Models of network routing and resource allocation

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Outline

- The processor sharing queue
- Sharing in networks – proportional fairness
- Multipath routing
- Markov chain description, and heavy traffic

Processor sharing discipline

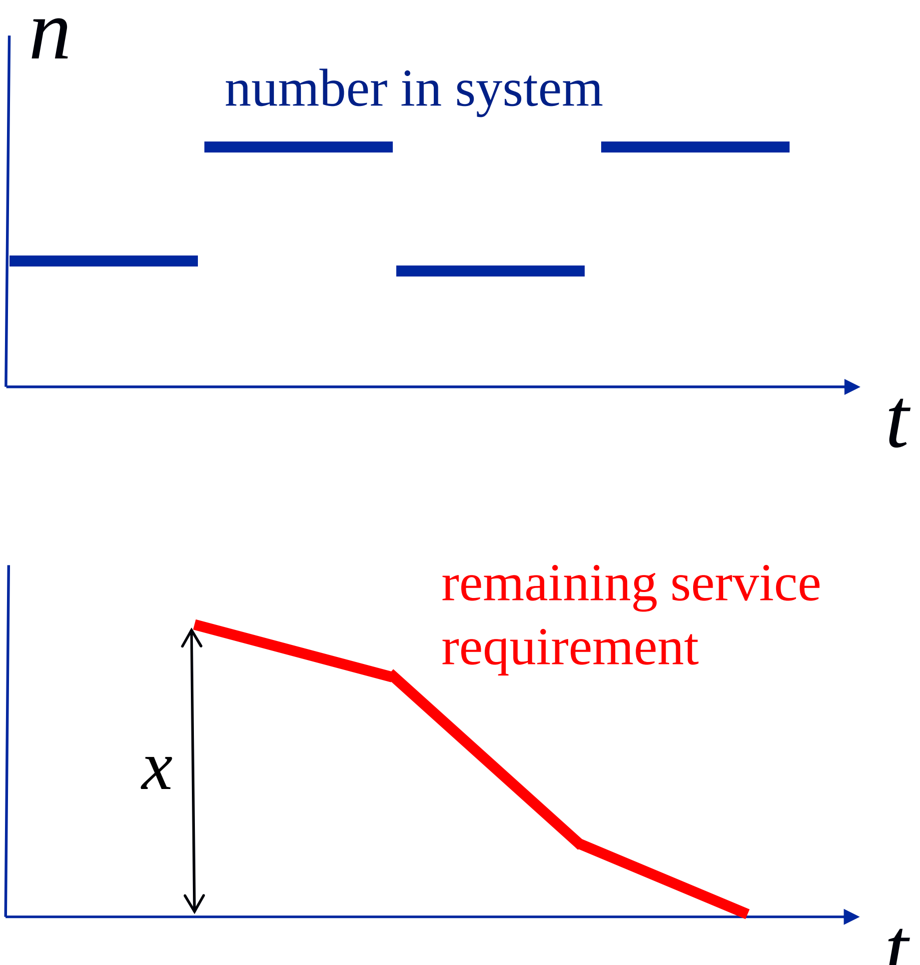
Kleinrock, 1967, 1976; Boxma tutorial, informs 2005

- Often attractive in practice, since gives
  - rapid service for short jobs
  - the appearance of a processor continuously available (albeit of varying capacity)
- Tractable analytically – a symmetric discipline. E.g. for M/G/1 PS

$$E\left[\text{sojourn time}, S \mid \text{job size}, x\right] = \frac{x}{C - \rho}$$

(similar tractability for LCFS, Erlang loss system, networks of symmetric queues)

The M/G/1 processor sharing queue



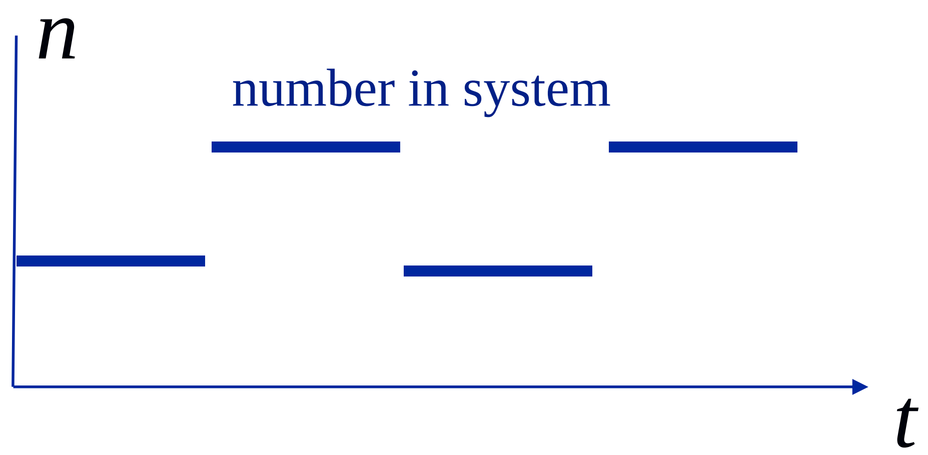
The M/G/1 processor sharing queue

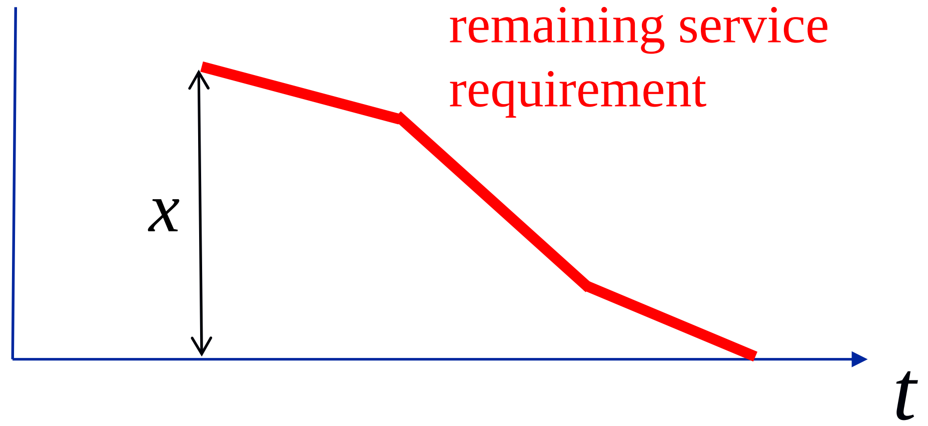
$$[S \mid x] \cong \frac{x}{C-\rho} + o(1/x)$$

if x is large;

$$[S \mid x] \cong x.\frac{n+1}{C} + o(x)$$

if x is small, where n is a geometric random variable.





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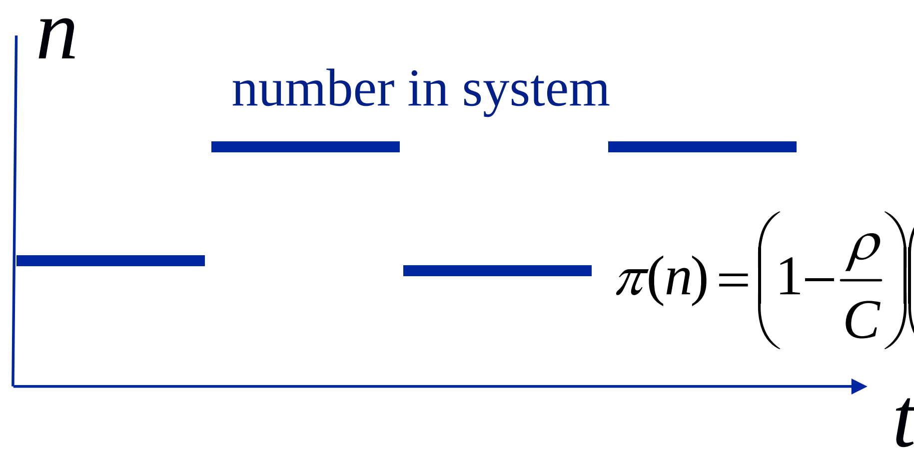
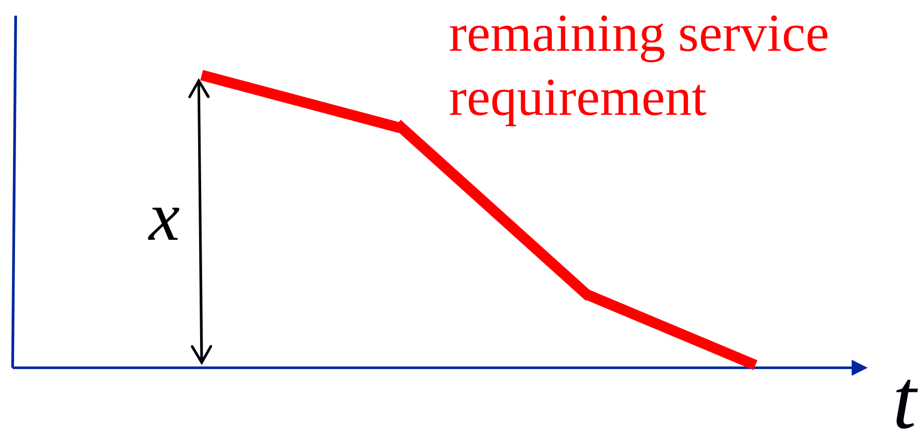


$$\pi(n) = \left(1-\frac{\rho}{C}\right)\left(\frac{\rho}{C}\right)^n$$



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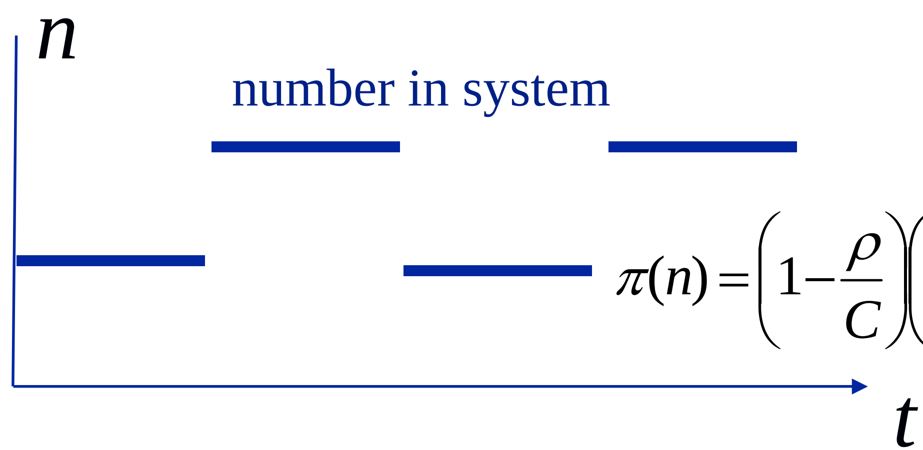

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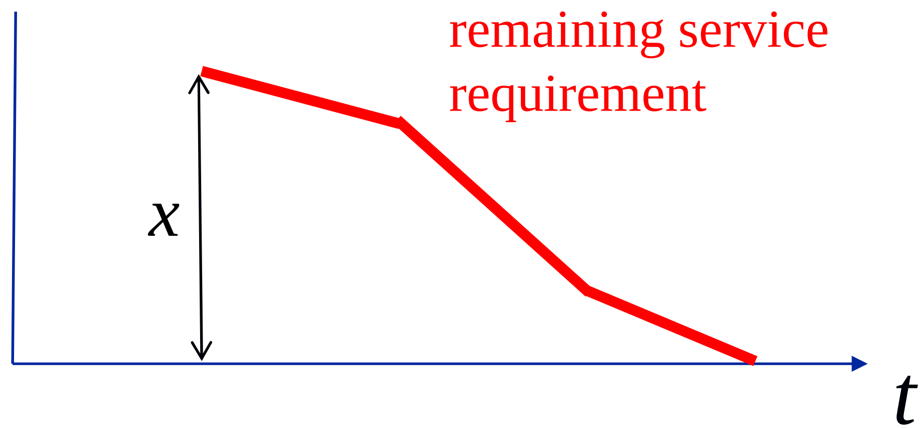


$$E[S \mid x] = \frac{x}{C-\rho}$$ in both cases, of course!

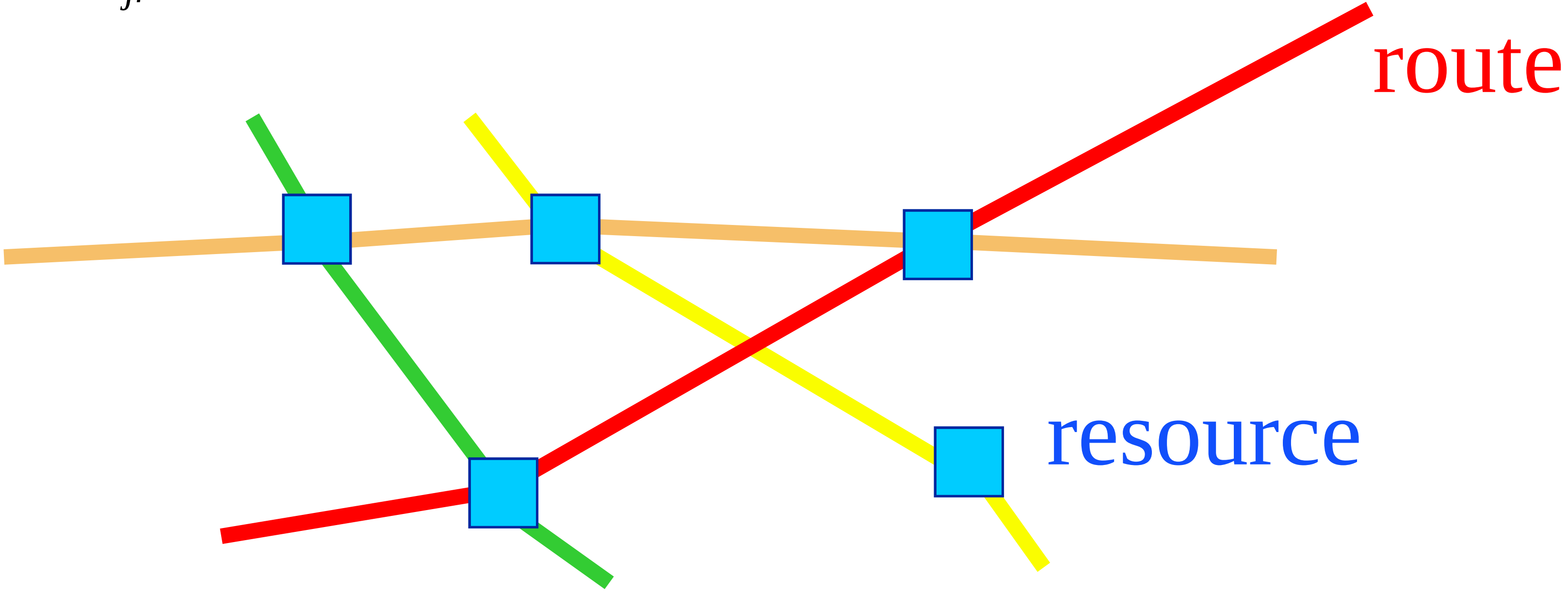
What is the network equivalent?

J - set of resources

R - set of routes

$A_{jr} = 1$ - if resource j is on route r

$A_{jr} = 0$ - otherwise



Rate allocation

n_r - number of flows on route r

x_r - rate of each flow on route r

Given the vector how are the rates chosen ?

$$n = (n_r, r \in R)$$

$$x = (x_r, r \in R)$$

Optimization formulation

Suppose $x = x(n)$ is chosen to

maximize $\sum_r w_r n_r \frac{x_r^{1-\alpha}}{1-\alpha}$

subject to $\sum_r A_{jr} n_r x_r \le C_j \quad j \in J$

$$x_r \ge 0 \quad r \in R$$

(weighted α-fair allocations, Mo and Walrand 2000)

$0 < \alpha < \infty$ (replace $\frac{x_r^{1-\alpha}}{1-\alpha}$ by $\log(x_r)$ if $\alpha = 1$)

Solution

$$x_r = \left(\frac{w_r}{\sum_j A_{jr} p_j(n)} \right)^{1/\alpha} \qquad r \in R$$

where

$$\sum_r A_{jr} n_r x_r \le \quad C_j \quad j \in J; \quad x_r \ge 0 \quad r \in R$$

$$p_j(n) \ge 0 \quad j \in J$$

$$p_j(n) \left(C_j - \sum_r A_{jr} n_r x_r \right) \ge 0 \quad j \in J$$

KKT conditions

$p_j(n)$ - *shadow price* (Lagrange multiplier) for the resource j capacity constraint

Examples of α-fair allocations

maximize $\sum_r w_r n_r \frac{x_r^{1-\alpha}}{1-\alpha}$

subject to $\sum_r A_{jr} n_r x_r \le C_j \quad j \in J$

$x_r \ge 0 \quad r \in R$

$$x_r = \left(\frac{w_r}{\sum_j A_{jr} p_j(n)} \right)^{1/\alpha} \quad r \in R$$

$\alpha \to 0 \quad (w = 1)$ - maximum flow

$\alpha \to 1 \quad (w = 1)$ - proportionally fair

$\alpha = 2 \quad (w_r = 1/T_r^2)$ - TCP fair

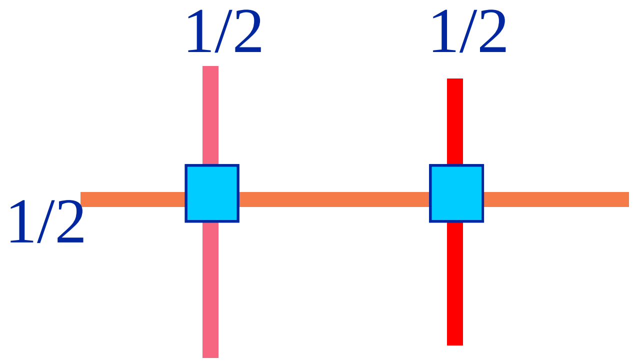
$\alpha \to \infty \quad (w = 1)$ - max-min fair

Example

$$n_r = 1,\ \ w_r = 1\ \ r \in R,$$
$$C_j = 1\ \ j \in J$$

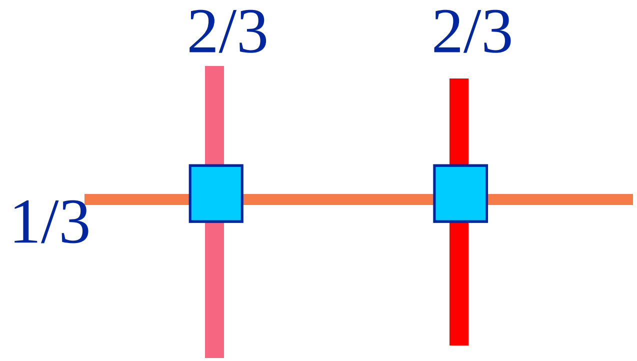
max-min fairness:

$$\alpha \to \infty$$



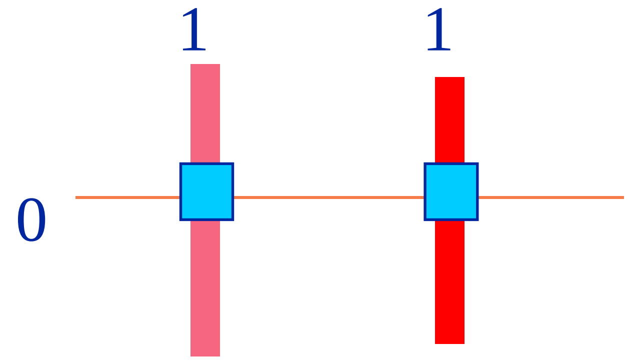
proportional fairness:

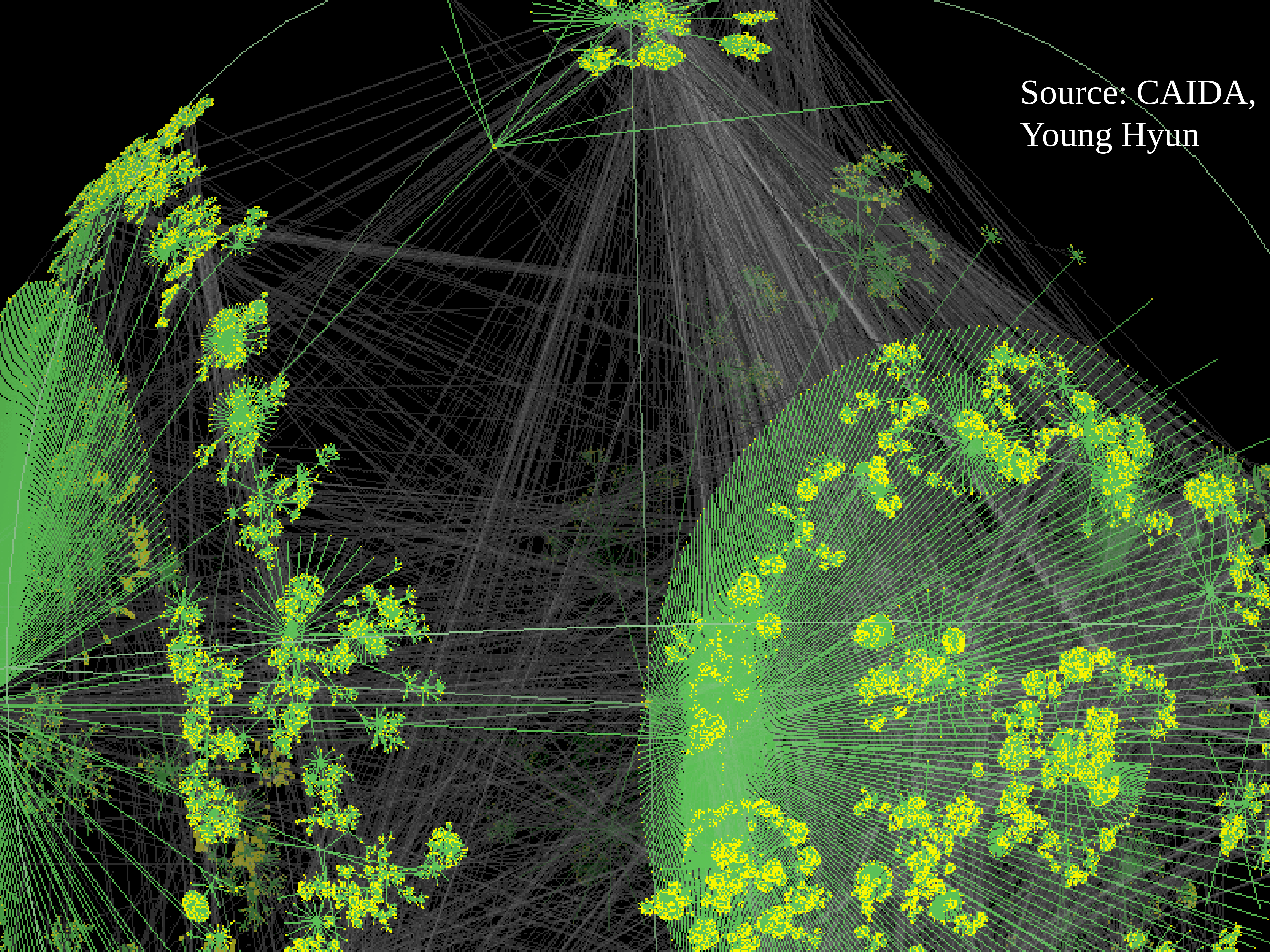
$$\alpha = 1$$

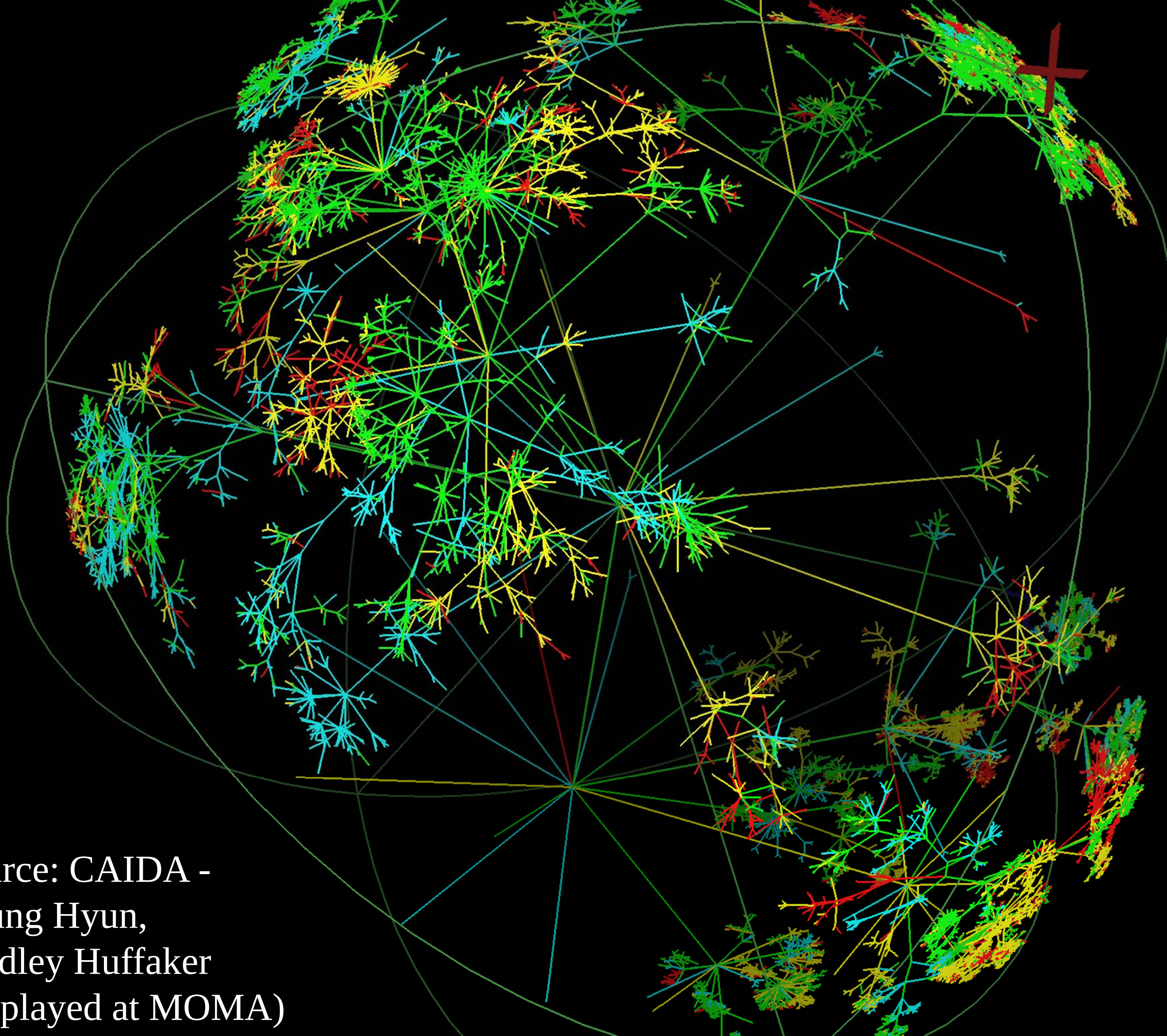


maximum flow:

$$\alpha \to 0$$




Source: CAIDA,
Young Hyun



Source: CAIDA -
Young Hyun,
Bradley Huffaker
(displayed at MOMA)

Flow level model

Define a Markov process $n(t) = (n_r(t), r \in R)$
with transition rates

$$n_r \to n_r + 1 \quad \text{at rate} \quad \nu_r \qquad r \in R$$

$$n_r \to n_r - 1 \quad \text{at rate} \quad n_r \, x_r(n) \, \mu_r \quad r \in R$$

- Poisson arrivals, exponentially distributed file sizes

Roberts and Massoulié 1998

Stability

Let $$\rho_r = \frac{\nu_r}{\mu_r} \quad r \in R$$

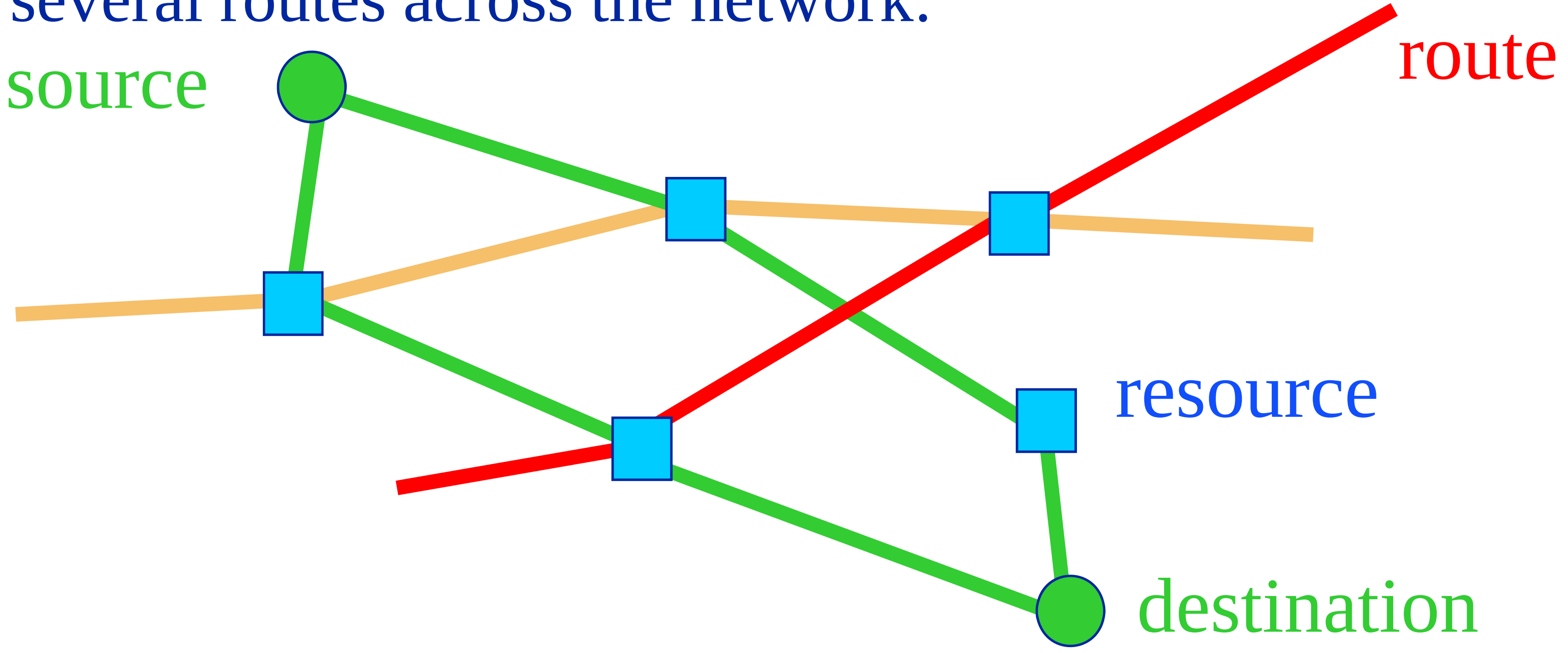
If $$\sum_r A_{jr}\, \rho_r \quad < \quad C_j \quad j \in J$$

then the Markov chain $n(t) = (n_r(t), r \in R)$
is positive recurrent

De Veciana, Lee & Konstantopoulos 1999;
Bonald & Massoulié 2001

Multipath routing

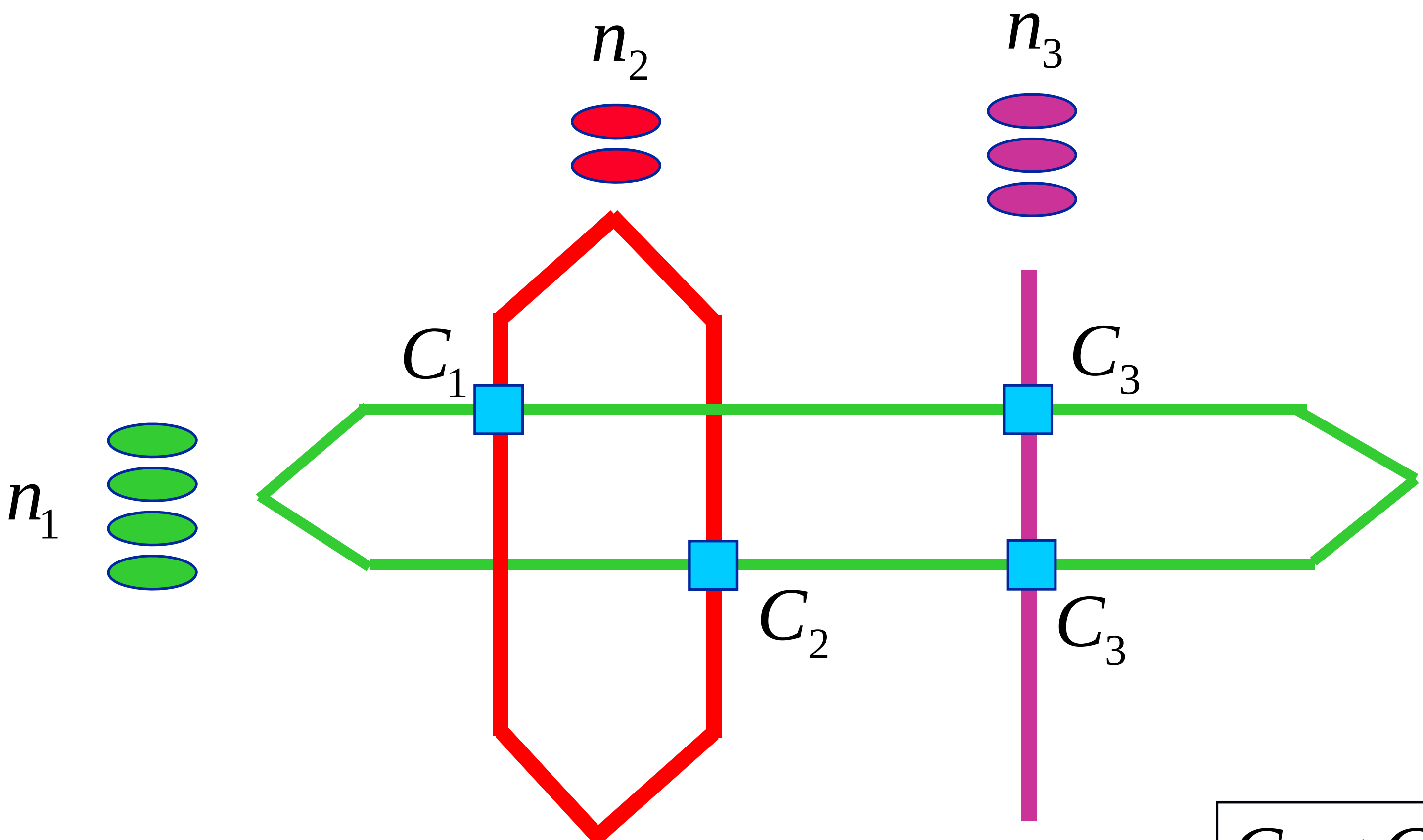
Suppose a source-destination pair has access to several routes across the network:



S - set of source-destination pairs

$r \in s$ - route r serves s-d pair s

Example of multipath routing

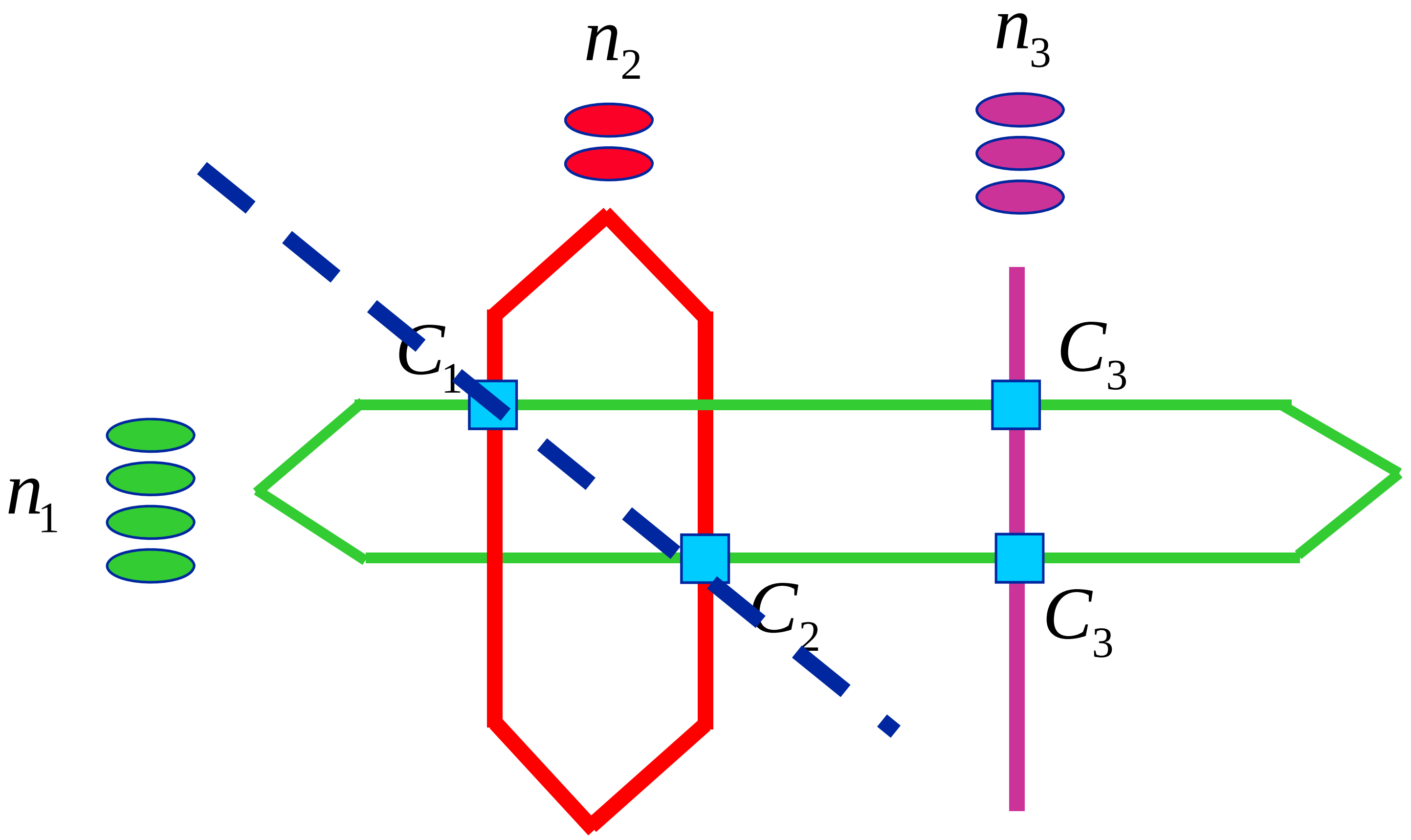


$$C_3 < C_1, C_2$$

Routes, as well as flow rates, are chosen to optimize

$\sum_s n_s \log(x_s)$ over source-destination pairs s

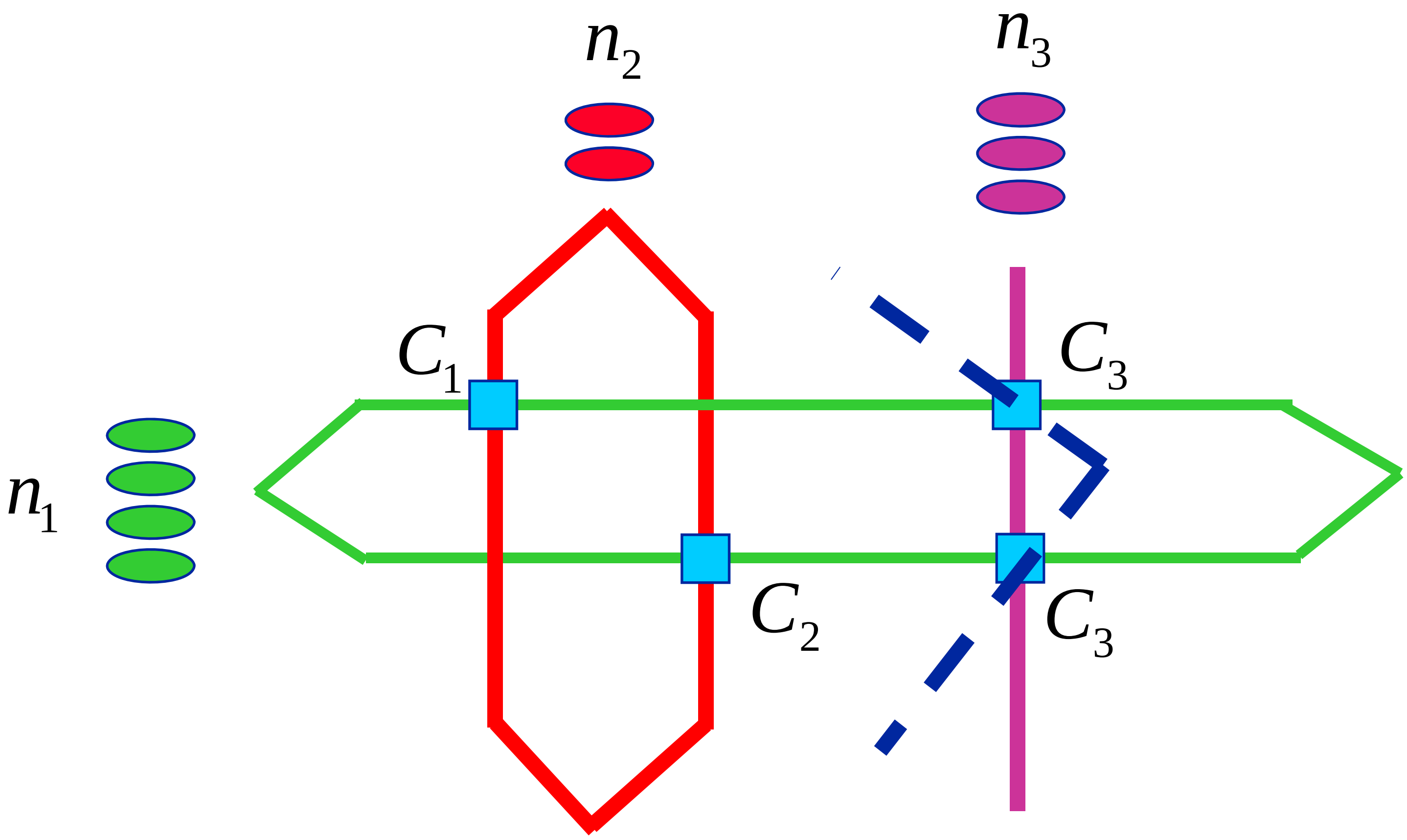
First cut constraint



$$n_1 x_1 + n_2 x_2 \leq C_1 + C_2$$

Cut defines a single *pooled resource*

Second cut constraint



$$\frac{1}{2} n_1 x_1 + n_3 x_3 \leq C_3$$

Cut defines a *second* pooled resource

Product form

$$\alpha = 1, \, w_r = 1, \;\; r \in R$$

In heavy traffic, and subject to some technical conditions, the (scaled) components of the shadow prices p for the pooled resources are independent and exponentially distributed. The corresponding approximation for n is

$$n_s \approx \rho_s \sum_j p_j A_{js} \qquad s \in S$$

where

$$p_j \sim \mathrm{Exp}(\overline{C}_j - \sum_s \overline{A}_{js} \rho_s) \qquad j \in \overline{J}$$

Dual random variables are independent and exponential

Kang, K, Lee and Williams 2009

The future?

- Many mathematical challenges, associated with the combination of network flow and the stochastic model of resource possession
- Applications to controlled motorways, router design, yield management.....