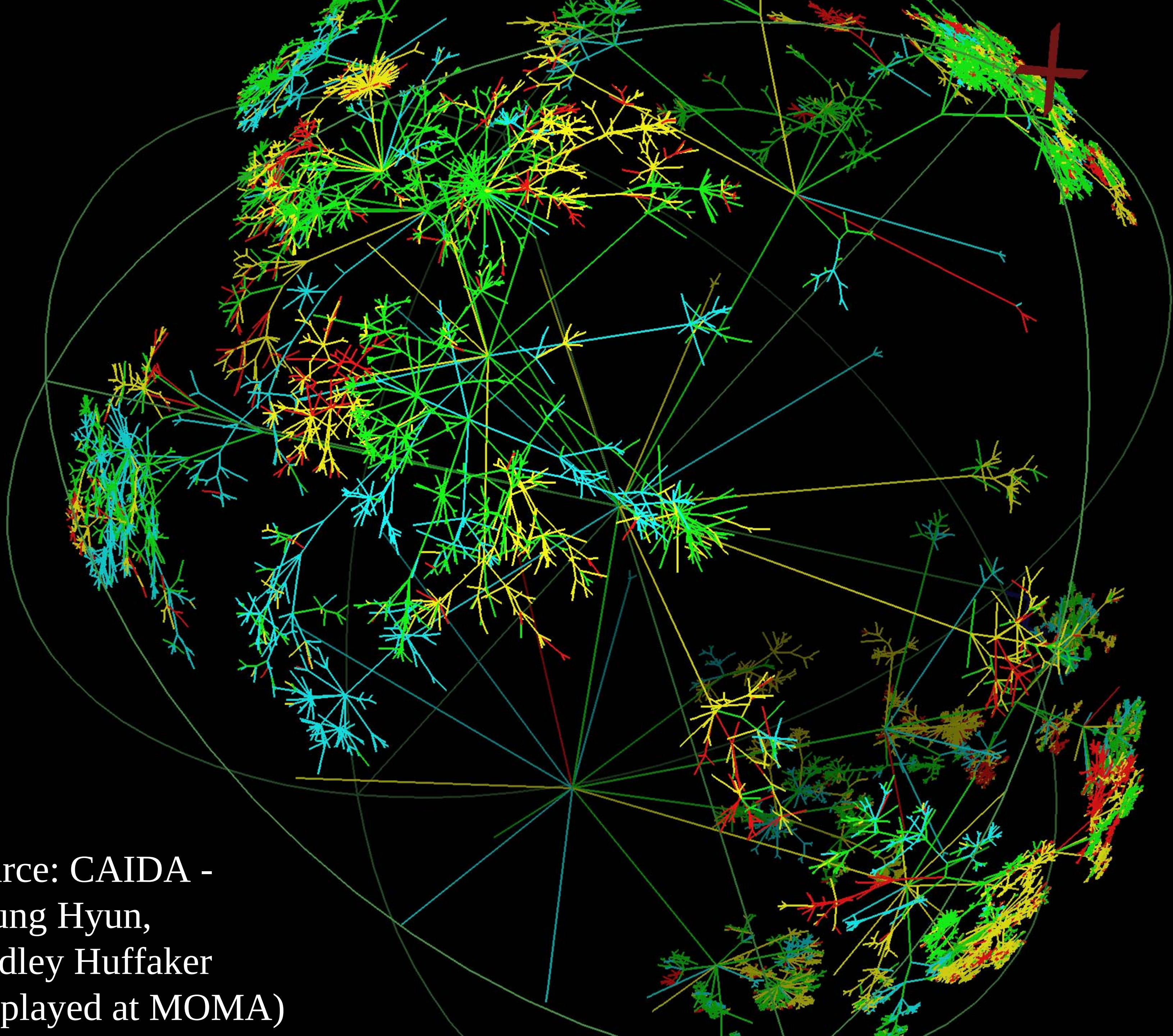
maximum flow:

$$\alpha \to 0$$





Source: CAIDA,
Young Hyun



Source: CAIDA -
Young Hyun,
Bradley Huffaker
(displayed at MOMA)

Flow level model

Define a Markov process $n(t) = (n_r(t), r \in R)$
with transition rates

$$n_r \to n_r + 1 \quad \text{at rate} \quad \nu_r \qquad r \in R$$

$$n_r \to n_r - 1 \quad \text{at rate} \quad n_r \, x_r(n) \, \mu_r \qquad r \in R$$

- Poisson arrivals, exponentially distributed file sizes

Roberts and Massoulié 1998

Stability

Let $$\rho_r = \frac{\nu_r}{\mu_r} \quad r \in R$$

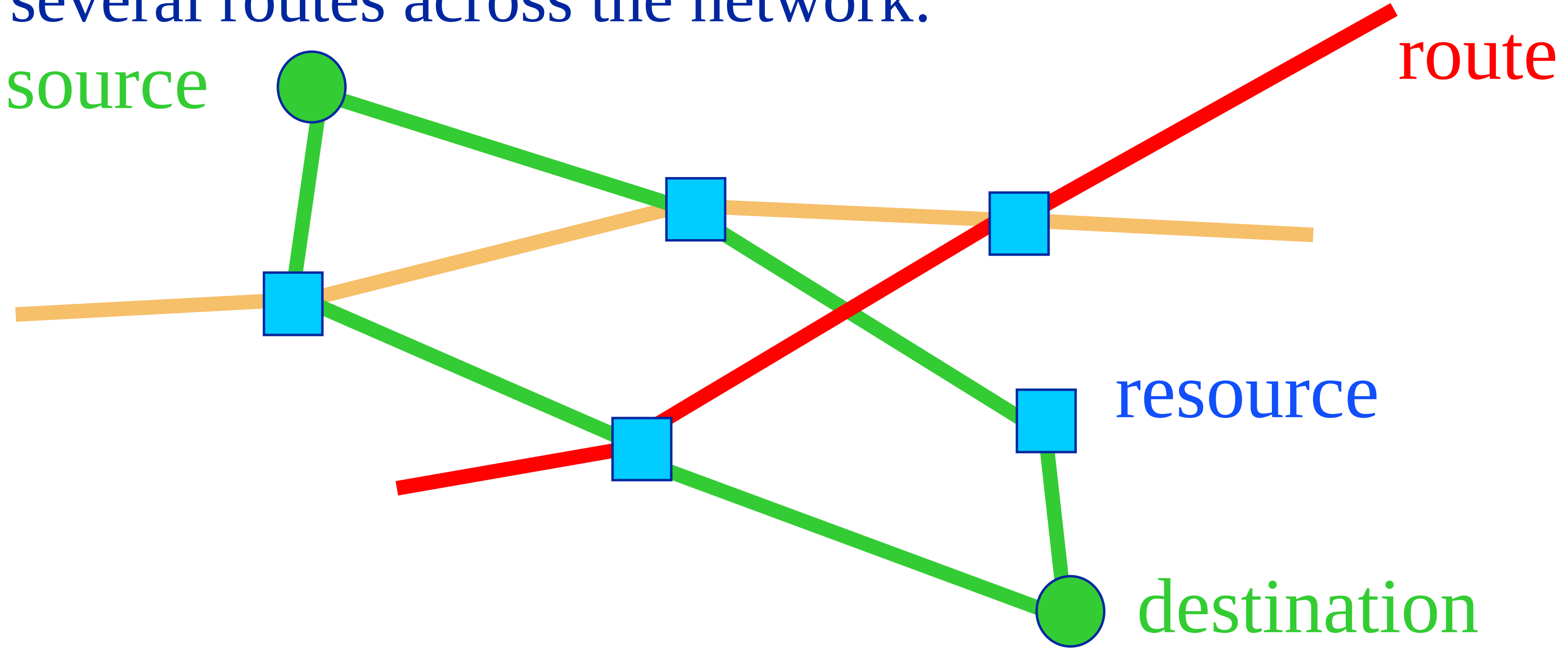
If $$\sum_r A_{jr}\, \rho_r \quad < \quad C_j \qquad j \in J$$

then the Markov chain is positive recurrent $$n(t) = (n_r(t), r \in R)$$

De Veciana, Lee & Konstantopoulos 1999;
Bonald & Massoulié 2001

Multipath routing

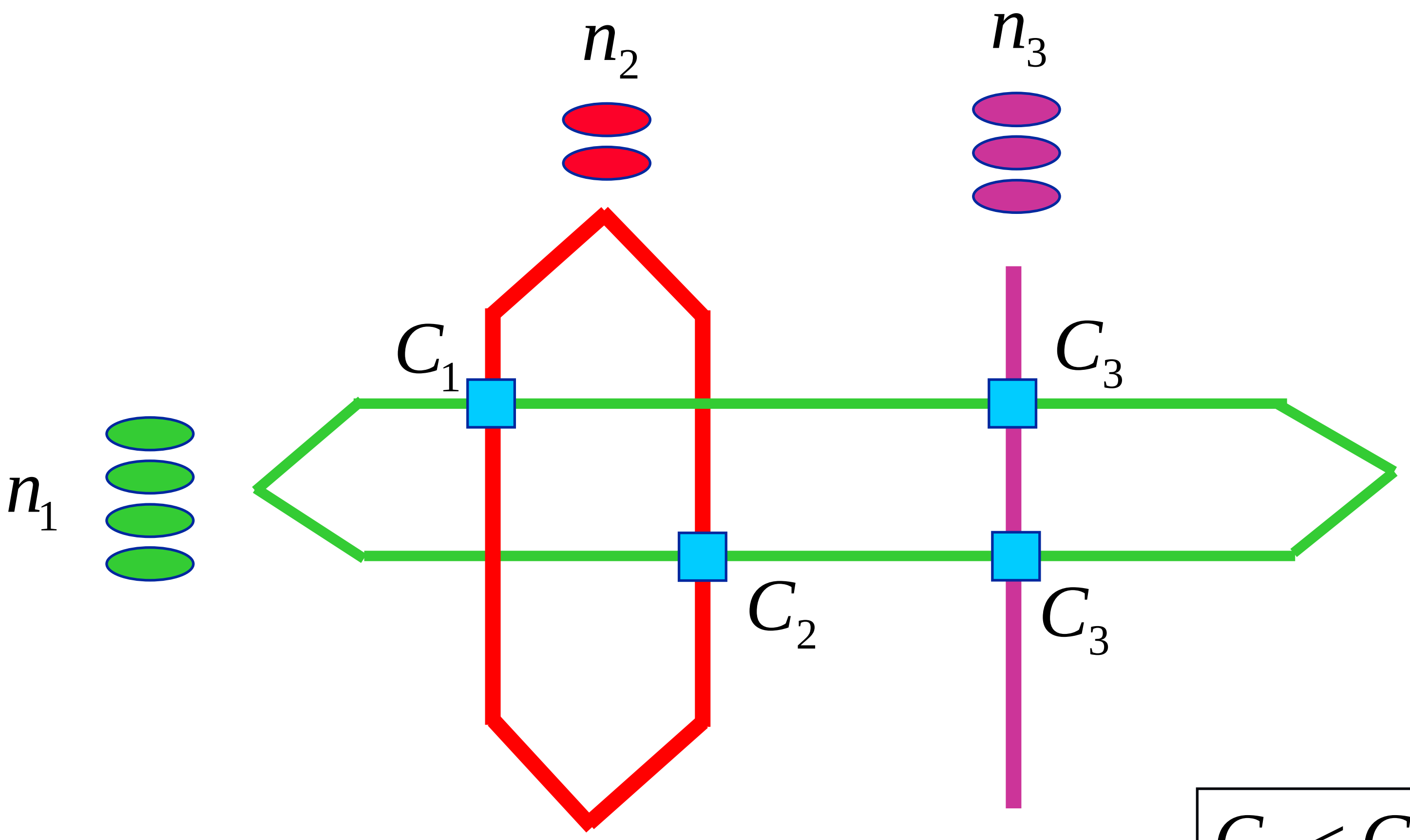
Suppose a source-destination pair has access to several routes across the network:



S - set of source-destination pairs

$r \in s$ - route r serves s-d pair s

Example of multipath routing

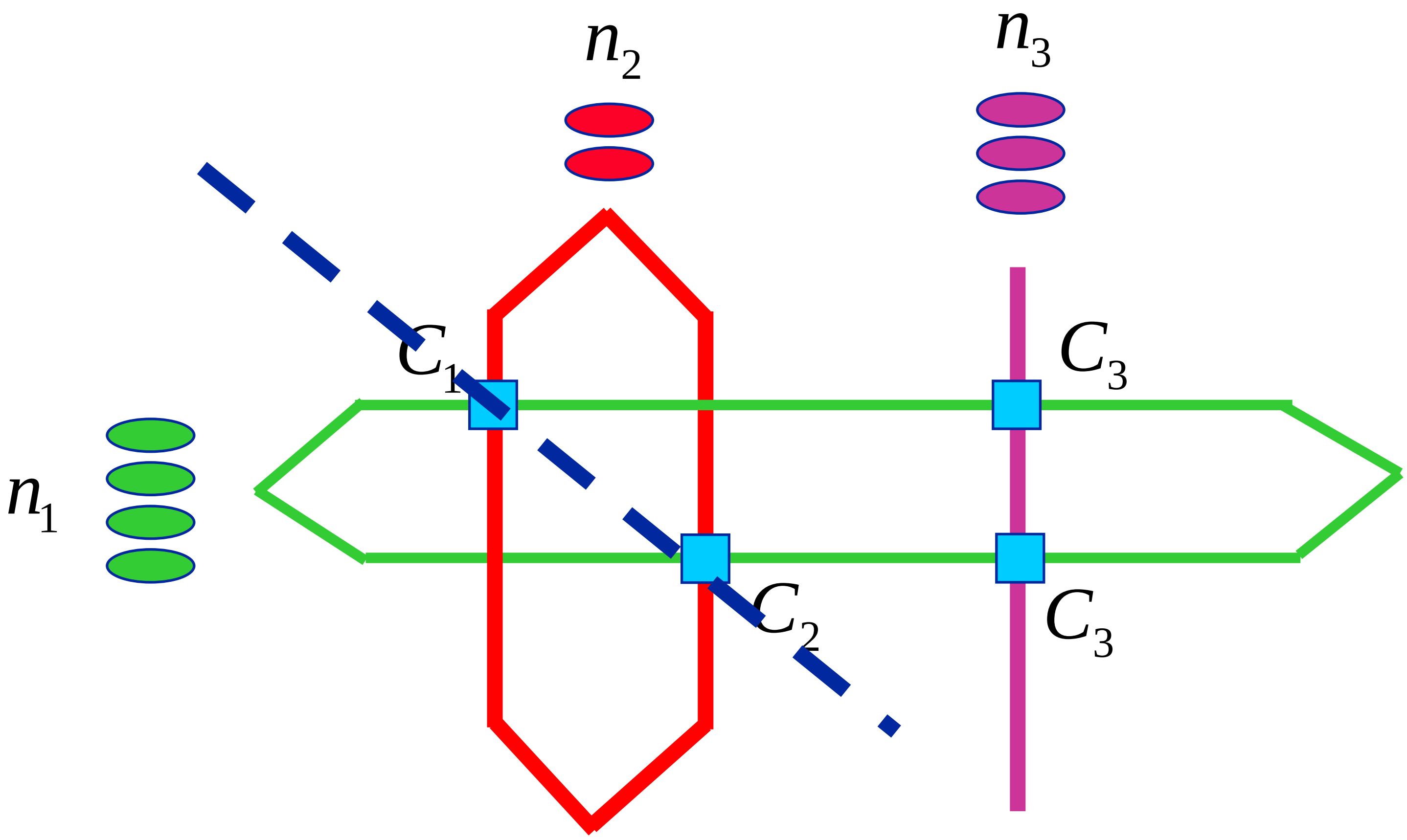


$C_3 < C_1, C_2$

Routes, as well as flow rates,
are chosen to optimize

$\sum_s n_s \log(x_s)$ over source-destination pairs s

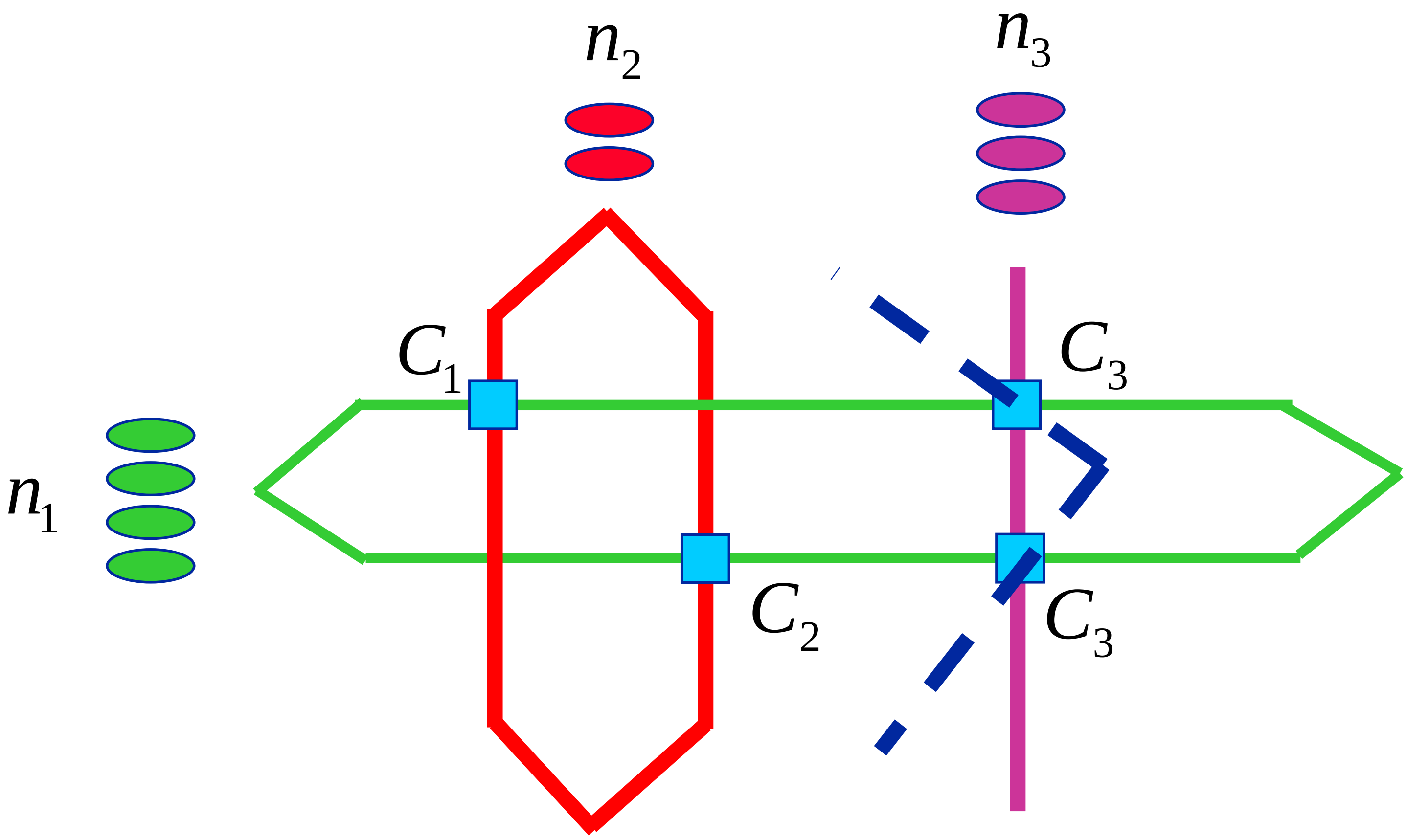
First cut constraint



$$n_1 x_1 + n_2 x_2 \leq C_1 + C_2$$

Cut defines a single *pooled resource*

Second cut constraint



$$\frac{1}{2}n_1 x_1 + n_3 x_3 \leq C_3$$

Cut defines a *second* pooled resource

Product form

$$\alpha = 1, w_r = 1, \ r \in R$$

In heavy traffic, and subject to some technical conditions, the (scaled) components of the shadow prices p for the pooled resources are independent and exponentially distributed. The corresponding approximation for n is

$$n_s \approx \rho_s \sum_j p_j A_{js} \quad s \in S$$

where

$$p_j \sim \mathrm{Exp}(\overline{C}_j - \sum_s \overline{A}_{js} \rho_s) \quad j \in \overline{J}$$

Dual random variables are independent and exponential

Kang, K, Lee and Williams 2009

Outline

- The processor sharing queue
- Sharing in networks – proportional fairness
- Multipath routing
- Markov chain description, and heavy traffic
- Ramp metering
- Resource pooling

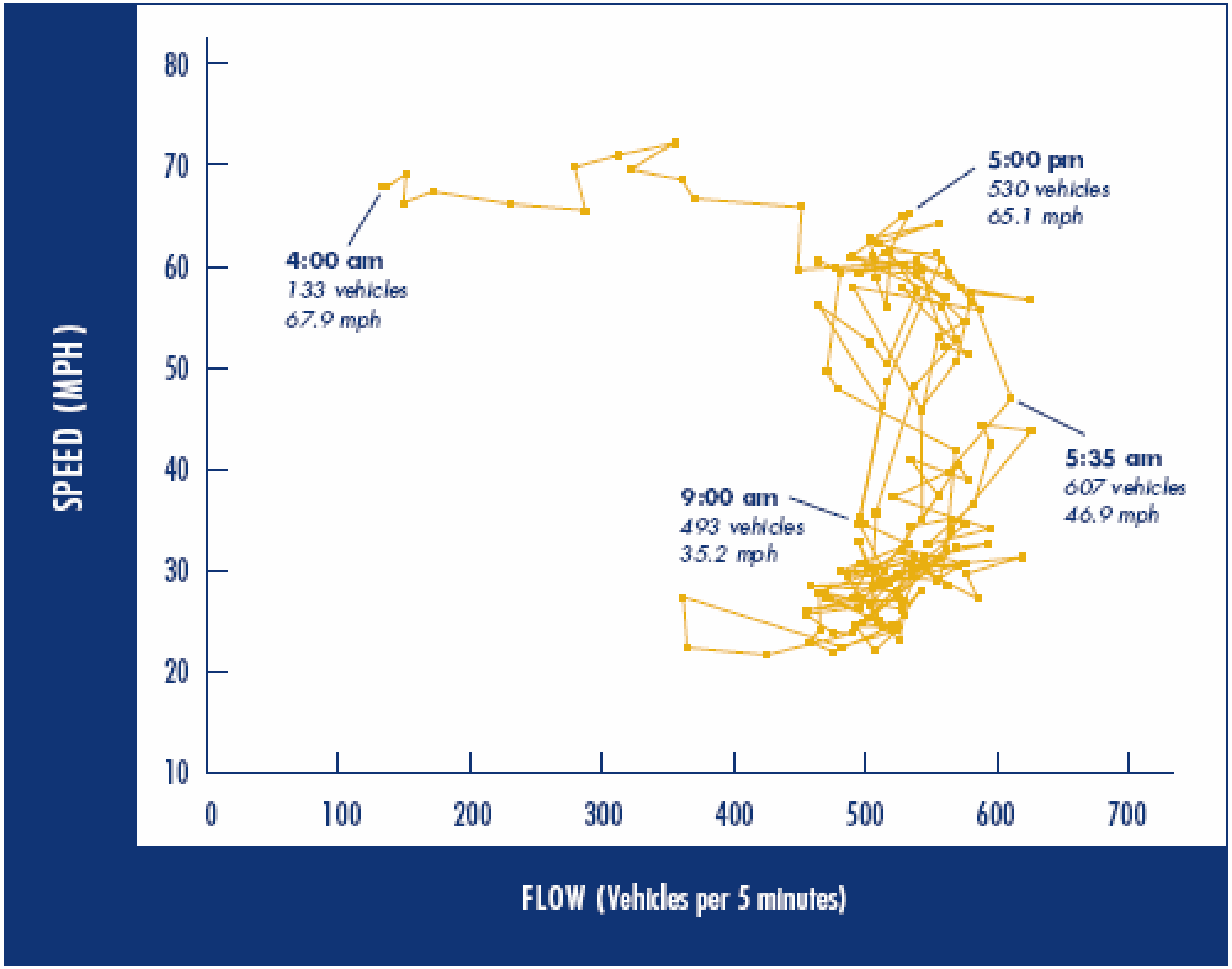


FIGURE 1

Speed vs. flow on I-10 westbound in 5 minute intervals from 4:00 am to 6:00 pm

[What we've learned about highway congestion](#)

P. Varaiya, Access 27, Fall 2005, 2-9.

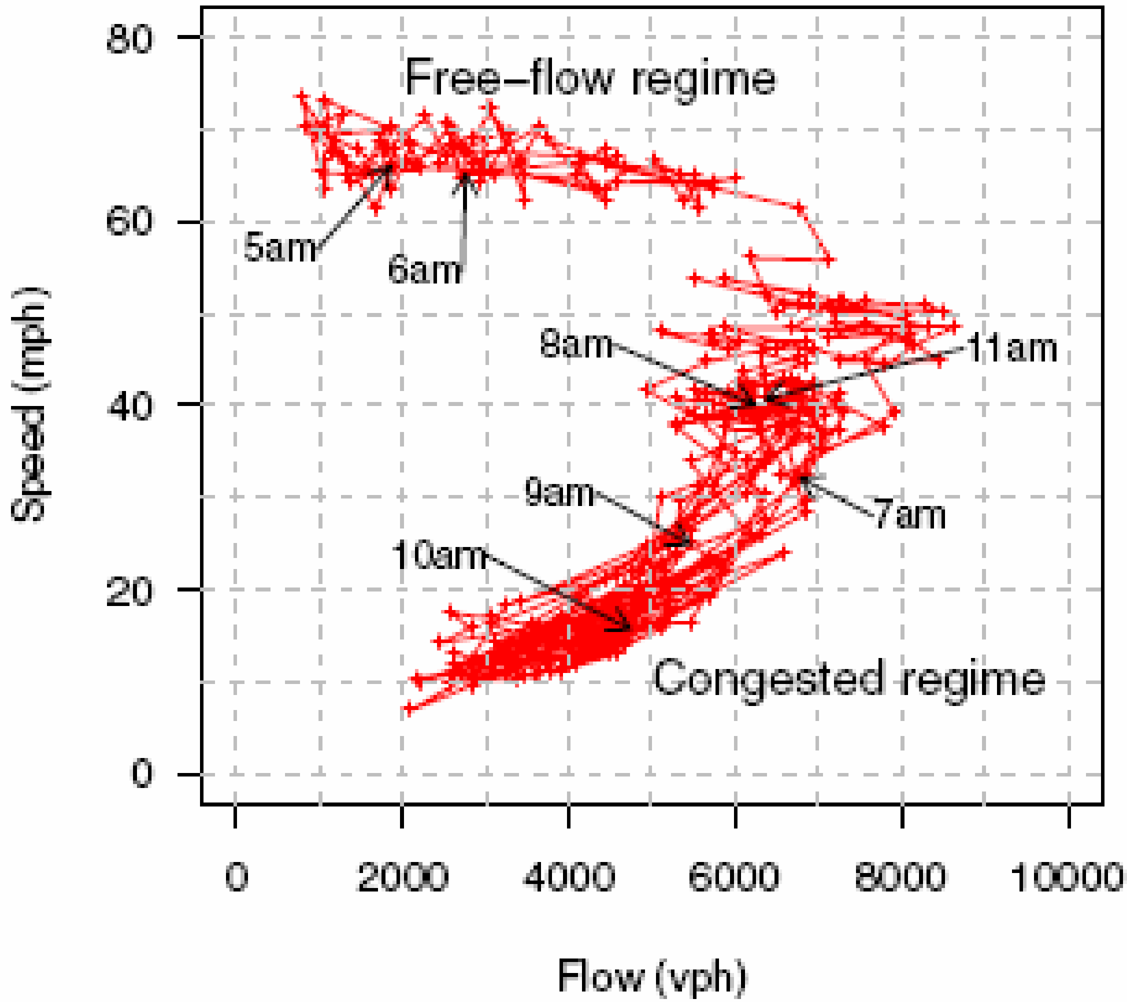


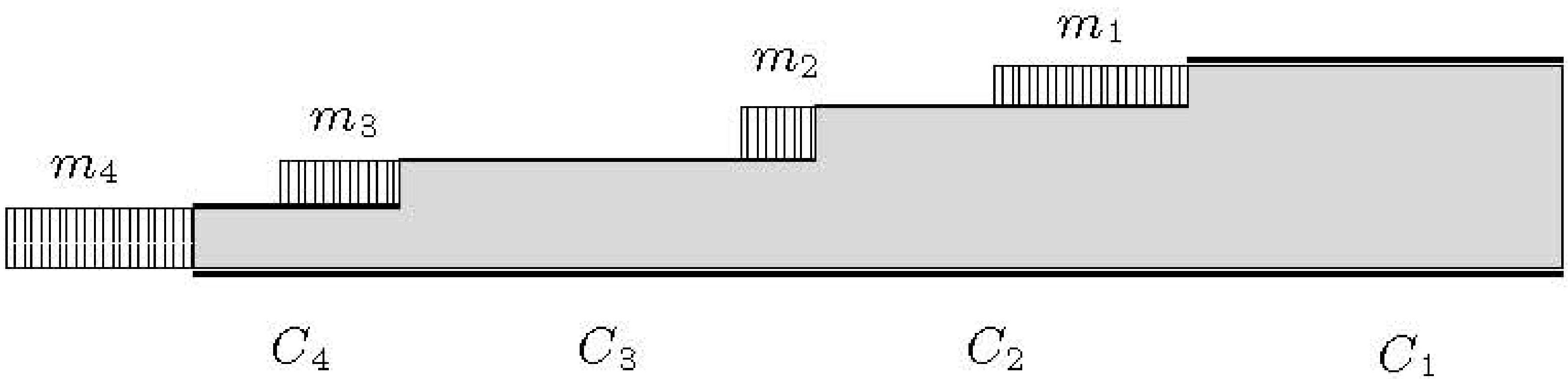
Figure 2. The relationship between the speed and flow of vehicles observed on the morning of Wednesday, 14 July 2004 using the M25 midway between junctions 11 and 12 in the clockwise direction. In the free-flow regime, flow rapidly increases with only a modest decline in speeds. Above a critical occupancy of vehicles there is a marked drop in speed with little, if any, improvement in flow which is then followed by a severe collapse into a congested regime where both flow and speed are highly variable and attain very low levels. Finally, the situation recovers with a return to higher flows and an improvement in speeds

Data, modelling and inference in road traffic networks

R.J. Gibbens and Y. Saatci

Phil. Trans. R. Soc. A366 (2008), 1907-1919.

A linear network



$$m_i(t) = m_i(0) + e_i(t) - \int_0^t \Lambda_i(m(s))\mathrm{d}s, \quad t \geq 0$$

queue size

cumulative inflow

metering rate

K and Williams 2010

Metering policy

Suppose the metering rates can be chosen to be any vector $\Lambda = \Lambda(m)$ satisfying

$$\sum_i A_{ji}\,\Lambda_i \le C_j, \quad j \in J$$

$$\Lambda_i \ge 0, \quad i \in I$$

$$\Lambda_i = 0, \quad m_i = 0$$

and such that

$$m_i(t) = m_i(0) + e_i(t) - \int_0^t \Lambda_i(m(s))\mathrm{d}s \ge 0, \quad t \ge 0$$

Optimal policy?

For each of $i = I, I\text{-}1, \ldots\ldots 1$ in turn choose

$$\int_0^t \Lambda_i(m(s))\mathrm{d}s \geq 0$$

to be maximal, subject to the constraints.

This policy minimizes

$$\sum_i m_i(t)$$

for all times t

Proportionally fair metering

Suppose $\Lambda(m) = (\Lambda_i(m), i \in I)$ is chosen to

maximize $$\sum_i m_i \log \Lambda_i$$

subject to $$\sum_i A_{ji} \Lambda_i \leq C_j, \quad j \in J$$

$$\Lambda_i \geq 0, \quad i \in I$$

$$\Lambda_i = 0, \quad m_i = 0$$

Proportionally fair metering

$$\Lambda_i(m) = \frac{m_i}{\sum_j p_j A_{ji}}, \quad i \in I$$

where

$$\boxed{\begin{aligned} &\Lambda_i \geq 0, \quad i \in I \\ &\sum_i A_{ji}\,\Lambda_i \leq C_j, \quad j \in J \\ &p_j \geq 0, \quad j \in J \\ &p_j\left(C_j - \sum_i A_{ji}\,\Lambda_i\right) \geq 0, \quad j \in J \end{aligned}}$$

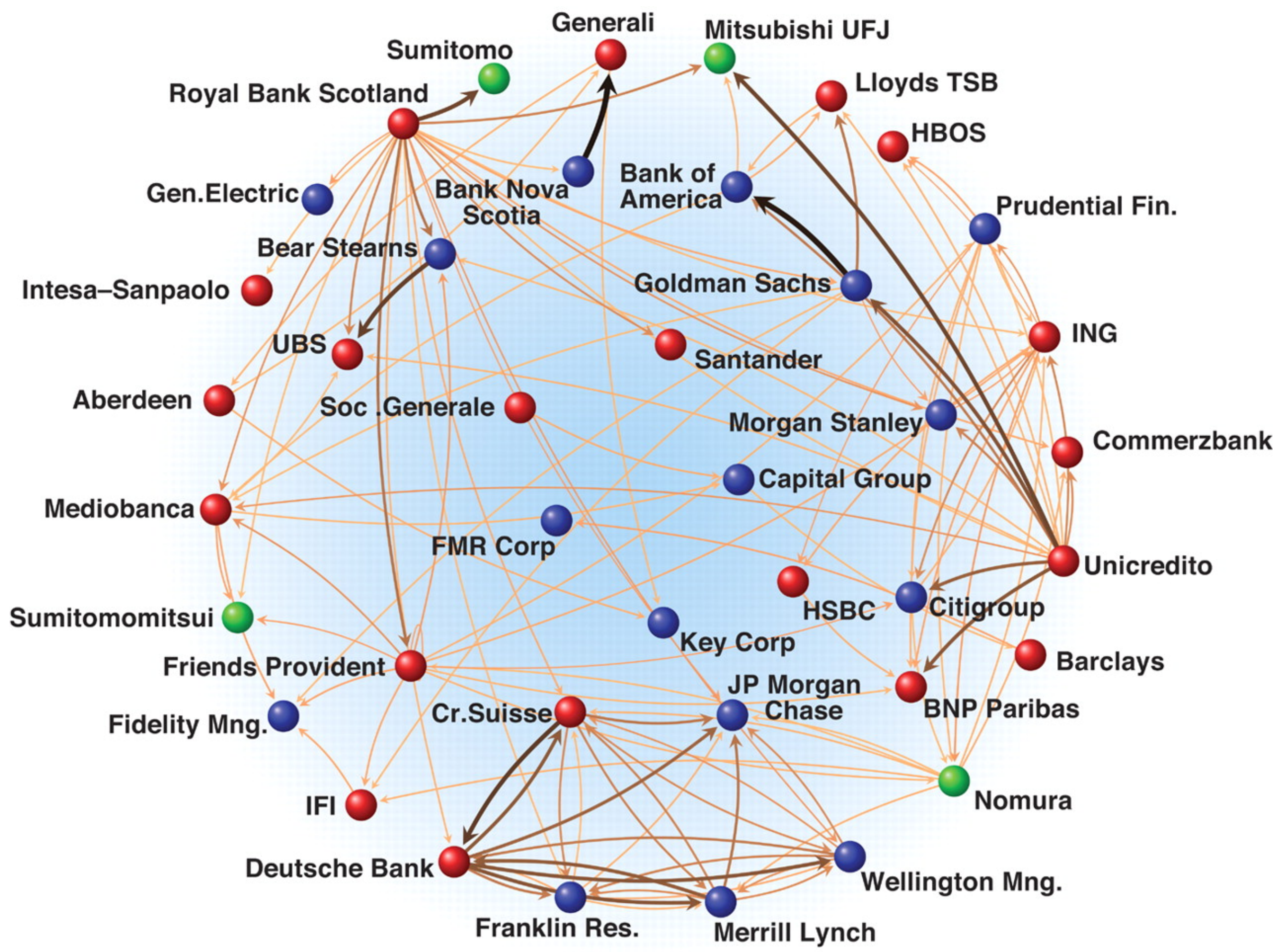
KKT conditions

p_j - *shadow price* (Lagrange multiplier) for the resource j capacity constraint

Outline

- The processor sharing queue
- Sharing in networks – proportional fairness
- Multipath routing
- Markov chain description, and heavy traffic
- Ramp metering
- Resource pooling

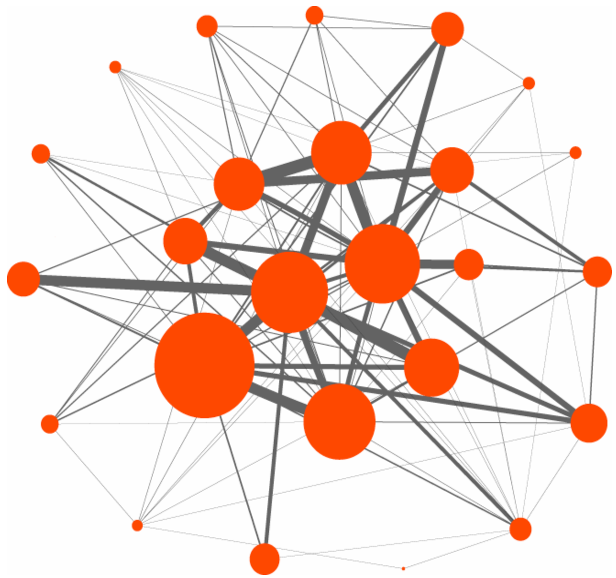
Fig. 2 A sample of the international financial network, where the nodes represent major financial institutions and the links are both directed and weighted and represent the strongest existing relations among them



F. Schweitzer et al., Science 325, 422 -425 (2009)

Published by AAAS

Chart 3.2 Network of large exposures between UK banks(a)(b)(c)



Source: FSA returns.

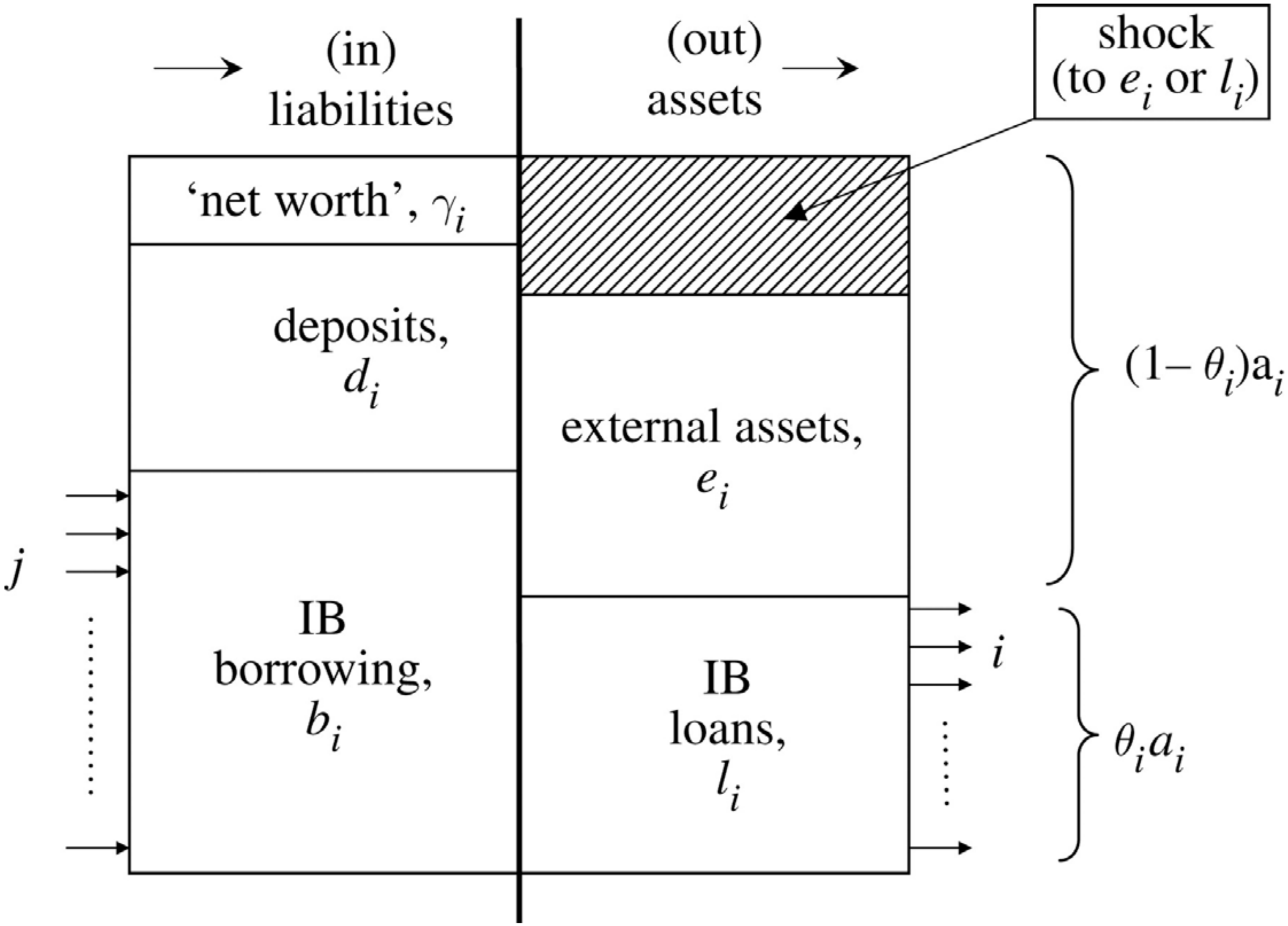
(a) A large exposure is one that exceeds 10% of a lending bank's eligible capital at the end of a period. Eligible capital is defined as Tier 1 plus Tier 2 capital, minus regulatory deductions.

(b) Each node represents a bank in the United Kingdom. The size of each node is scaled in proportion to the sum of (1) the total value of exposures to a bank, and (2) the total value of exposures of the bank to others in the network. The thickness of the line is proportional to the value of a single bilateral exposure.

(c) Based on 2009 Q2 data.

http://www.bankofengland.co.uk/financialstability/

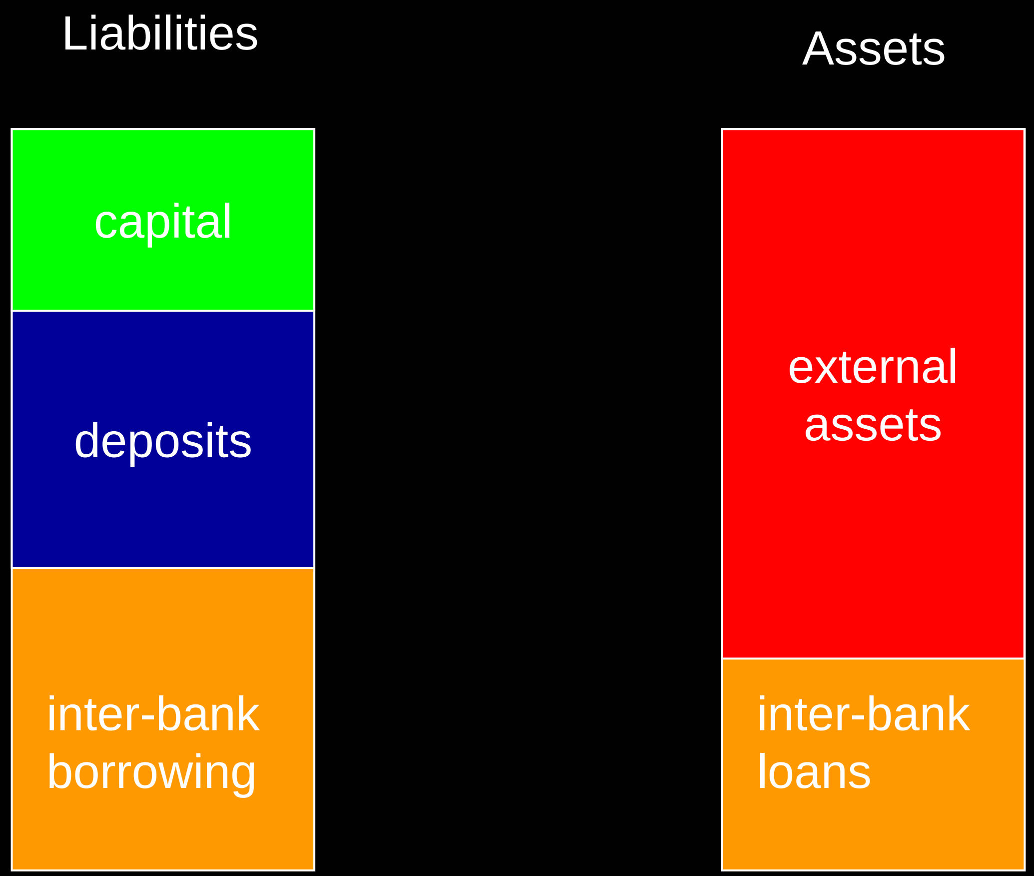
Schematic model for a 'node' in the IB network.



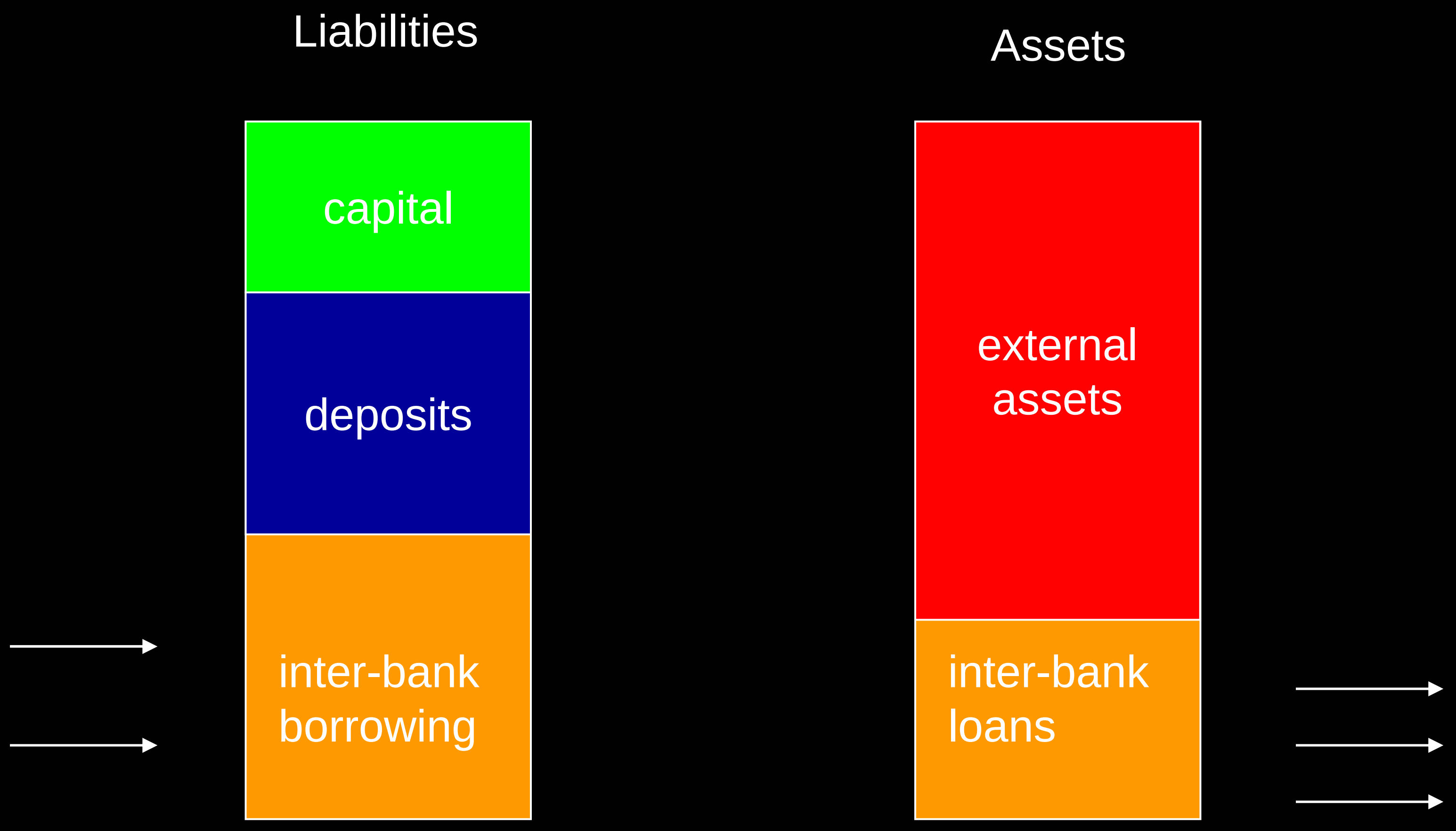
May R M , Arinaminpathy N J. R. Soc. Interface
doi:10.1098/rsif.2009.0359

(Also: Gai and Kapadia –
Contagion in Financial Markets)

Stylised bank balance sheet



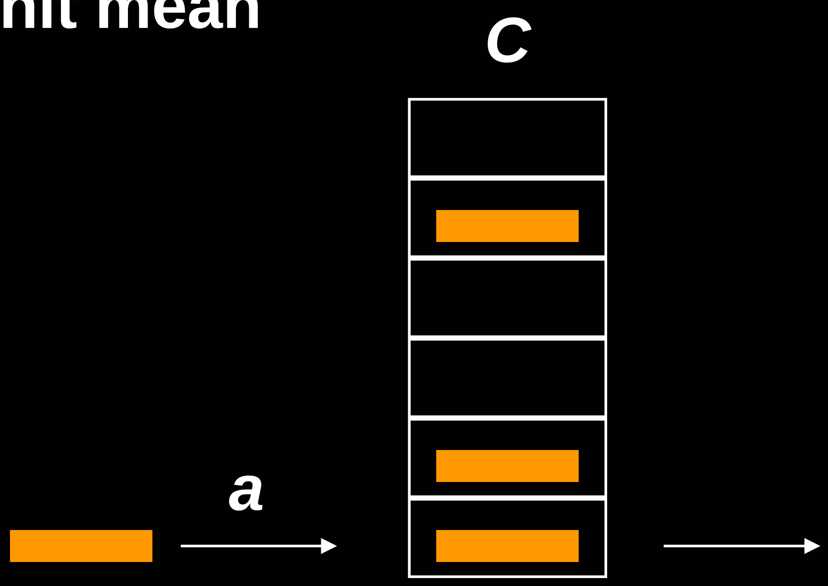
Stylised bank balance sheet



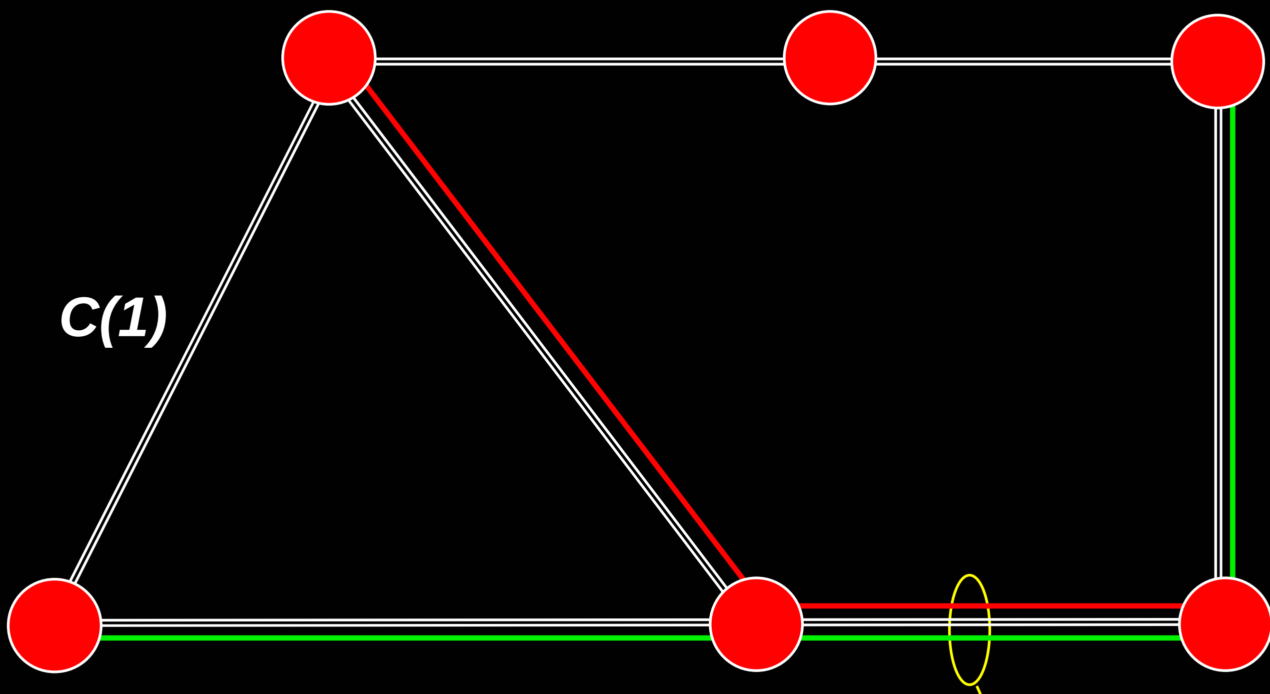
Erlang's formula

- **calls arrive randomly, at rate *a***
- **resource has *C* circuits**
- **accepted calls hold a circuit for a random holding time, with unit mean**
- **blocked calls are lost**
- **proportion of calls lost is:**

$$E(a, C) = \frac{a^{C}/C!}{\sum_{0}^{C}(a^{n}/n!)}$$



A loss network



Link constraint: $\sum_{r} A(j,r)\,n(r) \leq C(j)$

Resource pooling

Aims:

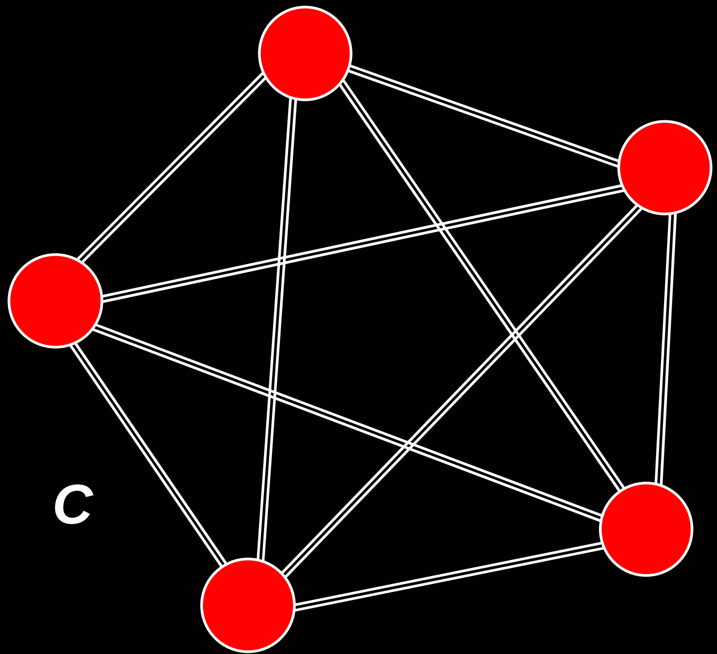
- respond robustly to failures and overloads
- lessen the impact of forecasting errors
- make use of spare capacity in the network
- permit flexible use of network resources

Problems:

- instability
- complexity

Example: alternative routing

- Complete graph
- All links have capacity C
- Call routed directly if possible; otherwise one randomly chosen alternative route may be tried



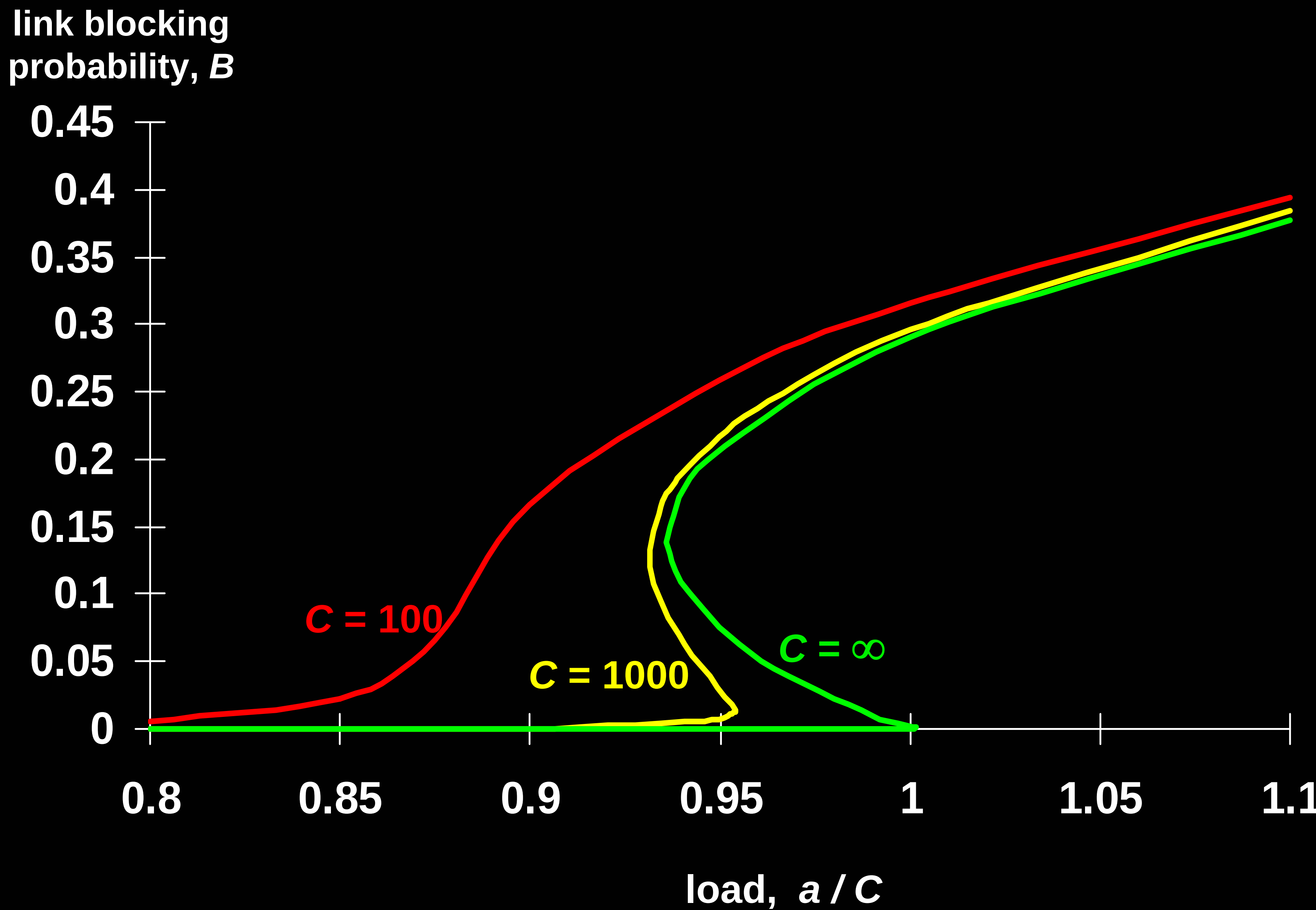
Marbukh 1984, Gibbens, Hunt, K 1990,
Crametz, Hunt 1991, Graham, Méléard 1993, 1994

alternative routing

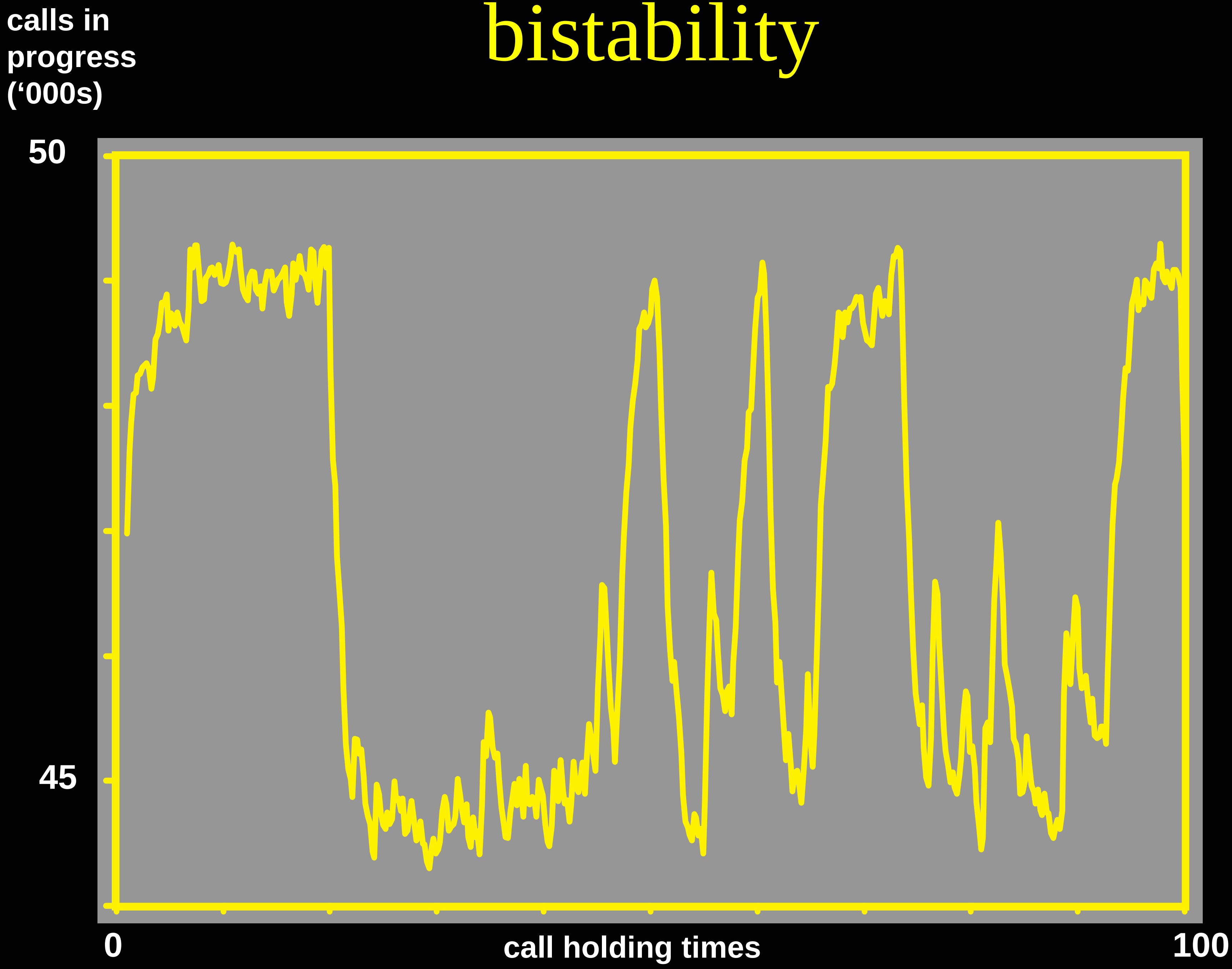
- Arrival rate per link - a
- Capacity per link - C
- Let B be the link blocking probability
- Then as the number of nodes grows, the blocking probability B approaches a solution of:

$$\boxed{B = \mathrm{E}(\mathrm{a}[1 + 2B(1 - B)], C)}$$

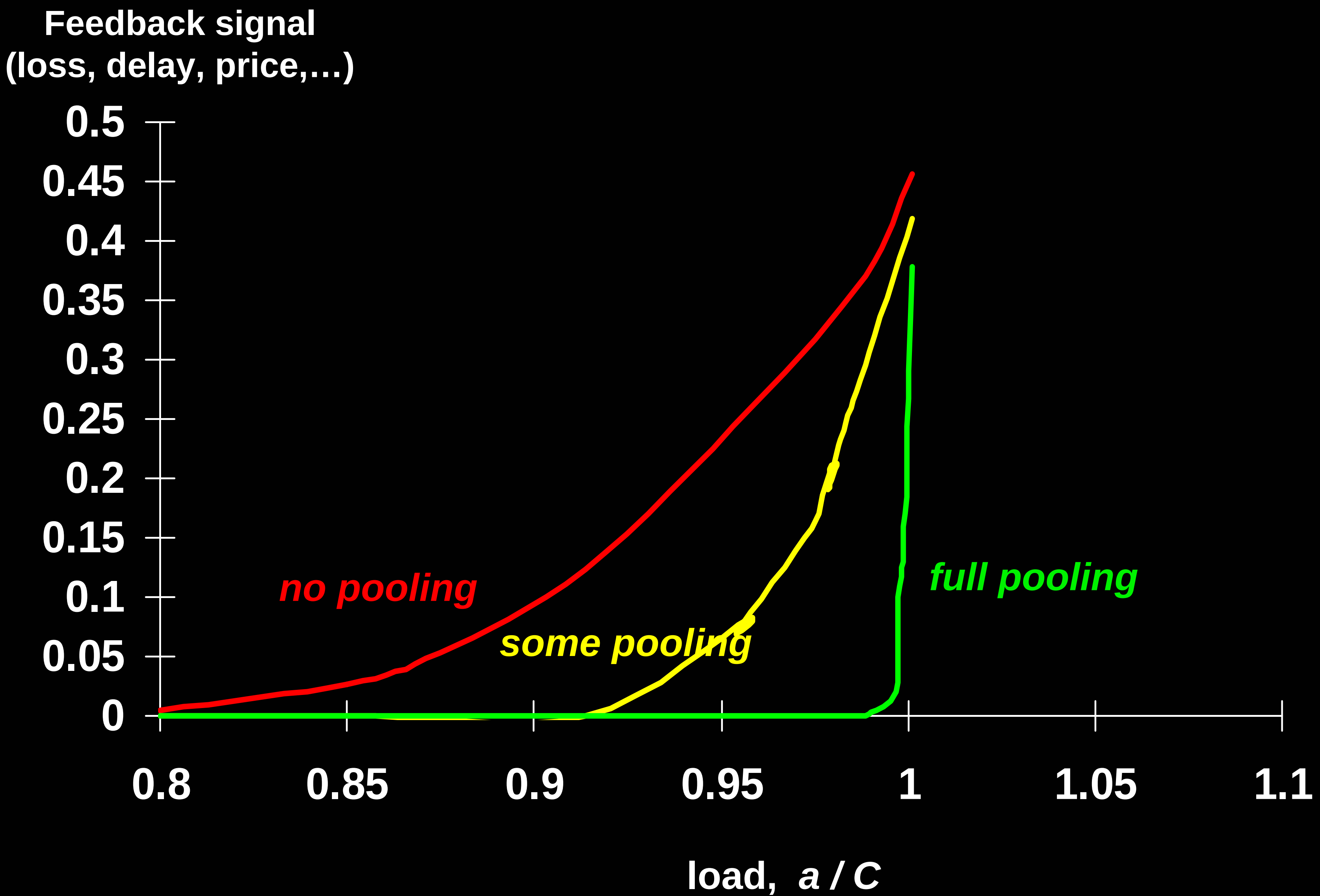
instability, and hysteresis



bistability



Sudden impact of capacity



Open questions on resource pooling

- Resource pooling does indeed
  - respond robustly to failures and overloads
  - lessen the impact of forecasting errors
  - make use of spare capacity in the network
  - permit flexible use of network resources
- But
  - can produce phase transitions if load amplified
  - obscures the approach of capacity overload
- Can decentralised control take account of system-wide risks?

The future?

- Many mathematical challenges, associated with the combination of network flow and stochastic models of resource possession
- Applications to controlled motorways, router design, systemic risk.....