Mathematical Models of Rate Control for Communication Networks

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Preface

The work in Chapter 2, Sections 3.2.1, 3.2.2 and 3.3 was done jointly with Professor F.P. Kelly, Statistical Laboratory, University of Cambridge and A.K. Maulloo, University of Mauritius. The joint work has also been published in the *Journal of the Operational Research Society* [40]. What remains is the result of my own work and includes nothing which is done in collaboration.

This dissertation is not substantially the same as any that I have submitted for any degree or diploma or other qualification at any University.

I wish to thank my supervisor Professor F.P. Kelly for his boundless enthusiasm and invaluable guidance. Also, thanks to the financial support of St. John's College, Cambridge. Last but not least, I am most grateful to my wife Sharon, and daughters Kathryn and Carolyn, who have been the main driving force behind my work.

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Summary

We consider the problem of congestion control in large-scale broadband communication networks such as the Internet, studying how available bandwidth might be shared among competing streams of elastic traffic, and related issues of stability, stochastic effects and time lags.

We first study the stability and fairness of two classes of rate control algorithms, which may be interpreted in terms of either congestion indication signals or explicit rates based on shadow prices. The algorithms, which operate on a decentralized basis, provide natural generalizations to large-scale networks of simple linear increase/multiplicative decrease schemes, and are shown to be stable about a system optimum characterized by a proportional fairness criterion using Lyapunov methods. The model is then extended to include stochastic perturbations and time lags.

We next study various aspects of user behaviour; in particular, where users are able to adapt quickly to their experience of congestion, where there are routeing choices and where the user population changes due to arrivals and departures. With adaptive users, under the assumption that each user tracks the optimum to his own optimization problem, the stability and convergence of the system may be established. The basic model extends naturally to include routeing. As for the models with variable numbers of users, the general conclusion is that adaptive fixed duration users are, in a sense, 'good traffic'.

We then explore the issue of time lags and stability, studying how the gain parameters relate to the time lags, price/feedback functions and the topology of the network. The stability for a series of simple networks is analysed using both discrete time difference equations and continuous time retarded differential equations. In general, the results scale well with the size of the network as the time lags and number of resources grow. Important insight is obtained on the design of the price/feedback function at each resource. Also, they indicate why self-clocking algorithms are likely to work well to maintain stability.

The results are illustrated using various simulation examples.

Contents

Chapter 1

Introduction

1.1 The problem

Developments in telecommunications technology are leading towards networks which will allow a wide variety of traffic streams to share the same broadband channel. A call, which might be a mixture of voice, video and data, would appear to the network as a stream of cells, and the aim is that these calls may be efficiently integrated, to share common resources. In particular, there are expected to be applications that are able to modify their data transfer rates according to the available bandwidth within the network. Traffic from such applications is termed *elastic* [60]; a typical current example is TCP (Transmission Control Protocol) traffic over the Internet [33], and others may include the controlled-load service of the Internet Engineering Task Force [66] and the ABR (Available Bit Rate) transfer capability of ATM (asynchronous transfer mode) networks [63].

The key issue we address in this dissertation concerns how the available bandwidth within a network should be shared among competing streams of elastic traffic; in particular, we present a tractable mathematical model and use it to analyse the stability and fairness of two classes of rate control algorithms. Two important aspects of decentralized systems, namely stochastic perturbations and time lags, will also be considered. Traditionally, stability has been considered an engineering issue, requiring an analysis of randomness and feedback operating on fast time-scales, while fairness and pricing have been considered economic issues, involving static comparisons of utility. In future networks, the intelligence embedded in end-systems, acting on behalf of human users, is likely to lessen the distinction between engineering and economic issues and increase the importance of an interdisciplinary view (for further discussion, see [37]).

We have assumed in this dissertation that pricing may be used to control demand and hence congestion in the network, and do not intend to discuss in detail the pros and cons of pricing. In the context of rate control algorithms, if the term 'price' is not preferred, an alternative interpretation in terms of congestion feedback signals in the network is also possible. In [19], Floyd and Fall discussed the possibility that unresponsive flows could contribute to congestion collapse in the Internet, since it is no longer possible to rely on all end-nodes to use end-to-end congestion control, or on developers to incorporate end-to-end congestion control in their applications. Usage-sensitive pricing is generally accepted as the best way to discourage usage when there is congestion and also to generate revenue for expansion, though it is still regarded by some as incompatible with the goals of the Internet. Some argue that the load-dependent costs are marginal, but forget that the costs of congestion to other users are very significant. The usage-based approach provides the direct information required for allocating scarce resources during times of congestion to those users who value them most. MacKie-Mason and Varian [47] showed that flat-pricing could lead to either higher congestion or lower capacity at equilibrium, both of which are undesirable effects. This is already evident in the case of the Internet and Odlyzko [54] has observed the evidence of repressed demand from available traffic statistics. (for more discussion on pricing, see also the collection of papers in [52].)

Any discussion of the performance of a rate control scheme must address the issue of fairness, since there exist situations where a given scheme might maximize network throughput, for example, while denying access to some users. The most commonly discussed fairness criterion is that of max-min fairness [3]: loosely, a set of rates is max-min fair if no rate may be increased without simultaneously decreasing another rate which is already smaller. In a network with a single bottleneck resource max-min fairness implies an equal share of the resource for each flow through it. Mazumdar et al. [51] have pointed out that from a game-theoretic standpoint such an allocation is not special, and have advocated instead the Nash bargaining solution, from cooperative game theory, as capturing natural assumptions as to what constitutes fairness.

The need for networks to operate in a public (and therefore potentially non-cooperative) environment has stimulated work on charging schemes for broadband networks. (See [36] for a scheme based on time and volume measurements for non-elastic traffic, MacKie-Mason and Varian [46] for a description of a 'smart market' based on a per-packet charge when the network is congested. See also [24] and [23] where Gibbens and Kelly discuss a simple pricing mechanism which provides end-nodes with the necessary information and correct incentive to use a network efficiently.) In [39], Kelly proposed a model in which a user chooses the charge per unit time that he is willing to pay; thereafter the user's rate is determined by the network according to a proportional fairness criterion applied to the rate per unit charge. It was shown in [39] that a system optimum is achieved when users' choices of charges and the network's choice of allocated rates are in equilibrium. There remained the question of how the proportional fairness criterion could be implemented in a large-scale model. In this dissertation we show that simple rate control algorithms, using additive increase/multiplicative decrease rules or explicit rates based on resource shadow prices, can provide stable convergence to proportionally fair pricing.

Mechanisms by which supply and demand reach equilibrium have, of course, long been a central concern of economists, and there exists a substantial body of theory on the stability of what are termed tatonnement processes ([1],[28],[64]). From this viewpoint, the rate control algorithms described in this dissertation are particular embodiments of a 'Walrasian auctioneer', searching for market clearing prices. The 'Walrasian auctioneer' of tatonnement theory is usually considered a rather implausible construct; we show that the structured nature of a communication network provides a very natural context within which to investigate the consequences for a tatonnement process of stochastic perturbations and time lags.

The organization of the dissertation is as follows. In the next chapter we describe our basic model of a network, and analyse two classes of rate control algorithms in terms of stability, rates of convergence, stochastic perturbations and time lags. Chapter 3 discusses various aspects of user behaviour; in particular we show how the basic model extends to the situation where users are able to adapt to their experience of congestion and where the user population changes due to arrivals and departures. Chapter 4 explores the issue of time lags and stability in greater depth, studying how the gain parameters relate to the time lags, price/feedback functions and the topology of the network. Chapter 5 illustrates the theoretical results with various simulation examples. Finally we conclude with some remarks and open issues.

1.2 Related work

There is a substantial literature on rate control algorithms, recently reviewed by Hernandez-Valencia et al. [31]. Key early papers of Jacobson [33] and Chiu and Jain [10] identified the advantages of adaptive schemes that either increase flows linearly or decrease flows multiplicatively, depending on the absence or presence of congestion. Jacobson's scheme has been implemented in TCP, the transmission control protocol of the Internet. Important recent papers of Bolot and Shankar [6], Fendick et al. [18] and Bonomi et al. [7] have analysed the stability of networks with a single bottleneck resource, where congestion is signalled by the build-up of a queue at the bottleneck's buffer, and where propagation delays are significant. Bolot and Shankar [6] modelled one and two connections involving a single bottleneck, obtaining closed-form solutions for the one connection case in both transient and steady state regimes. They also discussed the need for a trade-off between good steady-state behaviour and rapid convergence to equilibrium. Fendick et al. [18] analysed a class of feedback schemes using a system of retarded differential equations. Bonomi et al. [7] discussed the importance of gain and damping parameters for stabilizing behaviour and achieving fairness.

The framework we adopt in this dissertation is simpler than that analysed by these authors in that we directly model only rates and not queue lengths, but more complex in that we model a network with an arbitrary number of bottleneck resources. Theoretical work on queues serving the superposition of a large number of streams ([16], [38]) indicates several circumstances when the busy period preceding a buffer overflow may be relatively short, and several authors have argued the advantages of preventing queue build-up through the bounding of rates (see, for example, Charny et al. [9]). We also model two important aspects of decentralized systems, namely stochastic perturbations and time lags. Both these factors provide important insights into the behaviour of the network, and allow us to quantify the various relationships and trade-offs between speed of convergence, the magnitude of fluctuations about the equilibrium point, and the stability of the network.

The rate control algorithms which will be discussed provide natural generalizations to large-scale networks of simple additive increase/multiplicative decrease schemes, and are shown to be stable about a system optimum characterized by a proportional fairness criterion [39]. Stability is established by showing that, with an appropriate formulation of the overall optimization problem, the network's implicit objective function provides a Lyapunov function for the dynamical system defined by the rate control algorithm. Similar approaches have been used in the work by Mo and Walrand [53], Low and Lapsley [44][43], and Golestani and Bhattacharyya [26]. Mo and Walrand studied the convergence of a fair end-to-end window-based congestion control protocol which generalizes proportional fairness and includes arbitrarily close approximations of max-min fairness. In the scheme proposed by Low and Lapsley, users select transmission rates that maximize their own benefits, utility minus bandwidth cost, and network resources adjust bandwidth prices to coordinate the users' decisions. For Golestani and Bhattacharyya, the goal is to minimize a cost function of user rates using a gradient type algorithm. Massoulié and Roberts [50] discusses the design of distributed algorithms for sharing network bandwidth by comparing the criteria of max-min fairness, proportional fairness and minimum potential delay.

We will consider various aspects of user behaviour; in particular, adaptive users, routeing and user arrivals/departures from the system. Work on queues with prices and adaptive users is not new. References may be found in the paper of Stidham [62], in which he considered the questions of optimal pricing and capacity for a service facility formulated as a single-server queueing system. Rump and Stidham [59] examined further the stability of such a service facility with adaptive customer response. A more closely related paper is that by Gibbens and Kelly [24], where they considered various scenarios where the user population comprises a mix of elastic, intermittent and file transfer types. By using a simple charging mechanism, they show that end-nodes are provided with the necessary information and correct incentive to use the network efficiently. Each user may use their own transmission control algorithm to achieve a different user objective. However, the arrival and departure process of users is not treated.

Massoulié and Roberts, in [49], model the arrival and departure process of users by assuming that the traffic handled by the network appears as a succession of requests for the immediate transfer of a document (which we shall refer to as file transfer users). By studying the throughput performance, they show that quality of service can be improved by discriminating in favour of flows transferring smaller documents. Their model differs from those in this dissertation in at least two ways. Firstly, the objective is to maximize throughput performance, not utilities. Secondly, the traffic is entirely of the file transfer type whereas we shall consider a mix of users in one of our models.

The relationship between time lags and stability is not a new subject. The theory of continuous time differential-difference or retarded differential equations may be found in Bellman and Cooke [2], Hale [29], Marshall et al. [48] and Stépán [61]. Modelling the system using retarded differential equations is the approach used in [11], [12], [13] and [18]. Fendick et al. [18] showed that a class of feedback schemes is stable in the sense that the rate of transmission and queue length are rapidly confined to a small neighbourhood of the designed operating point. In [11], Chong proposed a second-order rate-based flow control algorithm and analysed the stability of a single resource and multiple users with diverse delays. In [13], Chong et al. proposed a first-order rate-based flow control (FRFC) algorithm where the rate allocated to each user is the difference between the observed resource queue length and queue threshold, multiplied by a control gain. They also derived a sufficient condition for the stability of a single bottleneck resource with multiple connections of diverse round-trip delays. Chong et al. [12] then showed the rate of convergence of the FRFC algorithm to be a function of control gain with its maximum being the inverse of the round-trip delay.

In this dissertation, we will first formulate the problem as a discrete time system, deriving local stability conditions for a sequence of simple networks. Parallel theorems are then obtained via retarded differential equations. In general, the results scale well with the size of the network as the time delays and number of resources grow. Important insight is obtained on the design of the price/feedback function at each resource. Also, the results indicate why self-clocking (see [33] or [24]) rate control algorithms are likely to work well to maintain stability. We will also show how the rate of convergence of the algorithms depends on the gain parameter, with its maximum being the inverse of the round-trip delay, a similar result to that obtained by Chong et al. [12].

Chapter 2

Network optimization

In this chapter, we will describe our basic model of a network, and analyse two classes of rate control algorithms in terms of stability, rates of convergence, stochastic perturbations and time lags. The algorithms, which operate on a decentralized basis, provide natural generalizations to large-scale networks of simple additive increase/multiplicative decrease schemes, and are shown to be stable about a system optimum characterized by a proportional fairness criterion [39]. Stability is established by showing that, with an appropriate formulation of the overall optimization problem, the network's implicit objective function provides a Lyapunov function for the dynamical system defined by the rate control algorithm. The network's optimization problem may be cast in primal or dual form: this leads naturally to two classes of algorithms, which may be interpreted in terms of either congestion indication feedback signals or explicit rates based on shadow prices. The model is then extended to include two important aspects of decentralized systems, namely stochastic perturbations and time lags. Both classes of algorithms provide natural implementations of proportionally fair pricing.

2.1 The basic model

We consider a network with a set J of *resources*, and let C_j be the finite capacity of resource j, for $j \in J$. Let a *route* r be a non-empty subset of J, and R be the set of possible routes. Set $A_{jr} = 1$ if $j \in r$, that is, resource j lies on route r, and set $A_{jr} = 0$ otherwise. This defines a $0-1$ matrix $A = (A_{jr}, j \in J, r \in R)$.

Associate a route r with a user, and suppose that if a rate x_r is allocated to user r, this has utility $U_r(x_r)$ to the user. Throughout this dissertation we shall assume that the utility $U_r(x_r)$ is an increasing, strictly concave and continuously differentiable function of x_r over the range $x_r \geq 0$ (following Shenker [60], we call traffic that leads to such a utility function *elastic* traffic), with $\lim_{x\to 0} U_r'(x) = \infty$ and $\lim_{x\to\infty} U_r'(x) = 0$. Assuming further that utilities are additive, the aggregate utility of rates $x = (x_r, r \in R)$ is $\sum_{r\in R} U_r(x_r)$. Let $U = (U_r(\cdot), r \in R)$ and $C = (C_j, j \in J)$. Under this model the system optimal rates solve the following problem.

$SYSTEM(U, A, C)$:

$$\begin{array}{ll} \text{maximize} & \sum_{r\in R} U_r(x_r) \\ \text{subject to} & Ax \leq C \\ \text{over} & x \geq 0. \end{array}$$

The above optimization problem is mathematically fairly tractable (with a strictly concave objective function and a convex feasible region) but it involves utilities U that are unlikely to be known by the network. Hence, in [39], Kelly proposed a decomposition into two simpler problems.

Now suppose user r may choose an amount to pay per unit time, w_r, and receives in return a rate x_r proportional to w_r, say $x_r = w_r/\lambda_r$. Then the utility maximization problem for user r is as follows.

$USER_r(U_r;\lambda_r)$:

$$\begin{array}{ll} \text{maximize} & U_r\left(\frac{w_r}{\lambda_r}\right) - w_r \\ \text{over} & w_r \geq 0. \end{array}$$

Suppose next that the network knows the vector $w = (w_r, r \in R)$, and attempts to maximize the logarithmic utility function $\sum_{r\in R} w_r \log x_r$. The network's optimization problem is then as follows.

$NETWORK(A, C; w)$:

$$\begin{array}{ll} \text{maximize} & \sum_{r\in R} w_r \log x_r \\ \text{subject to} & Ax \leq C \\ \text{over} & x \geq 0. \end{array}$$

In [39], Kelly showed that there exist vectors $\lambda = (\lambda_r, r \in R)$, $w = (w_r, r \in R)$ and $x = (x_r, r \in R)$, satisfying $w_r = \lambda_r x_r$ for $r \in R$, such that w_r solves $USER_r(U_r;\lambda_r)$ for $r \in R$ and x solves $NETWORK(A, C; w)$; further, the vector x is then the unique solution to $SYSTEM(U, A, C)$.

A vector of rates $x = (x_r, r \in R)$ is *proportionally fair* if it is feasible, that is $x \geq 0$ and $Ax \leq C$, and if for any other feasible vector x^*, the aggregate of proportional changes is either zero or negative:

$$\sum_{r\in R} \frac{x_r^* - x_r}{x_r} \leq 0. \tag{2.1}$$

If $w_r = 1, r \in R$, then a vector of rates x solves $NETWORK(A, C; w)$ if and only if it is proportionally fair. Such a vector is also the Nash bargaining solution (satisfying certain axioms of fairness [22]), and, as such, has been advocated in the context of telecommunications by Mazumdar et al. [51].

A vector x is such that the *rates per unit charge* are proportionally fair if x is feasible, and if for any other feasible vector x^*

$$\sum_{r\in R} w_r \frac{x_r^* - x_r}{x_r} \leq 0. \tag{2.2}$$

The relationship between the conditions (2.1) and (2.2) is well illustrated when $w_r, r \in R$, are all integral. For each $r \in R$, replace the single user r by w_r identical sub-users, construct the proportionally fair allocation over the resulting $\sum_r w_r$ users, and provide to user r the aggregate rate allocated to its w_r sub-users; then the resulting rates *per unit charge* are proportionally fair. This construction also illustrates the need to adapt the notion of fairness to a non-cooperative context, where it is possible for a single user to represent itself as several distinct users. It is straightforward to check [39] that a vector of rates x solves $NETWORK(A, C; w)$ if and only if the rates per unit charge are proportionally fair.

We note in passing that if, for a fixed set of users and arbitrary parameters $w = (w_r, r \in R)$, the network solves $NETWORK(A, C; w)$, then the resulting rates $x = (x_r, r \in R)$ solve a variant of $SYSTEM(U, A, C)$, with a weighted objective function $\sum_{r \in R} \alpha_r U_r(x_r)$ where $\alpha_r = w_r/(x_r U_r'(x_r))$ for $r \in R$. Thus a choice of the parameters $w = (w_r, r \in R)$ by the network (instead of the users) corresponds to an implicit weighting of the relative utilities of different users, with weights related to the users' various marginal utilities.

With the above decomposition, the utility function $U_r(x_r)$ is not required by the network and only appears in the optimization problem faced by user r. The Lagrangian [65] for the problem $NETWORK(A, C; w)$ is

$$L(x, z; \mu) = \sum_{r \in R} w_r \log x_r + \mu^T (C - Ax - z)$$

where $z \geq 0$ is a vector of slack variables and μ is a vector of Lagrange multipliers (or shadow prices). Then

$$\frac{\partial L}{\partial x_r} = \frac{w_r}{x_r} - \sum_{j \in J} A_{jr} \mu_j,$$

and so the unique optimum to the primal problem is given by

$$x_r = \frac{w_r}{\sum_{j \in J} A_{jr} \mu_j} \tag{2.3}$$

where $(x_r, r \in R)$, $(\mu_j, j \in J)$ solve

$$\mu \geq 0, \quad Ax \leq C, \quad \mu^T (C - Ax) = 0 \tag{2.4}$$

and relation (2.3).

We may derive the associated dual problem as follows. Maximizing the Lagrangian over x and z gives

$$\begin{aligned}
L(x, z; \mu) &= \sum_{r \in R} w_r \log\left(\frac{w_r}{\sum_{j \in J} A_{jr} \mu_j}\right) + \mu^T C - \mu^T A x \\
&= \sum_{r \in R} w_r \log w_r - \sum_{r \in R} w_r \log\left(\sum_{j \in J} A_{jr} \mu_j\right) + \sum_{j \in J} \mu_j C_j - \sum_{r \in R} w_r
\end{aligned}$$

After removing the terms not dependent on the shadow prices μ, and converting it to a maximization problem, we obtain the dual.

$DUAL(A, C; w)$:

$$\begin{aligned}
&\text{maximize} \quad \sum_{r \in R} w_r \log\left(\sum_{j \in J} A_{jr} \mu_j\right) - \sum_{j \in J} \mu_j C_j \\
&\text{over} \quad \mu \geq 0.
\end{aligned}$$

While the problems $NETWORK(A, C; w)$ and $DUAL(A, C; w)$ are mathematically tractable, it would be difficult to implement a solution in any centralized manner. A centralized processor, even if it were itself completely reliable and could cope with the complexity of the computational task involved, would have its lines of communication through the network vulnerable to delays and failures. Rather, interest focusses on algorithms which are decentralized and of a simple form: the challenge is to understand how such algorithms can be designed so that the network as a whole reacts intelligently to perturbations.

The network optimization problem $NETWORK(A, C; w)$ has an objective function which contains no penalties but has the capacity constraints $Ax \leq C$. A more general optimization problem could be obtained if the capacity constraints are replaced by penalties or costs, expressed perhaps in terms of delay or loss, that increase as the capacity of a resource is approached. In the context of a dynamic pricing model for a communication network with stochastic effects and time lags, this could be regarded as more appropriate. This motivates the following problem which may be regarded as a generalization or relaxation of $NETWORK(A, C; w)$:

$NETWORK^*(A, P; w)$:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{r\in R} w_r \log x_r - \sum_{j\in J} \int_0^{y_j} p_j(z)dz \\ \text{subject to} & Ax = y \\ \text{over} & x, y \geq 0. \end{array}$$

where $P = (p_j(\cdot), j \in J)$ and the function $p_j(z)$, $z \geq 0$, for $j \in J$, is a non-negative, continuous, increasing function of z, not identically zero.

We interpret the revised objective function as follows. Suppose that when a resource is heavily loaded the network incurs some cost which may be expressed in terms of a penalty function

$$c_j(y) = \int_0^y p_j(z)dz \tag{2.5}$$

that describes the rate at which cost is incurred at resource j when the load through it is y. Then the appropriate network optimization problem would be $NETWORK^*(A, P; w)$.

Consider the following example. Suppose that the total load y on a resource is Poisson and the time-axis is divided into non-overlapping slots each of length τ. Let the rate at which cost is incurred at resource j be

$$c_j(y) = \frac{1}{\tau} \sum_{n>N} (n - N) e^{-y\tau} \frac{(y\tau)^n}{n!}$$

when the load is y, that is, the expected number of cells per unit time that exceed a threshold N. Then a simple differentiation establishes that $p_j(y)$, determined by equation (2.5), is given by

$$p_j(y) = \sum_{n\geq N} e^{-y\tau} \frac{(y\tau)^n}{n!}, \tag{2.6}$$

Suppose we define the continuous functions

$$p_j(y) = (y - C_j + \varepsilon)^+ / \varepsilon^2$$

for $j \in J$. Then, as $\varepsilon \to 0$, the problem $NETWORK^*(A, P; w)$ approximates arbitrarily closely the original primal problem $NETWORK(A, C; w)$; in particular the vector x maximizing $NETWORK^*(A, P; w)$ approaches the solution x to relations (2.3) and (2.4). Note, however, that the derivative $p_j'(y)$ may become arbitrarily large as the approximation improves.

We proceed to derive the dual of $NETWORK^*(A, P; w)$. The Lagrangian for the problem is

$$L(x, y; \mu) = \sum_{r \in R} w_r \log x_r - \sum_{j \in J} \int_0^{y_j} p_j(z) dz - \mu^T (Ax - y)$$

Now

$$\begin{aligned} \frac{\partial L}{\partial x_r} &= \frac{w_r}{x_r} - \sum_{j \in J} A_{jr} \mu_j \\ \frac{\partial L}{\partial y_j} &= -p_j(y_j) + \mu_j \end{aligned}$$

Setting these derivatives to zero identifies the maximum. Substituting, we obtain

$$\begin{aligned} L(x, y; \mu) &= \sum_{r \in R} w_r \log\left(\frac{w_r}{\sum_{j \in J} A_{jr} \mu_j}\right) - \sum_{r \in R} w_r \\ &\quad + \sum_{j \in J} \left(\mu_j p_j^{-1}(\mu_j) - \int_0^{p_j^{-1}(\mu_j)} p_j(z) dz \right) \end{aligned}$$

where $\mu \geq 0$. The last term is just

$$\sum_{j \in J} \int_0^{\mu_j} q_j(\eta) d\eta$$

where the function $q_j(\cdot)$ is the inverse of $p_j(\cdot)$. Hence, after removing the terms not dependent on μ, we obtain the revised dual problem,

$DUAL^*(A, Q; w)$:

$$\begin{array}{ll} \text{maximize} & \sum_{r \in R} w_r \log\left(\sum_{j \in J} A_{jr} \mu_j\right) - \sum_{j \in J} \int_0^{\mu_j} q_j(\eta) d\eta \\ \text{over} & \mu \geq 0. \end{array}$$

where $Q = (q_j(\cdot), j \in J)$.

2.2 Rate control

In this section we discuss two classes of rate control algorithms for communication networks, designed to implement solutions to the optimization problems $NETWORK^*(A, P; w)$ and $DUAL^*(A, Q; w)$. The algorithms provide natural generalizations to large-scale networks of simple additive increase/multiplicative decrease schemes.

2.2.1 A primal algorithm

Consider the system of differential equations

$$\frac{d}{dt} x_r(t) = \kappa\Big(w_r - x_r(t) \sum_{j \in J} A_{jr} \mu_j(t)\Big) \tag{2.7}$$

where

$$\mu_j(t) = p_j\Big(\sum_{s \in R} A_{js} x_s(t)\Big). \tag{2.8}$$

(Here and throughout, unless otherwise mentioned, r ranges over the set R and j over the set J. Also $\kappa > 0$.) We may motivate the relations (2.7)–(2.8) in several ways. For example, suppose that $p_j(y)$ is a price charged by resource j, per unit flow through resource j, when the total load on resource j is $y = \sum_{s \in R} A_{js} x_s$. Then by adjusting the flow on route r, $x_r(t)$, in accordance with equations (2.7)–(2.8), the network attempts to equalize the aggregate cost of this flow, $x_r(t) \sum_{j \in J} A_{jr} \mu_j(t)$, with a target value w_r, for every $r \in R$. (For an enlightening description of the technological implementation of such algorithms in an ATM network, see [14]. See [17] on how proportional fairness could be used to implement an Internet with differentiated services.)

For an alternative motivation, suppose that resource j generates a continuous stream of feedback signals at rate $p_j(y)$ when the total load on resource j is $y = \sum_{s \in R} A_{js} x_s$. Suppose further that when resource j generates a feedback signal, a copy is sent to each user r whose route passes through resource j, where it is interpreted as a congestion indicator requiring some reduction in the flow x_r. Then equation (2.7) corresponds to a response by user r that comprises two components: a steady increase at rate proportional to w_r, and a multiplicative decrease at rate proportional to the stream of feedback signals received. (For early discussions of algorithms with additive increase and multiplicative decrease see [10] and [33]; [31] reviews several algorithms based on congestion indication feedback.)

We want to establish the global or asymptotic stability of the primal algorithm. Define the function

$$\mathcal{U}(x) = \sum_{r\in R} w_r \log x_r - \sum_{j\in J} \int_0^{\sum_{s\in R} A_{js}x_s} p_j(z)dz \tag{2.9}$$

where $w_r > 0$, $r \in R$, and suppose that, for $j \in J$, the function $p_j(z)$, $z \geq 0$, is a non-negative, continuous, increasing function of z, not identically zero.

Theorem 1 *The strictly concave function $\mathcal{U}(x)$ given by (2.9) is a Lyapunov function for the system of differential equations (2.7)–(2.8). The unique value x maximizing $\mathcal{U}(x)$ is a stable point of the system, to which all trajectories converge.*

Proof. The assumptions on $w_r > 0, r \in R$ and $p_j(\cdot), j \in J$, ensure that $\mathcal{U}(x)$ is strictly concave on $x \geq 0$ with an interior maximum; the maximizing value of x is thus unique. By Theorem 2.11 in [25], it suffices to show that $\dot{\mathcal{U}}(x) = 0$ at the maximum only and $\dot{\mathcal{U}}(x) > 0$ for all other x. Observe that

$$\frac{\partial}{\partial x_r}\mathcal{U}(x) = \frac{w_r}{x_r} - \sum_{j\in J} A_{jr} p_j(\sum_{s\in R} A_{js}x_s); \tag{2.10}$$

setting these derivatives to zero identifies the maximum. Further

$$\begin{aligned} \frac{d}{dt}\mathcal{U}\left(x(t)\right) &= \sum_{r\in R} \frac{\partial \mathcal{U}}{\partial x_r} \cdot \frac{dx_r}{dt} \\ &= \kappa \sum_{r\in R} \frac{1}{x_r(t)} \Big(w_r - x_r(t) \sum_{j\in J} A_{jr} p_j(\sum_{s\in R} A_{js}x_s(t))\Big)^2, \end{aligned}$$

establishing that $\mathcal{U}(x(t))$ is strictly increasing with t, unless $x(t) = x$, the unique x maximizing $\mathcal{U}(x)$. The function $\mathcal{U}(x)$ is thus a Lyapunov function for the system (2.7)–(2.8), and the theorem follows ([25], Chapter 2). □

We note that the theorem does not rely upon the property that p_j, $j \in J$, are increasing functions, but only upon the weaker property that $\mathcal{U}(x)$ is strictly concave with an interior maximum.

2.2.2 A dual algorithm

The equations (2.7)–(2.8) represents a system where rates vary gradually, and shadow prices are given as functions of the rates. Next we consider a system where shadow prices vary gradually, with rates given as functions of the shadow prices. Let

$$\frac{d}{dt}\mu_j(t) = \kappa\left(\sum_{r\in R} A_{jr}x_r(t) - q_j\left(\mu_j(t)\right)\right) \tag{2.11}$$

where

$$x_r(t) = \frac{w_r}{\sum_{k\in J} A_{kr}\mu_k(t)}. \tag{2.12}$$

The relationship between the algorithm (2.11)–(2.12) and the dual problem $DUAL^*(A, Q; w)$ parallels that between the primal algorithm (2.7)–(2.8) and the problem $NETWORK^*(A, P; w)$, and, again, we may motivate the algorithm in several ways. For example, suppose that $q_j(\eta)$ is the flow through resource j which generates a price at resource j of η. Then an economist would describe the right hand side of equation (2.11) as the vector of excess demand at prices $(\mu_j(t), j \in J)$, and would recognise equations (2.11)–(2.12) as a tatonnement process by which prices adjust according to supply and demand ([64], Chapter 21).

The global stability of the dual algorithm may be established using a similar approach. Define the function

$$\mathcal{V}(\mu) = \sum_{r\in R} w_r \log\left(\sum_{j\in J} A_{jr}\mu_j\right) - \sum_{j\in J}\int_0^{\mu_j} q_j(\eta)d\eta \tag{2.13}$$

where $w_r > 0$, $r \in R$, and suppose that, for $j \in J, q_j(0) = 0$ and $q_j(\eta), \eta \geq 0$, is a continuous, strictly increasing function of η.

Theorem 2 *The strictly concave function $\mathcal{V}(\mu)$ given by (2.13) is a Lyapunov function for the system of differential equations (2.11)–(2.12). The unique value μ maximizing $\mathcal{V}(\mu)$ is a stable point of the system, to which all trajectories converge.*

Proof. The assumptions on $w_r > 0, r \in R$ and on $q_j(\cdot), j \in J$, ensure that $\mathcal{V}(\mu)$ is strictly concave on $\mu \geq 0$ with an interior maximum; the maximizing value of μ is thus unique, and is determined by setting the derivatives

$$\frac{\partial}{\partial \mu_j} \mathcal{V}(\mu) = \sum_{r \in R} \frac{A_{jr} w_r}{\sum_{k \in J} A_{kr} \mu_k} - q_j(\mu_j) \tag{2.14}$$

to zero. Also,

$$\begin{aligned} \frac{d}{dt} \mathcal{V}\left(\mu(t)\right) &= \sum_{j \in J} \frac{\partial \mathcal{V}}{\partial \mu_j} \cdot \frac{d}{dt} \mu_j(t) \\ &= \kappa \sum_{j \in J} \left(\sum_{r \in R} \frac{A_{jr} w_r}{\sum_{k \in J} A_{kr} \mu_k(t)} - q_j\left(\mu_j(t)\right) \right)^2, \end{aligned}$$

establishing that $\mathcal{V}(\mu(t))$ is strictly increasing with t, unless $\mu(t) = \mu$, the unique value μ maximizing $\mathcal{V}(\mu)$. The function $\mathcal{V}(\mu)$ is thus a Lyapunov function for the system (2.11)–(2.12), and the theorem follows ([25], Chapter 2). □

The maximization of the Lyapunov function $\mathcal{V}(\mu)$ becomes the original dual problem $DUAL(A, C; w)$ if, for $j \in J$, $\eta > 0$, $q_j(\eta) = C_j$. These functions violate the assumption that $q_j(\eta)$ is continuous at $\eta = 0$, but they may be arbitrarily closely approximated, for example by the functions

$$q_j(\eta) = \frac{C_j \eta}{\eta + \varepsilon}$$

for small positive ε. Note, however, that the derivative $q_j'(\eta)$ may become arbitrarily large as the approximation improves.

The Lyapunov approach thus provides an enlightening analysis of the global stability of the primal and dual algorithms. However, the model has omitted two important aspects of decentralized systems, namely stochastic perturbations, and time lags.

2.3 Rate of convergence and stochastic analysis

In this section we determine the rate of convergence, and, by considering perturbations about the stable point, model stochastic effects of the system.

2.3.1 Rate of convergence - primal

We have seen, in Theorem 1, that the system (2.7)–(2.8) converges to a unique stable point. We now investigate the rate of convergence, by linearization about the stable point.

Let x identify the unique vector maximizing $\mathcal{U}(x)$, let $\mu_j = p_j(\sum_{s\in R} A_{js}x_s)$, and suppose p_j is differentiable at this point, with derivative p_j'. Let $x_r(t) = x_r + x_r^{1/2} y_r(t)$. Then, linearizing the system (2.7)–(2.8) about x, we obtain

$$\begin{aligned}\frac{d}{dt} y_r(t) &= -\kappa\left(y_r(t)\sum_{j\in J} A_{jr}\mu_j + x_r^{1/2}\sum_{j\in J} A_{jr}p_j' \sum_{s\in R} A_{js}x_s^{1/2}y_s(t)\right)\\ &= -\kappa\left(\frac{w_r}{x_r} y_r(t) + x_r^{1/2}\sum_{j\in J}\sum_{s\in R} p_j' A_{jr}A_{js}x_s^{1/2}y_s(t)\right).\end{aligned}$$

We may write this in matrix form as

$$\frac{d}{dt} y(t) = -\kappa(WX^{-1} + X^{1/2}A^T P' A X^{1/2})y(t) \tag{2.15}$$

where $y(t) = (y_r(t), r \in R)$, $X = \mathrm{diag}(x_r, r \in R)$, $W = \mathrm{diag}(w_r, r \in R)$ and $P' = \mathrm{diag}(p_j', j \in J)$.

Let

$$\Gamma^T \Phi \Gamma = WX^{-1} + X^{1/2}A^T P' A X^{1/2} \tag{2.16}$$

where Γ is an orthogonal matrix, $\Gamma^T\Gamma = I$, and $\Phi = \mathrm{diag}(\phi_r, r \in R)$ is the matrix of eigenvalues, necessarily positive, of the real, symmetric, positive definite matrix (2.16). Then

$$\frac{d}{dt} y(t) = -\kappa \Gamma^T \Phi \Gamma y(t), \tag{2.17}$$

and thus the rate of convergence to the stable point is determined by the smallest eigenvalue, $\phi_r, r \in R$, of the matrix (2.16). Note that speed of convergence increases both with the gain parameter κ and with the magnitude of the derivatives P'; we shall see that this conclusion requires qualification in the presence of either stochastic effects or time-lags.

Our early assumption that $p_j(z)$, $j \in J$, are increasing functions is convenient and often natural: it implies that $\mathcal{U}(x)$ is strictly concave with an interior maximum. If it is not required that the functions $p_j(z)$, $j \in J$, be increasing then there exist examples where $\mathcal{U}(x)$ does not have an interior maximum and examples where stationary points of $\mathcal{U}(x)$ are not unique. Provided only $p_j(z)$ is differentiable at a stationary point, the local behaviour near the stationary point is described by the equation (2.17).

2.3.2 Rate of convergence - dual

We proceed similarly for the dual algorithm. Let μ identify the unique vector maximizing $\mathcal{V}(\mu)$, let $x_r = w_r / \sum_{k \in J} A_{kr}\mu_k$, and suppose $q_j(\eta)$ is differentiable at the point $\eta = \mu_j$, with derivative q_j'. Let $\mu_j(t) = \mu_j + \xi_j(t)$. Then, linearizing the system (2.11)–(2.12) about μ, we obtain

$$\begin{aligned}
\frac{d}{dt}\xi_j(t) &= \kappa\left(\sum_{r\in R} A_{jr}w_r\Big(\sum_{k\in J} A_{kr}(\mu_k+\xi_k(t))\Big)^{-1} - \Big(q_j + q_j'(\mu_j)\xi_j(t)\Big)\right)\\
&= \kappa\left(\sum_{r\in R} A_{jr}x_r\Big(1 - \frac{\sum_{k\in J}A_{kr}\xi_k(t)}{\sum_{k\in J}A_{kr}\mu_k}\Big) - \Big(q_j + q_j'(\mu_j)\xi_j(t)\Big)\right)\\
&= -\kappa\left(\sum_{r\in R} A_{jr}x_r^2 w_r^{-1}\sum_{k\in J}A_{kr}\xi_k(t) + q_j'(\mu_j)\xi_j(t)\right).
\end{aligned}$$

We may write this in matrix form as

$$\frac{d}{dt}\,\xi(t) = -\kappa(AXW^{-1}XA^T + Q')\xi(t)$$

where $\xi(t) = (\xi_j(t), j \in J)$, $X = \mathrm{diag}(x_r, r \in R)$, $W = \mathrm{diag}(w_r, r \in R)$ and $Q' = \mathrm{diag}(q_j', j \in J)$. Let

$$\Theta^T\Psi\Theta = AXW^{-1}XA^T + Q' \tag{2.18}$$

where Θ is an orthogonal matrix, $\Theta^T\Theta = I$, and $\Psi = \mathrm{diag}(\psi_j, j \in J)$ is the matrix of eigenvalues, necessarily non-negative, of the real, symmetric, positive semi-definite matrix (2.18). Then

$$\frac{d}{dt}\,\xi(t) = -\kappa\Theta^T\Psi\Theta\xi(t), \tag{2.19}$$

and thus the rate of convergence to the stable point is determined by the smallest eigenvalue of the matrix (2.18). Note that speed of convergence increases both with the gain parameter κ and with the magnitude of the derivatives Q'. Similarly, this conclusion requires qualification in the presence of either stochastic effects or time-lags.

2.3.3 Stochastic analysis - primal

Stochastic perturbations within the network may well arise from a resource's method of sensing its load. Equation (2.8) represents the response $\mu_j(t)$ of resource j as a continuous function of a load, $y = \sum_{s\in R} A_{js}x_s$, which is assumed known. In practice, a resource may assess its load by an error-prone measurement mechanism, and then choose a feedback signal from a small set of possible signals. (See [7] and [31] for more detailed descriptions of binary feedback and congestion indication rate control algorithms.)

We consider a stochastic perturbation of the linearized equation (2.17). Let

$$dy(t) = -\kappa \left(\Gamma^T \Phi \Gamma y(t)dt + F dB(t)\right) \tag{2.20}$$

where F is an arbitrary $|R| \times |I|$ matrix and $B(t) = (B_i(t), i \in I)$ is a collection of independent standard Brownian motions, extended to $-\infty < t < \infty$. (For a reference on stochastic processes, see [35]. Later we describe how the modelling of different sources of randomness may lead to various explicit forms for the matrix F.)

The stationary solution to the system (2.20) is

$$\begin{aligned} y(t) &= -\kappa \int_{-\infty}^{t} e^{-\kappa(t-\tau)\Gamma^T\Phi\Gamma} F dB(\tau) \\ &= -\kappa \int_{-\infty}^{0} e^{\kappa\tau\Gamma^T\Phi\Gamma} F dB(\tau) \end{aligned} \tag{2.21}$$

by substituting τ for $(\tau - t)$. We can check by differentiating both sides of equation (2.21) with respect to t, which gives

$$dy(t) = -\kappa\Gamma^T\Phi\Gamma y(t)dt - \kappa e^{-\kappa(t-t)\Gamma^T\Phi\Gamma} F dB(t)$$

as required. The solution (2.21) is a linear transformation of the Gaussian process $(B(\tau), \tau < t)$; hence $y(t)$ has a multivariate normal distribution, $y(t) \sim N(0, \Sigma)$, where

$$\begin{aligned}\Sigma &= \mathbb{E}\left[y(t)y(t)^T\right] \\ &= \kappa^2 \int_{-\infty}^{0} e^{\kappa\tau\Gamma^T\Phi\Gamma} F F^T e^{\kappa\tau\Gamma^T\Phi\Gamma} d\tau\end{aligned}$$

since the $B_j(t)$'s are independent. Now

$$\begin{aligned}e^{\kappa\tau\Gamma^T\Phi\Gamma} &= I + \kappa\tau\Gamma^T\Phi\Gamma + \frac{\kappa^2\tau^2}{2!}\Gamma^T\Phi^2\Gamma + \ldots \\ &= \Gamma^T e^{\kappa\tau\Phi}\Gamma\end{aligned}$$

so substituting $\tau = \kappa\tau$, we obtain

$$\Sigma = \kappa\Gamma^T \left[\int_{-\infty}^{0} e^{\tau\Phi}\Gamma F F^T \Gamma^T e^{\tau\Phi} d\tau\right] \Gamma.$$

But $e^{\tau\Phi} = \mathrm{diag}(e^{\tau\phi_i})$, where ϕ_i are the diagonal elements of the diagonal matrix Φ. Define the symmetric matrix $[\Gamma F; \Phi]$ by

$$\begin{aligned}[\Gamma F; \Phi]_{rs} &= \left[\int_{-\infty}^{0} e^{\tau\Phi}\Gamma F F^T \Gamma^T e^{\tau\Phi} d\tau\right]_{rs} \\ &= \int_{-\infty}^{0} e^{\tau\phi_r} \left[\Gamma F F^T \Gamma^T\right]_{rs} e^{\tau\phi_s} d\tau \\ &= \frac{[\Gamma F F^T \Gamma^T]_{rs}}{\phi_r + \phi_s}.\end{aligned}$$

Then

$$\Sigma = \kappa\Gamma^T[\Gamma F; \Phi]\Gamma. \tag{2.22}$$

Note that the covariance matrix (2.22) increases linearly with the gain parameter κ; as κ increases, the faster convergence to equilibrium described by relation (2.17) is at the cost of a greater spread at equilibrium. Varying the derivatives P' has a more subtle effect, through relation (2.16) and the construction (2.22), on the covariance matrix; broadly, as P' increases, not only is convergence to equilibrium faster, but also the spread at equilibrium decreases. However, we shall see later that, in the presence of time-lags, increasing P' may compromise stability.

We next illustrate how various sources of randomness may lead to different covariance structures.

Congestion indication with joint feedback. Consider the following stochastic version of the system (2.7)–(2.8). Let $(N_j(\tau), \tau \geq 0)$, for $j \in J$, be a collection of independent unit rate Poisson processes, and let

$$dx_r(t) = \kappa \left(w_r dt - x_r(t) \sum_{j \in J} A_{jr} \varepsilon_j dN_j \left(\varepsilon_j^{-1} \int_0^t \mu_j(\tau) d\tau \right) \right), \tag{2.23}$$

where the functions $\mu_j(\cdot)$, for $j \in J$, are given by equation (2.8). Equation (2.23) would describe the following model: resource j generates feedback signals indicating congestion as a time-dependent Poisson process at rate $\varepsilon_j^{-1}\mu_j(t)$; when resource j generates a feedback signal, a copy is sent to each user r whose route passes through resource j; and user r reacts to such a feedback signal by reducing $x_r(t)$ by an amount $\kappa\varepsilon_j x_r(t)$.

Now as $\varepsilon \to 0$, the normalized Poisson process $((\varepsilon N_j(\tau/\varepsilon) - \tau)\varepsilon^{-1/2}, \tau \geq 0)$ converges in distribution to a standard Brownian motion [4]. This motivates the approximation, valid when ε_j are small, of the Poisson driving equation (2.23) by its Brownian version

$$dx_r(t) = \kappa \left(w_r dt - x_r(t) \sum_{j \in J} A_{jr} \left(\mu_j(t) dt + \varepsilon_j^{1/2} \mu_j(t)^{1/2} dB_j(t) \right) \right) \tag{2.24}$$

where $(B_j(t), t \geq 0)$, for $j \in J$, are a collection of independent standard Brownian motions.

The corresponding Brownian version of the linearized system (2.17) is just equation (2.20) where $B(t) = (B_j(t), j \in J)$, and F is an $|R| \times |J|$ matrix with elements

$$F_{rj} = \varepsilon_j^{1/2} \mu_j^{1/2} A_{jr} x_r^{1/2}. \tag{2.25}$$

Then

$$FF^T = X^{1/2} A^T EPA X^{1/2}, \tag{2.26}$$

where $E = \text{diag}(\varepsilon_j, j \in J)$ and $P = \text{diag}(\mu_j, j \in J)$, and hence the stationary covariance matrix Σ may be calculated from expression (2.22).

Congestion indication with individual feedback. Consider next the Poisson driving equation

$$dx_r(t) = \kappa \left(w_r dt - \sum_{j \in J} A_{jr} \varepsilon_j dN_{jr} \left(\varepsilon_j^{-1} \int_0^t x_r(\tau) \mu_j(\tau) d\tau \right) \right) \tag{2.27}$$

where $(N_{jr}(\tau), \tau \geq 0)$, for $j \in J, r \in R$, are a collection of independent unit rate Poisson processes. This would describe the following model: feedback signals from resource j to user r arise at rate $\varepsilon_j^{-1} x_r(t) \mu_j(t)$; and user r reacts to such a feedback signal by reducing $x_r(t)$ by an amount $\kappa \varepsilon_j$. The Brownian approximation, valid when ε_j are small, becomes

$$dx_r(t) = \kappa \left(w_r dt - x_r(t) \sum_{j \in J} A_{jr} \left(\mu_j(t) dt + \varepsilon_j^{1/2} x_r(t)^{-1/2} \mu_j(t)^{1/2} dB_{jr}(t) \right) \right),$$

whose linearization is equation (2.20) where the $|R| \times |J|.|R|$ matrix F is given by

$$F_{r,(j,s)} = \varepsilon_j^{1/2} \mu_j^{1/2} A_{jr} I[r = s]. \tag{2.28}$$

Thus FF^T is the matrix $\mathrm{diag}(\sum_{j \in J} A_{jr} \mu_j \varepsilon_j, r \in R)$, and the stationary covariance matrix Σ may be calculated from expression (2.22).

Source fluctuations. Consider the Brownian driving equation

$$dx_r(t) = \kappa \left(w_r dt - x_r(t) \sum_{j \in J} A_{jr} \mu_j(t) dt + \varepsilon_r^{1/2} x_r(t)^{1/2} dB_r(t) \right)$$

which might correspond to fluctuations arising at sources, rather than within the network. For this system the $|R| \times |R|$ matrix F is the matrix $\mathrm{diag}(\varepsilon_r^{1/2}, r \in R)$.

2.3.4 Stochastic analysis - dual

Next consider a stochastic perturbation of the linearized equation (2.19). Let

$$d\xi(t) = -\kappa \left(\Theta^T \Psi \Theta \xi(t) dt - G dB(t) \right) \tag{2.29}$$

where $B(t) = (B_i(t), i \in I)$ is a collection of independent standard Brownian motions, and G is a $|J| \times |I|$ matrix.

A similar analysis to that of the last section determines the stationary covariance matrix Σ of $\xi(t)$. Define the symmetric matrix $[\Theta G; \Psi]$ by

$$[\Theta G; \Psi]_{jk} = \frac{[\Theta G G^T \Theta^T]_{jk}}{\psi_j + \psi_k}.$$

where Θ and Ψ are given by equation (2.18). Then

$$\Sigma = \kappa \Theta^T [\Theta G; \Psi] \Theta. \tag{2.30}$$

Note that the covariance matrix increases linearly with the gain parameter κ; as κ increases, the faster convergence to equilibrium described by relation (2.19) is at the cost of a greater spread at equilibrium.

Next we describe an example illustrating how a model of the form (2.29) might arise.

Shadow prices inferred from fluctuating flow rates. Consider the Poisson driving equation

$$d\mu_j(t) = \kappa \left(\sum_{r \in R} A_{jr} \varepsilon_r dN_r \left(\varepsilon_r^{-1} \int_0^t x_r(\tau) d\tau \right) - q_j(\mu_j(t)) dt \right) \tag{2.31}$$

where $(N_r(\tau), \tau \geq 0)$, for $r \in R$, are a collection of independent unit rate Poisson processes. This would describe a model where, on a very fine time-scale, the flow on route r takes the form of a time-dependent Poisson process of rate $x_r(t)/\varepsilon_r$, with each point of the process containing a workload of size ε_r. The Brownian approximation, valid when ε_r are small, becomes

$$d\mu_j(t) = \kappa \left(\sum_{r \in R} A_{jr} \left(x_r(t) dt + \varepsilon_r^{1/2} x_r(t)^{1/2} dB_r(t) \right) - q_j(\mu_j(t)) dt \right) \tag{2.32}$$

whose linearization is equation (2.29) where G is a $|J| \times |R|$ matrix with elements

$$G_{jr} = \varepsilon_r^{1/2} x_r^{1/2} A_{jr}$$

and thus $GG^T = AXEA^T$ where $E = \operatorname{diag}(\varepsilon_r, r \in R)$.

2.3.5 Variants

Several variants of the primal (2.7)–(2.8) and dual (2.11)–(2.12) algorithms allow a similar analysis. For example, if the right hand side of equation (2.7) is multiplied by a positive function $f_r(x(t), \mu(t))$ then Theorem 1 remains valid. Similarly, if the right hand side of equation (2.11) is multiplied by a positive function $f_j(x(t), \mu(t))$ then Theorem 2 remains valid. As a simple example, we could divide the right hand side of equation (2.7) by w_r, or of equation (2.11) by $q_j(\mu_j(t))$.

A more subtle variation is obtained if equation (2.11) is replaced by

$$\frac{d}{dt}\mu_j(t) = \kappa\left(p_j(\sum_{r\in R} A_{jr}x_r(t)) - \mu_j(t)\right), \tag{2.33}$$

where p_j is the inverse function of q_j, and $x_r(t)$ is again given by equation (2.12). Note that the expression (2.33) is of the same sign as expression (2.11), and so the proof of Theorem 2 goes through as before. Suppose $p_j(y)$ is differentiable at the stable point with derivative p'_j, and let $\mu_j(t) = \mu_j + \xi_j(t)$. Then, linearizing the system (2.33) about the equilibrium point, we obtain

$$\frac{d}{dt}\xi(t) = -\kappa(I + P'AXW^{-1}XA^T)\xi(t)$$

allowing the local convergence properties of the algorithm (2.33) to be studied.

2.4 Time lags

2.4.1 Primal

Consider next the lagged, discrete time system

$$x_r[t+1] = x_r[t] + \kappa\left(w_r - x_r[t]\sum_{j\in J} A_{jr}\mu_j[t - d(j,r)]\right) \tag{2.34}$$

where

$$\mu_j[t] = p_j\left(\sum_{s\in R} A_{js}x_s[t - d(j,s)]\right), \tag{2.35}$$

and $d(j,r)$, $j \in J$, $r \in R$, are non-negative integers. This might correspond to a model of congestion indication with joint feedback, where there is a delay of

$d(j,r)$ between resource j generating a feedback signal and user r receiving it, and another delay of $d(j,r)$ between user r changing its rate and the altered flow reaching resource j. Say that a vector x is an *equilibrium point* of the system (2.34)–(2.35) if $x_r[t] = x_r$, for $t = \ldots, 0, 1, 2, \ldots$, satisfies these equations.

Theorem 3 *The vector x maximizing the strictly concave function $\mathcal{U}(x)$ given by (2.9) is the unique equilibrium point of the system (2.34)–(2.35).*

Proof. The vector x is an equilibrium point if and only if it solves

$$w_r = x_r \sum_{j\in J} A_{jr} p_j \Big(\sum_{s\in R} A_{js} x_s \Big).$$

But this is precisely the stationarity condition implied by the partial derivatives (2.10) of the function $\mathcal{U}(x)$, a strictly concave function with a unique maximum. □

For small enough values of κ the equilibrium point will be asymptotically stable, since if we replace κ by $\kappa\delta$ in equation (2.34) and let $x_r[t] = x_r[t/\delta]$, then as $\delta \to 0$ we may approximate arbitrarily closely a solution to equations (2.7)–(2.8). But for small values of κ convergence to the equilibrium point is slow, and so it is of interest to investigate the local stability of the equilibrium point for general values of κ.

Let $\mu_j = p_j(\sum_{s\in R} A_{js} x_s)$, and suppose p_j is differentiable at this point, with derivative p_j'. Let $x_r[t] = x_r + x_r^{1/2} y_r[t]$. Then, linearizing the system (2.34)–(2.35) about x, we obtain

$$\begin{aligned} & y_r[t+1] \qquad\qquad (2.36) \\ = \;\; & y_r[t] - \kappa \left(\frac{w_r}{x_r} y_r[t] + \sum_{j\in J} \sum_{s\in R} p_j' A_{jr} A_{js} x_r^{1/2} x_s^{1/2} y_s[t - d(j,r) - d(j,s)] \right). \end{aligned}$$

Define the $|R| \times |R|$ matrices $(L[d], d = 0, 1, \ldots, D)$ by

$$(L[d])_{rs} = \sum_{j\in J} p_j' A_{jr} A_{js} x_r^{1/2} x_s^{1/2} I[d(j,r) + d(j,s) = d]$$

where $D = \max_{j,r,s}\{d(j,r) + d(j,s)\}$ and I is the indicator function. Thus

$$\sum_{d=0}^{D} L[d] = X^{1/2} A^T P' A X^{1/2},$$

the second term of the key matrix (2.16). Define the vector $y[t] = (y_r[t], r \in R)$. Then we can rewrite equation (2.36) in the matrix form

$$\begin{pmatrix} y[t+1] \\ y[t] \\ \vdots \\ y[t-D+1] \end{pmatrix} = L \begin{pmatrix} y[t] \\ y[t-1] \\ \vdots \\ y[t-D] \end{pmatrix} \tag{2.37}$$

where

$$L = \begin{pmatrix} I - \kappa(WX^{-1} + L[0]) & -\kappa L[1] & -\kappa L[2] & \dots & -\kappa L[D] \\ I & 0 & 0 & \dots & 0 \\ 0 & I & 0 & & \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2.38}$$

The equilibrium point x of the system (2.34)–(2.35) is stable if the spectral radius of the matrix L is less than unity ([45], Section 9.4). Recall that in our model of stochastic effects, increasing the derivatives P' had broadly the same reductive effect on the covariance matrix (2.22) as *decreasing* the gain parameter κ; in contrast the destabilising effect on the matrix (2.38) of increasing P' is broadly the same as *increasing* κ.

For simplicity of notation we have used the same gain parameter κ for each $r \in R$. If κ is replaced by κ_r in equation (2.34), then we again obtain relations (2.37)–(2.38), but now with κ interpreted as the matrix $\mathrm{diag}(\kappa_r, r \in R)$. An interesting topic concerns how the time delays within a network affect the choice of gain parameters; we might for example study the problem of choosing $\mathrm{diag}(\kappa_r, r \in R)$ in order to minimize the spectral radius of the matrix L. This topic will be taken up again in Chapter 4.

There exist other natural discrete time versions of the equations (2.7)–(2.8), and these too may be analysed in a similar manner. For example, consider the method of repeated substitution, $x_r[t+1] = w_r / \sum_{j\in J} A_{jr}\mu_j[t-d(j,r)]$ or its damped version

$$x_r[t+1] = (1-\kappa)x_r[t] + \kappa\frac{w_r}{\sum_{j\in J} A_{jr}\mu_j[t-d(j,r)]}$$

where $\mu_j[t]$ is again given by equation (2.35). Then the linearized relations (2.37)–(2.38) are altered in that the top row of the matrix (2.38) becomes

$$(I-\kappa(I+XW^{-1}L[0]), -\kappa XW^{-1}L[1], \ldots, -\kappa XW^{-1}L[D]).$$

2.4.2 Dual

Consider next the system

$$\mu_j[t+1] = \mu_j[t] + \kappa\Big(\sum_{r\in R} A_{jr}x_r[t-d(j,r)] - q_j(\mu_j[t])\Big) \tag{2.39}$$

where

$$x_r[t] = \frac{w_r}{\sum_{k\in J} A_{kr}\mu_k[t-d(k,r)]}\,. \tag{2.40}$$

A vector μ is an equilibrium point of the system (2.39)–(2.40) if $\mu_j[t] = \mu_j$, for $t = \ldots, 0, 1, 2, \ldots$, satisfies these equations.

Theorem 4 *The vector μ maximizing the strictly concave function $\mathcal{V}(\mu)$ given by (2.13) is the unique equilibrium point of the system (2.39)–(2.40).*

Proof. The vector μ is an equilibrium point if and only if it solves

$$\sum_{r\in R}\frac{A_{jr}w_r}{\sum_{k\in J} A_{kr}\mu_k} = q_j(\mu_j).$$

But this is precisely the stationarity condition implied by the partial derivatives (2.14) of the function $\mathcal{V}(\mu)$, a strictly concave function with a unique maximum. The result follows. □

Next we investigate the stability of the equilibrium point by linearizing the system (2.39)–(2.40) about μ. Let $x_r = w_r/\sum_{k\in J} A_{kr}\mu_k$, and suppose q_j is differentiable at the point $y = \mu_j$, with derivative q_j'. Let $\mu_j[t] = \mu_j + \xi_j[t]$. Then, linearizing the system (2.39)–(2.40) about μ, we obtain

$$\xi_j[t+1] = \xi_j[t] - \kappa\Big(\sum_{r\in R} A_{jr} x_r^2 w_r^{-1} \sum_{k\in J} A_{kr}\xi_k[t - d(j,r) - d(k,r)] + q_j'\xi_j[t]\Big). \quad (2.41)$$

Define the $|J| \times |J|$ matrices $(M[d], d = 0, 1, \ldots, D)$ by

$$(M[d])_{jk} = \sum_{r\in R} x_r^2 w_r^{-1} A_{jr} A_{kr} I[d(j,r) + d(k,r) = d] \quad (2.42)$$

where now $D = \max_{j,k,r}\{d(j,r) + d(k,r)\}$ and I is the indicator function. Thus

$$\sum_{d=0}^{D} M[d] = AXW^{-1}XA^T.$$

Define the vector $\xi[t] = (\xi_j[t], j \in J)$. Then we can rewrite equation (2.41) in the matrix form

$$\begin{pmatrix} \xi[t+1] \\ \xi[t] \\ \vdots \\ \xi[t-D+1] \end{pmatrix} = M \begin{pmatrix} \xi[t] \\ \xi[t-1] \\ \vdots \\ \xi[t-D] \end{pmatrix}$$

where

$$M = \begin{pmatrix} I - \kappa(M[0] + Q') & -\kappa M[1] & -\kappa M[2] & \ldots & -\kappa M[D] \\ I & 0 & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}. \quad (2.43)$$

The equilibrium point μ of the system (2.39)–(2.40) is stable if the spectral radius of the matrix M is less than unity ([45], Section 9.4). With stochastic effects, increasing the derivatives Q' has broadly the same reductive effect on the covariance matrix (2.30) as *decreasing* the gain parameter κ; in contrast the destabilising effect on the matrix (2.43) of increasing Q' is broadly the same as *increasing* κ.

2.5 Example: Congestion indication in a four-node network

In this section we illustrate the results of this chapter through a discussion of some examples. We illustrate how the functions $p_j, j \in J$, may be determined by the detailed stochastic behaviour of resource j; a simple four-node network is used to facilitate comparisons between feedback mechanisms. The results of this chapter are, of course, intended to apply to large-scale networks, and a discussion on the behaviour of the dual algorithm in a random network with 100 resources and 1000 routes may be found in [40].

Suppose that the total load y on a resource takes the form, on a very fine time-scale, of a Poisson stream of cells at rate y/ε. Suppose that the time-axis is divided into non-overlapping slots each of length $\tau\varepsilon$, and that a feedback signal is generated for a slot if the total number of cells arriving in that slot exceeds a threshold N. (While there may well be a queue at a resource, we suppose for the moment that the feedback signals are generated by the process just described, rather than, for example, by the queue size exceeding a threshold.) Suppose that when a feedback signal is generated, it is sent to each user r whose route passes through resource j, where it is interpreted as a congestion indicator requiring a reduction in the rate $x_r(t)$ of size $\kappa\varepsilon x_r(t)$. If the probability that a signal is generated in any single slot is small, then this model corresponds to equation (2.23), with $\varepsilon_j = \varepsilon$ and

$$p_j(y) = \frac{1}{\tau} \sum_{n>N} e^{-y\tau} \frac{(y\tau)^n}{n!} . \tag{2.44}$$

Consider the network illustrated in Figure 2.1, where $|J| = |R| = 4$. Let $w_r = 0.0002, r \in R$, suppose $p_j(y)$ is given by equation (2.44), and choose $N = 128$, $\tau = 50$, so that the equilibrium point is $x_r = 1.0, r \in R$, $\mu_j = 0.0001, j \in J$. Then, from relations (2.22) and (2.26), the covariance matrix of the rates $(x_r(t), r \in R)$

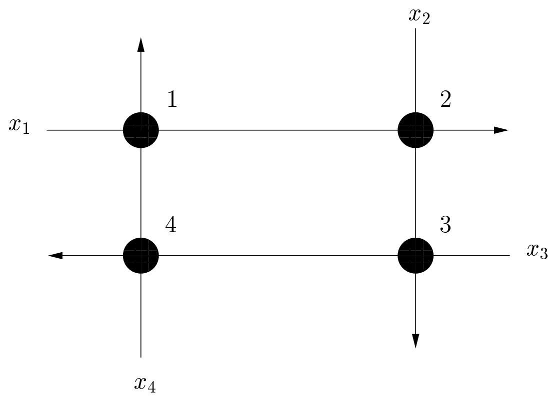


Figure 2.1: A four-node network.

for this joint feedback model is calculated to be

$$\Sigma_1 = \kappa\varepsilon 10^{-2} \begin{pmatrix} 2.4 & 0.8 & -0.8 & 0.8 \\ 0.8 & 2.4 & 0.8 & -0.8 \\ -0.8 & 0.8 & 2.4 & 0.8 \\ 0.8 & -0.8 & 0.8 & 2.4 \end{pmatrix},$$

a matrix whose form we shall discuss shortly.

For a second example, suppose again that a feedback signal is generated by a slot when the total number of cells arriving in that slot exceeds a threshold, N. But now suppose that when a feedback signal is generated at resource j, it is directed at a random route r with probability x_r/y (for example, the signal might be sent to the route responsible for the last cell arriving during the slot that generated the feedback signal). If user r receives a feedback signal, then the rate $x_r(t)$ is reduced by an amount $\kappa\varepsilon$. This model corresponds to equation (2.27), with $\varepsilon_j = \varepsilon$ and

$$p_j(y) = \frac{1}{\tau y} \sum_{n>N} e^{-y\tau} \frac{(y\tau)^n}{n!}. \tag{2.45}$$

Again let $w_r = 0.0002, r \in R$, and now choose $N = 125, \tau = 50$ so that, using equation (2.45), $x_r = 1.0, r \in R, \mu_j = 0.0001, j \in J$, is once again an equilibrium point. Then, from relations (2.22) and (2.28), the covariance matrix of the rates

$(x_r(t), r \in R)$ for this individual feedback model is calculated to be

$$\Sigma_2 = \kappa\varepsilon 10^{-1} \begin{pmatrix} 1.5 & -1.2 & 1.1 & -1.2 \\ -1.2 & 1.5 & -1.2 & 1.1 \\ 1.1 & -1.2 & 1.5 & -1.2 \\ -1.2 & 1.1 & -1.2 & 1.5 \end{pmatrix}.$$

(The function (2.45) is *not* an increasing function of y for all values of y. But it is increasing at the point $y = 2$, hence the matrix (2.16) is positive definite and this implies that the equilibrium point is locally stable. We note in passing that function (2.45) provides an example where, if the parameters $w_r, r \in R$, are set too large, the function $\mathcal{U}(x)$ has no interior maximum and the system (2.7)–(2.8) has no equilibrium point).

It is interesting to compare the magnitudes and structures of the matrices Σ_1 and Σ_2. That Σ_2 is larger in magnitude is expected, since with individual feedback there are additional sources of variation in the random choice of which rate is to be reduced by a feedback signal. Note also that rates on routes sharing a node are positively correlated for the joint feedback model. The explanation is that, with joint feedback, congestion indication at a node causes both routes through that node to decrease their rates simultaneously. However, for individual feedback, routes sharing a node are negatively correlated. In this case, a decrease in the flow on a route will allow increases on routes sharing a common node with it.

We have simulated equations (2.23) and (2.27), with $\varepsilon = 1.0$, $\kappa = 0.01$, with results that agree well with the matrices Σ_1 and Σ_2.

Suppose next that when the number of cells within a slot exceeds N, each of the cells within the slot causes a feedback signal to be sent to the user responsible for that cell. Suppose a user responds to each feedback signal by reducing its rate by $\kappa\varepsilon$. Then the expected number of feedback signals generated per slot is

$$\sum_{n>N} n e^{-y\tau} \frac{(y\tau)^n}{n!} = y\tau \sum_{n \geq N} e^{-y\tau} \frac{(y\tau)^n}{n!},$$

and so

$$p_j(y) = \sum_{n \geq N} e^{-y\tau} \frac{(y\tau)^n}{n!}, \tag{2.46}$$

an expression rather similar to the form (2.44). The covariance structure of the model depends on the size of N relative to the number of routes through a resource. If N is large and most or all routes through a resource receive a feedback signal when overload occurs, then the covariance structure resembles that of the joint feedback model; if N is small, and an essentially random set of routes receives a feedback signal, the covariance structure resembles more closely that of the individual feedback model.

Many other mechanisms for signal generation are possible. For example, suppose that cells pass through a buffer which acts as a single server queue, and signals are generated whenever a cell arrives to find the buffer above a threshold level. Under Poisson arrival assumptions the rate of signal generation, and hence $p_j(y)$, may be determined from the analysis of an M/D/1 queue. Note that the buffer may well be a virtual buffer, with service rate lower than that of a real buffer at the resource, in order to signal congestion before the onset of cell loss. The M/D/1 model of a resource will be discussed again in Sections 3.4.1 and 5.1. Another example is the proposed Random Early Detection or RED [21] gateway which detects incipient congestion by computing the average queue size. When a preset threshold is exceeded, the gateway drops or marks each arriving packet with a certain probability which is a function of the average queue size.

Poisson streams and non-overlapping slots allow simple calculations, and are suggestive of the results that may be obtained with more complex models. It is important to note, however, that the main results of earlier sections do not depend upon Poisson assumptions; the derivation of the covariance structure (2.22) was based on a more general central limit approximation, while Theorems 1 and 2 rely only upon rather weak properties of the functions p_j, q_j, $\mathcal{U}$ or $\mathcal{V}$.

Chapter 3

User behaviour

In this chapter, we will consider various aspects of user behaviour; in particular, adaptive users, routeing and user arrivals/departures from the system.

We first study the stability of the system when users are able to adapt quickly to their experience of congestion and show how our methods extend to this situation. In Chapter 2, our analyses of the primal algorithm (2.7)–(2.8) and the dual algorithm (2.11)–(2.12) have assumed that the parameters $(w_r, r \in R)$ chosen by the users are fixed, at least on the time scales concerned in the analyses. With increasing intelligence embedded in end-systems, users may well be able to vary the parameters $(w_r, r \in R)$ even within these short time scales. We will show global stability for two models with adaptive users. The first model, without the parameters $(w_r, r \in R)$, assumes that each user employs a gradient-type primal algorithm to maximize its own objective function. For the second model, each user is assumed to choose $w_r(t)$ in order to track the optimum to its own optimization problem, which we term as *accurate tracking*.

The possibility that a user may be served by several routes is then considered. We revise the system, network and user problems to include routeing, generalize the primal and dual algorithms and their corresponding Lyapunov functions, thus showing that routeing may be naturally integrated with proportionally fair pricing.

The second half of this chapter extends the accurate tracking model to allow for fluctuations in the user population due to arrivals and departures. For the first model, we consider each route r as a superposition of sub-users who stay in the network for a fixed duration e.g. watching a video, independent of the allocated rate. We first establish stability and convergence for the system and then show how our methods may be used to study the stochastic effects due to the rate and arrival/departure processes, the relative importance of which is dependent on their speeds of convergence. For the second model, we consider a single resource with a mix of two user types, some are fixed duration users (as in the first case) while others are file transfer users i.e. each has a fixed job size to perform before leaving the network. The general conclusion is that adaptive fixed duration users are, in a sense, 'good traffic'.

3.1 A model without w_r's

We first look at a revised model of the system without the parameters $(w_r, r \in R)$. Suppose resource j tags a charge $p_j(y)$, per unit flow through it, onto packets as they are served when the total load on resource j is y. Suppose also that user r is able to monitor the price per unit rate it is charged, $\lambda_r(t) = \sum_{j \in J} A_{jr}\mu_j(t)$, where $\mu_j(t)$ is given by equation (2.8), and may vary $x_r(t)$ in order to solve the utility maximization problem,

$USER_r(U_r; \lambda_r)$:

$$\begin{array}{ll} \text{maximize} & U_r(x_r) - \lambda_r x_r \\ \text{over} & x_r \geq 0 \end{array}$$

Let user r vary its rate $x_r(t)$ according to the differential equation

$$\frac{d}{dt}x_r(t) = \kappa\Big(U_r'(x_r(t)) - \sum_{j \in J} A_{jr}\mu_j(t)\Big) \tag{3.1}$$

or zero when this expression is negative and $x_r(t) = 0$.

We shall establish stability of the system by using a revision of the argument leading to Theorem 1. Consider the revised objective function

$$\mathcal{U}(x) = \sum_{r\in R} U_r(x_r) - \sum_{j\in J} \int_0^{\sum_{s\in R} A_{js}x_s} p_j(z)dz \tag{3.2}$$

where the function $p_j(z)$, $z \geq 0$, for $j \in J$, is a non-negative, continuous, increasing function of z, not identically zero. Then we can prove the following theorem.

Theorem 5 *The strictly concave function $\mathcal{U}(x)$ given by (3.2) is a Lyapunov function for the system of differential equations (3.1) and the unique value x maximizing $\mathcal{U}(x)$ is a stable point of the system, to which all trajectories converge.*

Proof. By assumption, $\lim_{x\to 0} U_r'(x) = \infty$ and $\lim_{x\to\infty} U_r'(x) = 0$, $r \in R$, and $p_j(\cdot), j \in J$ are non-negative, continuous, increasing functions, not identically zero. These ensure that $\mathcal{U}(x)$ is strictly concave on $x \geq 0$ with an interior maximum. We note that

$$\frac{\partial}{\partial x_r}\mathcal{U}(x) = U_r'(x_r) - \sum_{j\in J} A_{jr} p_j(\sum_{s\in R} A_{js}x_s),$$

Together with equation (3.1) we have

$$\begin{aligned} \frac{d}{dt}\mathcal{U}(x(t)) &= \sum_{r\in R} \frac{\partial \mathcal{U}}{\partial x_r} \cdot \frac{dx_r}{dt} \\ &= \kappa \sum_{r\in R} \Big(U_r'(x_r(t)) - \sum_{j\in J} A_{jr} p_j(\sum_{s\in R} A_{js}x_s(t))\Big)^2 \end{aligned}$$

establishing that $\mathcal{U}(x(t))$ is strictly increasing with t, unless $x(t) = x$, the unique x maximizing $\mathcal{U}(x)$. Thus $\mathcal{U}(x)$ is a Lyapunov function for the above system, and the unique x maximizing $\mathcal{U}(x)$ is a stable point of the system, to which all trajectories converge ([25], Chapter 2). Note that the maximizing x also solves the user's utility maximization problem as well. □

As before, note that if the right hand side of equation (3.1) is multiplied by a positive function $f_r(x(t), \mu(t))$, then Theorem 5 remains valid. This provides a hint on how a similar result might be obtained for a model with parameters $(w_r, r \in R)$.

3.2 Accurate tracking

3.2.1 Primal

Consider now the model with parameters $(w_r, r \in R)$. We shall see that a slight modification of the previous section's approach will yield the same result. Suppose that user r is able to monitor its rate $x_r(t)$ continuously, and to vary smoothly $w_r(t)$ so as to track the optimum to $USER_r(U_r; \lambda_r)$, where $\lambda_r(t) = w_r(t)/x_r(t)$ is the charge per unit rate to user r at time t. That is, user r solves the optimization problem

$$\begin{aligned} \text{maximize} \quad & f_r(w_r) = U_r(w_r/\lambda_r) - w_r \\ \text{over} \quad & w_r \geq 0 \end{aligned}$$

A simple differentiation gives

$$f_r'(w_r) = \frac{1}{\lambda_r} U_r'(w_r/\lambda_r) - 1$$

Since by assumption $\lim_{x \to 0} U_r'(x) = \infty$ and $\lim_{x \to \infty} U_r'(x) = 0$, the equation $f_r'(w_r) = 0$ always has a positive solution $w_r = x_r U_r'(x_r)$, where $x_r = w_r/\lambda_r$. Thus, under *accurate tracking* by user r, the parameter $w_r(t)$ will satisfy

$$w_r(t) = x_r(t) U_r'(x_r(t)), \tag{3.3}$$

while, for the primal algorithm, $x_r(t)$ evolves according to the revised differential equation

$$\frac{d}{dt} x_r(t) = \kappa \Big(w_r(t) - x_r(t) \sum_{j \in J} A_{jr} \mu_j(t) \Big) \tag{3.4}$$

where $\mu_j(t)$ is given by equation (2.8). Then we have the following result.

Theorem 6 *Suppose $x_r(0) > 0$, $r \in R$. Under accurate tracking defined by (3.3), the strictly concave function $\mathcal{U}(x)$ given by (3.2) is a Lyapunov function for the system of differential equations (3.4). The unique x maximizing (3.2) is a stable point of the system, to which all trajectories converge.*

Proof. The assumptions on the utility functions $U_r, r \in R$, and $p_j, j \in J$, ensure that $\mathcal{U}(x)$ is strictly concave on $x \geq 0$ with a unique interior maximum. Note that

$$\frac{\partial}{\partial x_r}\mathcal{U}(x) = U_r'(x_r) - \sum_{j\in J} A_{jr} p_j(\sum_{s\in R} A_{js}x_s);$$

Then

$$\begin{aligned} \frac{d}{dt}\mathcal{U}(x(t)) &= \sum_{r\in R} \frac{\partial \mathcal{U}}{\partial x_r} \cdot \frac{dx_r}{dt} \\ &= \kappa \sum_{r\in R} x_r(t)\Big(U_r'(x_r(t)) - \sum_{j\in J} A_{jr} p_j(\sum_{s\in R} A_{js}x_s(t))\Big)^2, \end{aligned}$$

using relation (3.3) to substitute for $w_r(t)$. This establishes that $\mathcal{U}(x(t))$ is strictly increasing with t, unless $x(t) = x$, the unique x maximizing $\mathcal{U}(x)$. Hence, $\mathcal{U}(x)$ is a Lyapunov function for the above system, and the unique x maximizing $\mathcal{U}(x)$ is a stable point of the system, to which all trajectories converge ([25], Chapter 2). □

The conditions imposed on the utility functions may be viewed as technical conditions. It is certainly reasonable to assume that $\lim_{x\to\infty} U_r'(x) = 0$ corresponding to decreasing marginal utility. Also, for x large enough, the form of $U_r(x)$ is actually hypothetical if users do not have infinite amounts of wealth and are limited by the peak rates of their devices. The condition $\lim_{x\to 0} U_r'(x) = \infty$ is more subtle. It ensures that the function $\mathcal{U}(x)$ given by (3.2) has an interior maximum and keeps $x_r(t)$ away from zero. If not, convergence to the maximum will not occur should a trajectory cause one of the rates to hit zero prematurely (hence the initial condition for x in the theorem as well). Since this is a local condition, it does not really restrict the choice of utility functions. Notice however that the condition $U_r'(0) = \infty$ is not necessary for Theorem 5. The weaker condition that the function $\mathcal{U}(x)$ given by (3.2) has a maximum is sufficient in that case. The difference between the algorithms given by equations (3.1) and (3.4) is due essentially to a factor of $x_r(t)$.

Linearization may also be used to investigate behaviour near the stable point; for example, the revised form of equation (2.15) becomes

$$\frac{d}{dt}y(t) = -\kappa X^{1/2}(A^T P' A - U'')X^{1/2}y(t) \tag{3.5}$$

where $U'' = \mathrm{diag}(U_r''(x_r), r \in R)$. This reduces to the original equation (2.15) in the case of the logarithmic utility function.

The adaptive behaviour of users is an interesting subject in itself and other models are possible. See [24] for other models of user behaviour and [15] on an intelligent agent that replaces the user in choosing the willingness to pay, w_r, by learning the user's preferences.

3.2.2 Dual

A similar analysis is possible for the dual algorithm. Under accurate tracking by user r of the optimum to $USER_r(U_r; \lambda_r)$ the parameter $w_r(t)$ will be given by equation (3.3), and, for the dual algorithm,

$$x_r(t) = \frac{w_r(t)}{\sum_{j \in J} A_{jr} \mu_j(t)} \tag{3.6}$$

where $\mu_j(t)$ evolves according to the differential equation (2.11). To find a Lyapunov function for the system, it is helpful to construct the dual to the relaxation of the problem $SYSTEM(U, A, C)$:

$SYSTEM^*(U, A, P)$**:**

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} U_r(x_r) - \sum_{j \in J} \int_0^{y_j} p_j(z) dz \qquad (3.7) \\ \text{subject to} \quad & Ax = y \\ \text{over} \quad & x, y \geq 0 \end{aligned}$$

The Lagrangian for the problem is

$$L(x, y; \mu) = \sum_{r \in R} U_r(x_r) - \sum_{j \in J} \int_0^{y_j} p_j(z) dz - \mu^T (Ax - y)$$

Now

$$\frac{\partial L}{\partial x_r} = U_r'(x_r) - \sum_{j \in J} A_{jr} \mu_j \tag{3.8}$$

$$\frac{\partial L}{\partial y_j} = -p_j(y_j) + \mu_j \tag{3.9}$$

Hence, the dual to the relaxed problem is

$$\begin{aligned}\text{minimize} \quad & \sum_{r\in R} \max_{x_r\geq 0} \left(U_r(x_r) - x_r \sum_{j\in J} A_{jr}\mu_j\right) \\ & + \sum_{j\in J}\left(\mu_j p_j^{-1}(\mu_j) - \int_0^{p_j^{-1}(\mu_j)} p_j(z)dz\right) \\ \text{over} \quad & \mu \geq 0\end{aligned}$$

The second term is just

$$\sum_{j\in J}\int_0^{\mu_j} q_j(\eta)d\eta$$

where the function $q_j = p_j^{-1}$.

For the first term, let $D_r(\lambda_r) = x_r$, where x_r is the positive solution to the equation $\lambda_r = U_r'(x_r)$, since $\lim_{x\to 0} U_r'(x) = \infty$ and $\lim_{x\to\infty} U_r'(x) = 0$. Then

$$\begin{aligned}\max_{x_r\geq 0}\left(U_r(x_r) - \lambda_r x_r\right) &= \int_0^{D_r(\lambda_r)} D_r^{-1}(y)dy - \lambda_r D_r(\lambda_r) + \text{const.} \qquad (3.10)\\ &= \int_{\lambda_r}^{\infty} D_r(\zeta)d\zeta + \text{const.}\end{aligned}$$

where $\lambda = \mu^T A$. Thus the dual reduces to

$$\begin{aligned}\text{maximize} \quad & \mathcal{V}(\mu) = \sum_{r\in R}\int^{\sum_{j\in J} A_{jr}\mu_j} D_r(\zeta)d\zeta - \sum_{j\in J}\int_0^{\mu_j} q_j(\eta)d\eta \qquad (3.11)\\ \text{over} \quad & \mu \geq 0\end{aligned}$$

The lower limit in the integral of the function D_r can be chosen to be any fixed value in the range $(U_r'(\infty), U_r'(0))$. We may interpret $D_r(\lambda_r)$ as the demand of user r when confronted with a price per unit flow of λ_r; under accurate tracking by user r,

$$x_r(t) = D_r(\sum_{j\in J} A_{jr}\mu_j(t)). \qquad (3.12)$$

Theorem 7 *Suppose the function $\mathcal{V}(\mu)$ given by (3.11) is strictly concave with a maximum. Under accurate tracking defined by (3.12), $\mathcal{V}(\mu)$ is a Lyapunov function for the system of differential equations (2.11) and (3.6). The unique μ maximizing $\mathcal{V}(\mu)$ is a stable point of the system, to which all trajectories converge.*

Proof. Now $\mathcal{V}(\mu)$ has a maximum by assumption; the maximum is unique by strict concavity. Note that

$$\frac{\partial}{\partial \mu_j}\mathcal{V}(\mu) = \sum_{r\in R} A_{jr} D_r(\sum_{k\in J} A_{kr}\mu_k) - q_j(\mu_j)$$

and thus

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(\mu) &= \sum_{j\in J}\frac{\partial \mathcal{V}}{\partial \mu_j}\cdot\frac{d\mu_j}{dt} \\ &= \kappa\sum_{j\in J}\left(\sum_{r\in R} A_{jr} D_r(\sum_{k\in J} A_{kr}\mu_k(t)) - q_j(\mu_j(t))\right)^2\end{aligned}$$

using equation (2.11) and relation (3.12). Hence $\mathcal{V}(\mu)$ provides a Lyapunov function for the system (2.11) and (3.6), and the unique μ maximizing $\mathcal{V}(\mu)$ is a stable point of the system, to which all trajectories converge ([25], Chapter 2). □

Notice that in contrast to Theorem 6, there is no need for the function $\mathcal{V}(\mu)$ to have an interior maximum. This is because the RHS of the dual algorithm (2.11), using relation (3.12), is exactly the derivative of $\mathcal{V}(\mu)$ with respect to μ_j multiplied by κ. In the above, we have used a Lyapunov method to derive the stability of the dual algorithm. The dual approach is also used by Low and Lapsley in [44] where they proposed a gradient projection algorithm to solve the dual problem.

It would appear from (3.11) that the optimized function seems to minimize the aggregate *consumer surplus* ([64], Chapter 10) which runs counter to the usual aim of maximizing social welfare. We will next show the relation between the primal algorithm and social welfare.

3.2.3 Social welfare

How might we link the analysis in the preceding sections with the need to optimize social welfare e.g. in the context of pricing the Internet? Consider the following optimization problem, which looks at choosing the right prices and rates in order to maximize the sum of consumer surpluses and network revenue minus costs,

$WELFARE(D,p)$

$$\begin{array}{ll} \text{maximize} & \sum_{r\in R}\left(\int_{\lambda_r}^{\infty} D_r(\zeta)d\zeta + \lambda_r D_r(\lambda_r)\right) - \sum_{j\in J}\int_0^{y_j} p_j(z)dz \\ \text{subject to} & AD(\lambda) = y \\ \text{over} & y, \lambda \geq 0 \end{array}$$

where $D(\lambda) = (D_r(\lambda_r), r \in R)$. The Lagrangian for the problem is

$$\begin{aligned} L(y,\lambda;\mu) &= \sum_{r\in R}\left(\int_{\lambda_r}^{\infty} D_r(\zeta)d\zeta + \lambda_r D_r(\lambda_r)\right) - \sum_{j\in J}\int_0^{y_j} p_j(z)dz \\ &\quad -\mu^T(AD(\lambda) - y) \end{aligned}$$

Differentiating with respect to λ_r and y_j gives

$$\frac{\partial L}{\partial \lambda_r} = \lambda_r D_r'(\lambda_r) - \sum_{j\in J} A_{jr}\mu_j D_r'(\lambda_r) \tag{3.13}$$

$$\frac{\partial L}{\partial y_j} = -p_j(y_j) + \mu_j \tag{3.14}$$

for each $r \in R$ and $j \in J$.

Assuming that $D_r'(\lambda_r)$ is non-zero for $0 < \lambda_r < \infty$, and using the relation $D_r(\lambda_r) = x_r \Leftrightarrow \lambda_r = U_r'(x_r)$, then equations (3.13)–(3.14) are just the conditions (3.8)–(3.9) for the primal optimum. Also, together with relation (3.10), we see that apart from a constant, the social welfare optimum is the same as the primal optimum (3.7).

We now use a revision of the argument discussed earlier in this chapter. Define the Lyapunov function

$$\mathcal{U}(x) = \sum_{r\in R}\left(\int_{\lambda_r}^{\infty} D_r(\zeta)d\zeta + \lambda_r D_r(\lambda_r)\right) - \sum_{j\in J}\int_0^{\sum_{s\in R} A_{js}D_s(\lambda_s)} p_j(z)dz$$

Note that

$$\begin{aligned} \frac{\partial}{\partial x_r}\mathcal{U}(x) &= \frac{\partial \mathcal{U}}{\partial \lambda_r}\cdot\frac{\partial \lambda_r}{\partial x_r} \\ &= \left(\lambda_r D_r'(\lambda_r) - \sum_{j\in J} A_{jr}p_j D_r'(\lambda_r)\right)\cdot\frac{1}{D_r'(\lambda_r)} \\ &= \lambda_r - \sum_{j\in J} A_{jr}p_j(\sum_{s\in R} A_{js}D_s(\lambda_s)) \end{aligned}$$

Under the primal algorithm (3.4), we have $\lambda_r(t) = w_r(t)/x_r(t), r \in R$. Then

$$\begin{aligned}
\frac{d}{dt}\mathcal{U}(x) &= \sum_{r\in R} \frac{\partial \mathcal{U}}{\partial x_r} \cdot \frac{dx_r}{dt} \\
&= \sum_{r\in R} \kappa \Big(\lambda_r(t) - \sum_{j\in J} A_{jr}\mu_j(t)\Big)\Big(w_r(t) - x_r(t)\sum_{j\in J} A_{jr}\mu_j(t)\Big) \\
&= \kappa \sum_{r\in R} x_r(t) \Big(U_r'(x_r(t)) - \sum_{j\in J} A_{jr} p_j(\sum_{s\in R} A_{js} x_s(t))\Big)^2
\end{aligned}$$

using relation (3.3). This establishes stability and convergence for the primal algorithm provided the same conditions on the utilities hold as in Theorem 6. Hence, the primal algorithm moves in the direction of increasing social welfare.

3.3 Routeing

Let $s \in S$ now label a user, and suppose s is identified with a subset of R, the routes available to serve the user s. Set $H_{sr} = 1$ if $r \in s$, so that route r serves user s, and set $H_{sr} = 0$ otherwise. This defines a 0–1 matrix $H = (H_{sr}, s \in S, r \in R)$. For each $r \in R$ let $s(r)$ identify a value $s \in S$ such that $H_{sr} = 1$, and suppose this value is unique; view $s(r)$ as the user served by route r.

Now let y_r be the flow on route r, and suppose that resource j incurs a cost $c_j(\sum_{r\in R} A_{jr} y_r)$ dependent on the flow through that resource, where $c_j(\cdot)$ is a strictly convex function. Consider the following optimization problems.

$SYSTEM(U,H,A,C)$:

$$\begin{array}{ll}
\text{maximize} & \sum_{s\in S} U_s(x_s) - \sum_{j\in J} c_j(\sum_{r\in R} A_{jr} y_r) \\
\text{subject to} & Hy = x \\
\text{over} & x, y \geq 0.
\end{array}$$

$NETWORK(H,A,C;w)$:

$$\begin{array}{ll}
\text{maximize} & \sum_{s\in S} w_s \log(\sum_{r\in R} H_{sr} y_r) - \sum_{j\in J} c_j(\sum_{r\in R} A_{jr} y_r) \\
\text{over} & y \geq 0.
\end{array}$$

$USER_s(U_s; \lambda_s)$:

$$\begin{aligned} \text{maximize} \quad & U_s(w_s/\lambda_s) - w_s \\ \text{over} \quad & w_s \geq 0 \end{aligned}$$

Then, using the approach of [39], it is possible to prove the following theorem.

Theorem 8 *There exist vectors $\lambda = (\lambda_s, s \in S)$, $w = (w_s, s \in S)$ and $y = (y_r, r \in R)$ satisfying $Hy = x$ and $w_s = \lambda_s x_s$, $s \in S$, such that w_s solves $USER_s(U_s; \lambda_s)$ and y solves $NETWORK(H, A, C; w)$; further x is then the unique vector with the property that there exists a vector y such that (x, y) solves $SYSTEM(U, H, A, C)$.*

Proof. The Lagrangian for the problem $SYSTEM(U, H, A, C)$ is

$$L(x, y; l) = \sum_{s \in S} U_s(x_s) - \sum_{j \in J} c_j(\sum_{r \in R} A_{jr} y_r) + l^T(Hy - x)$$

Then

$$\begin{aligned} \frac{\partial L}{\partial x_s} &= U_s'(x_s) - l_s \\ \frac{\partial L}{\partial y_r} &= -\sum_{j \in J} A_{jr} p_j(\sum_{a \in R} A_{ja} y_a) + l_{s(r)} \end{aligned}$$

using relation (2.5). By assumption, $\lim_{x \to 0} U_s'(x) = \infty$ and $\lim_{x \to \infty} U_s'(x) = 0$, $s \in S$. Hence, the following conditions hold at a maximum of L:

$$U_s'(x_s) = l_s\,, \qquad x_s > 0 \tag{3.15}$$

$$l_{s(r)} = \sum_{j \in J} A_{jr} p_j(\sum_{a \in R} A_{ja} y_a) \quad \text{if} \quad y_r > 0 \tag{3.16}$$

$$l_{s(r)} \leq \sum_{j \in J} A_{jr} p_j(\sum_{a \in R} A_{ja} y_a) \quad \text{if} \quad y_r = 0$$

Similarly, the conditions at the maximum for the problems $NETWORK(H, A, C; w)$ and $USER_s(U_s; \lambda_s)$ are

$$w_{s(r)} = \Big(\sum_{a \in R} H_{s(r)a} y_a\Big)\Big(\sum_{j \in J} A_{jr} p_j(\sum_{a \in R} A_{ja} y_a)\Big) \quad \text{if} \quad y_r > 0 \tag{3.17}$$

$$w_{s(r)} \leq \Big(\sum_{a \in R} H_{s(r)a} y_a\Big)\Big(\sum_{j \in J} A_{jr} p_j(\sum_{a \in R} A_{ja} y_a)\Big) \quad \text{if} \quad y_r = 0$$

and

$$U_s'(w_s/\lambda_s) = \lambda_s\,, \qquad w_s > 0 \tag{3.18}$$

Now the triple (l, x, y) which satisfies the conditions (3.15)–(3.16) and relation $Hy = x$ identifies a solution to (3.17) and (3.18) with $\lambda_s = l_s$ and $w_s = \lambda_s x_s, s \in S$. This establishes the existence of the claimed vectors λ, w and y.

Conversely, suppose y and w_s solve $NETWORK(H, A, C; w)$ and $USER_s(U_s; \lambda_s)$ respectively. Using the relations $x = Hy$ and $w_s = \lambda_s x_s$, we see that equations (3.17)–(3.18) are equivalent to equations (3.15)–(3.16) if we set $\lambda_s = l_s, s \in S$. This implies that x and y solve $SYSTEM(U, H, A, C)$. The uniqueness of x follows from strict concavity. □

Thus both the network and system problems may be generalized to include route-ing choices. We now sketch the natural generalizations of the primal and dual algorithms, and their corresponding Lyapunov functions.

Generalize the primal algorithm (2.7)–(2.8) to become

$$\frac{d}{dt}\,y_r(t) = \kappa\left(w_{s(r)} - \Big(\sum_{a\in R} H_{s(r)a} y_a(t)\Big)\sum_{j\in J} A_{jr}\mu_j(t)\right) \tag{3.19}$$

(or zero if this expression is negative and $y_r(t) = 0$) where

$$\mu_j(t) = p_j(\sum_{a\in R} A_{ja} y_a(t)), \tag{3.20}$$

and let

$$\mathcal{U}(y) = \sum_{s\in S} w_s \log(\sum_{r\in R} H_{sr} y_r) - \sum_{j\in J}\int_0^{\sum_{r\in R} A_{jr} y_r} p_j(z)dz.$$

Then observe that

$$\frac{\partial}{\partial y_r}\mathcal{U}(y) = \frac{w_{s(r)}}{\sum_{a\in R} H_{s(r)a} y_a} - \sum_{j\in J} A_{jr} p_j(\sum_{a\in R} A_{ja} y_a)$$

and thus the dynamical system (3.19)–(3.20) has the property that

$$\frac{d}{dt}\mathcal{U}(y(t)) > 0$$

unless y solves $NETWORK(H, A, C; w)$.

Similarly the dual algorithm (2.11)–(2.12) may be generalized to incorporate a form of least cost routeing. For $s \in S$ let

$$\sum_{r \in R} H_{sr} y_r(t) = x_s(t) = \frac{w_s}{\min_{r \in s} \sum_{j \in J} A_{jr} \mu_j(t)},$$

and suppose $y_r(t)$ is only positive on routes r that attain the minimum in the denominator. Then the dynamical system

$$\frac{d}{dt}\mu_j(t) = \kappa\Big(\sum_{r \in R} A_{jr} y_r(t) - q_j(\mu_j(t))\Big)$$

has the property that $\mathcal{V}(\mu(t))$ is an increasing function of t, where

$$\mathcal{V}(\mu) = \sum_{s \in S} w_s \log(\min_{r \in s} \sum_{j \in J} A_{jr} \mu_j) - \sum_{j \in J} \int_0^{\mu_j} q_j(\eta) d\eta.$$

To see this, let $w_s = \sum_{r \in s} w_r(t)$ where by assumption

$$w_r(t) = y_r(t) \sum_{j \in J} A_{jr} \mu_j(t)$$

is positive only on routes r of minimum cost. Then rewrite the function $\mathcal{V}(\mu)$ as

$$\mathcal{V}(\mu) = \sum_{r \in R} w_r \log(\sum_{j \in J} A_{jr} \mu_j) - \sum_{j \in J} \int_0^{\mu_j} q_j(\eta) d\eta.$$

Notice that

$$\frac{d}{dt}\mathcal{V}(\mu) = \sum_{j \in J} \frac{d\mathcal{V}}{d\mu_j} \cdot \frac{d\mu_j}{dt} + \sum_{r \in R} \frac{d\mathcal{V}}{dw_r} \cdot \frac{dw_r}{dt}$$

The first term on the right-hand side is positive as usual. It suffices to show that the second term is zero. Now

$$\begin{aligned}
\sum_{r \in R} \frac{d\mathcal{V}}{dw_r} \cdot \frac{dw_r}{dt} &= \sum_{s \in S} \sum_{r \in s} \frac{d\mathcal{V}}{dw_r} \cdot \frac{dw_r}{dt} \\
&= \sum_{s \in S} \sum_{r \in s} \log(\sum_{j \in J} A_{jr} \mu_j) \frac{dw_r}{dt} \\
&= \sum_{s \in S} \log(\sum_{j \in J} A_{jr} \mu_j) \sum_{r \in s} \frac{dw_r}{dt} \\
&= 0
\end{aligned}$$

since $\sum_{j \in J} A_{jr} \mu_j$, for $r \in s$, are equal on routes of minimum cost and $\sum_{r \in s} \dot{w}_r = 0$. Thus routeing, as well as rate control, may be naturally integrated with proportionally fair pricing.

3.4 User arrivals and departures

We now consider changes in the user population due to arrivals and departures. While the network optimization process is expected to happen on a finer time-scale than the user population process, the presence of time lags gives rise to the possibility that it might not be completed before a change in the user population occurs. (A simple calculation, taking the earth's circumference as $40000\,km$ and speed of light to be $3 \times 10^5\,kms^{-1}$, will show that light takes about 1/7 seconds to go round the earth and thus it is not unlikely that a user might arrive within the optimization time-scale of the network.)

In the first sub-section, we consider each route r as a superposition of sub-users, each staying in the network for a fixed duration e.g. watching a video. In this case, stability and convergence for the dynamical system may be established again with a function which is essentially the system's implicit objective function. Using a simple example, we illustrate how our methods may be used to study the stochastic effects due to the rate and arrival/departure processes, the relative importance of which is dependent on their speeds of convergence. For the second sub-section, we consider a single resource with a mix of two user types, some are fixed duration users (as in the first case) while others are file transfer users i.e. each has a fixed job size to perform before leaving the network. Here, we show that the system exhibits bad or good behaviour depending on whether the number of fixed duration users is small (approaching the regime considered by Massoulié and Roberts [49]) or large respectively.

3.4.1 Fixed duration users

Suppose for route r, user r is in fact a superposition of sub-users who arrive and depart, each with the same utility function $U_r(\cdot)$, which may not be logarithmic. Let $x_r^{sub}(t)$ and $n_r(t)$ be the allocated rate to a sub-user and the number of sub-users using route r at time t respectively. Consider the following system of differential equations,

$$\frac{d}{dt}x_r^{sub}(t) = \kappa x_r^{sub}(t)\Big(U_r'(x_r^{sub}(t)) - \sum_{j\in J} A_{jr}\mu_j(t)\Big) \tag{3.21}$$

$$\frac{d}{dt}n_r(t) = \alpha_r - \beta_r n_r(t) \tag{3.22}$$

where now $\mu_j(t) = p_j(\sum_{r\in R} A_{jr}n_r(t)x_r^{sub}(t))$.

The first equation (3.21) is obtained by assuming that all sub-users are adaptive and adopt accurate tracking described by equation (3.3). The second equation (3.22) describes the following user process: arrivals on route r occur at rate α_r while departures occur at rate $\beta_r n_r(t)$. The latter might arise when each sub-user stays for a fixed duration before departing (independent of the allocated rate) e.g. watching a video.

Define the strictly concave function

$$\mathcal{U}(x) = \sum_{r\in R}\frac{\alpha_r U_r(x_r^{sub})}{\beta_r} - \sum_{j\in J}\int_0^{\sum_{s\in R}\frac{A_{js}\alpha_s x_s^{sub}}{\beta_s}} p_j(z)dz \tag{3.23}$$

Then we have the following theorem.

Theorem 9 *Suppose $x_r(0) > 0$, $r \in R$. The unique x maximizing $\mathcal{U}(x)$ given by (3.23) is a stable point of the system (3.21)–(3.22), to which all trajectories converge.*

Proof. We note that by the assumptions on the utility functions $U_r, r \in R$ and $p_j, j \in J$ that $\mathcal{U}(x)$ is strictly concave on $x \geq 0$ with a unique interior maximum. The solution to equation (3.22) is clearly

$$n_r(t) = \frac{\alpha_r}{\beta_r} + b_r e^{-\beta_r t} \tag{3.24}$$

where b_r is a constant of integration and thus for each $r \in R$, $n_r(t)$ converges to α_r/β_r as $t \to \infty$. Differentiating (3.23) with respect to x_r^{sub} gives

$$\frac{\partial}{\partial x_r^{sub}}\mathcal{U}(x) = \frac{\alpha_r}{\beta_r}\left(U_r'(x_r^{sub}) - \sum_{j\in J} A_{jr}p_j(\sum_{s\in R}\frac{A_{js}\alpha_s x_s^{sub}}{\beta_s})\right);$$

Hence, using the convergence of $n_r(t)$ in (3.24),

$$\begin{aligned}\frac{d}{dt}\mathcal{U}(x(t)) &= \sum_{r\in R}\frac{\partial\mathcal{U}}{\partial x_r^{sub}}\cdot\frac{dx_r^{sub}}{dt}\\ &\to \kappa\sum_{r\in R}\frac{\alpha_r x_r^{sub}(t)}{\beta_r}\left(U_r'(x_r^{sub}(t)) - \sum_{j\in J}A_{jr}p_j(\sum_{s\in R}\frac{A_{js}\alpha_s x_s^{sub}(t)}{\beta_s})\right)^2\end{aligned}$$

as $t \to \infty$. Thus $\mathcal{U}(x)$ is eventually a Lyapunov function for the above system as $t \to \infty$. Hence the unique $(x_r^{sub}, n_r, r \in R)$ maximizing $\mathcal{U}(x)$ is a stable point of the system, to which all trajectories converge ([25], Chapter 2). □

We note that the objective function (3.23) is closely related to that given by (3.2). In fact, it is just multiplied by the limiting number of sub-users on each route. Let the stable point be $(x_r^{sub}, n_r, r \in R)$ where $U_r'(x_r^{sub}) = \sum_{j\in J} A_{jr}\mu_j$ and $n_r = \alpha_r/\beta_r$. Linearizing with $n_r(t) = n_r + m_r(t)$ and $x_r^{sub}(t) = x_r^{sub} + (x_r^{sub})^{1/2}y_r^{sub}(t)$, we have

$$\begin{aligned}\frac{d}{dt}y_r^{sub}(t) &= \kappa(x_r^{sub})^{1/2}\Big(U_r''(x_r^{sub})^{1/2}y_r^{sub}(t) - \sum_{j\in J}A_{jr}p_j'\sum_{s\in R}A_{js}x_s^{sub}m_s(t)\\ &\qquad - \sum_{j\in J}A_{jr}p_j'\sum_{s\in R}A_{js}n_s(x_s^{sub})^{1/2}y_s^{sub}(t)\Big)\\ \frac{d}{dt}m_r(t) &= -\beta_r m_r(t)\end{aligned}$$

which gives the matrix equation

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ m(t)\end{pmatrix} = -L\begin{pmatrix} y(t) \\ m(t)\end{pmatrix} \tag{3.25}$$

where

$$L = \begin{pmatrix} \kappa X^{1/2}(A^TP'AN - U'')X^{1/2} & \kappa X^{1/2}A^TP'AX \\ 0 & \beta \end{pmatrix} \tag{3.26}$$

with $y(t) = (y_r^{sub}(t), r \in R)$, $m(t) = (m_r(t), r \in R)$, $X = \mathrm{diag}(x_r^{sub}, r \in R)$, $N = \mathrm{diag}(n_r, r \in R)$, $P' = \mathrm{diag}(p'_j, j \in J)$, $U'' = \mathrm{diag}(U''_r, r \in R)$ and $\beta = \mathrm{diag}(\beta_r, r \in R)$.

Suppose L is diagonalizable (for example, when all its eigenvalues are distinct). Let $L = \Theta^{-1}\Psi\Theta$ where $\Psi = \mathrm{diag}(\psi_r, r \in R)$ is the matrix of eigenvalues of L and the column vectors of Θ^{-1} are the eigenvectors of L. Then

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ m(t) \end{pmatrix} = -\Theta^{-1}\Psi\Theta \begin{pmatrix} y(t) \\ m(t) \end{pmatrix} \tag{3.27}$$

and thus the rate of convergence is determined by the eigenvalue with the smallest real part, $\psi_r, r \in R$ of the matrix (3.26). Suppose each of the parameters κ, α and β are increased by some positive factor. Then the eigenvalues of the matrix (3.26) are multiplied by the same factor and hence the speed of convergence will be increased by equation (3.27). However, we expect that the presence of time lags will bound the increase of these parameters if stability is not to be compromised.

Stochastic analysis

We now consider a stochastic perturbation of the linearized equation (3.27). Let

$$d\begin{pmatrix} y(t) \\ m(t) \end{pmatrix} = -\left(\Theta^{-1}\Psi\Theta \begin{pmatrix} y(t) \\ m(t) \end{pmatrix} dt + M dB(t)\right) \tag{3.28}$$

where M is an arbitrary $2|R| \times |I|$ matrix and $B(t) = (B_i(t), i \in I)$ is a collection of independent standard Brownian motions, extended to $-\infty < t < \infty$.

The stationary solution to the system (3.28) is

$$\begin{pmatrix} y(t) \\ m(t) \end{pmatrix} = -\int_{-\infty}^{t} e^{-(t-\tau)\Theta^{-1}\Psi\Theta} M dB(\tau), \tag{3.29}$$

as can be checked by differentiating both sides of equation (3.29) with respect to t. The solution is a linear transformation of the Gaussian process $(B(\tau), \tau < t)$; hence

$(y(t), m(t))^T$ has a multivariate normal distribution, $(y(t), m(t))^T \sim N(0, \Sigma)$, where

$$\begin{aligned}\Sigma &= \mathbb{E}\Big[\begin{pmatrix} y(t) \\ m(t) \end{pmatrix}\begin{pmatrix} y(t) \\ m(t) \end{pmatrix}^T\Big] \\ &= \int_{-\infty}^{0} e^{\tau\Theta^{-1}\Psi\Theta} M M^T e^{\tau\Theta^T\Psi(\Theta^{-1})^T} d\tau \\ &= \Theta^{-1}\left[\int_{-\infty}^{0} e^{\tau\Psi}\Theta M M^T \Theta^T e^{\tau\Psi} d\tau\right](\Theta^{-1})^T.\end{aligned}$$

Define the symmetric matrix $[\Theta M; \Psi]$ by

$$\begin{aligned}[\Theta M; \Psi]_{rs} &= \left[\int_{-\infty}^{0} e^{\tau\Psi}\Theta M M^T \Theta^T e^{\tau\Psi} d\tau\right]_{rs} \\ &= \frac{[\Theta M M^T \Theta^T]_{rs}}{\psi_r + \psi_s}.\end{aligned}$$

Then

$$\Sigma = \Theta^{-1}[\Theta M; \Psi](\Theta^{-1})^T. \tag{3.30}$$

We next illustrate the above results with an example.

Poisson arrivals and departures with individual feedback. Consider the following stochastic version of the system (3.21)–(3.22). Let

$$dx_r^{sub}(t) = \kappa\left(x_r^{sub}(t)U_r'(x_r^{sub}(t))dt - \sum_{j\in J} A_{jr}\varepsilon_j dN_{jr}\Big(\int_0^t \frac{x_r^{sub}(\tau)\mu_j(\tau)}{\varepsilon_j}d\tau\Big)\right) \tag{3.31}$$

$$dn_r(t) = \eta\left(dN_r^1\Big(\eta^{-1}\alpha_r t\Big) - dN_r^2\Big(\eta^{-1}\beta_r\int_0^t n_r(\tau)d\tau\Big)\right) \tag{3.32}$$

where $(N_{jr}(\tau), \tau \geq 0)$, for $j \in J$, $r \in R$, $(N_r^1(\tau), \tau \geq 0)$ and $(N_r^2(\tau), \tau \geq 0)$, for $r \in R$, are collections of independent unit rate Poisson processes, $(\varepsilon_j, j \in J)$ and η are scaling parameters.

The above equations describe the following system. The arrival of sub-users on route r is a Poisson process of constant rate $\eta^{-1}\alpha_r$, while departures occur as a time-dependent Poisson process at rate $\eta^{-1}\beta_r n_r(t)$. Feedback signals from resource j to a sub-user on route r arise at rate $\varepsilon_j^{-1}x_r^{sub}(t)\mu_j(t)$; a sub-user reacts to such a feedback by reducing $x_r^{sub}(t)$ by an amount $\kappa\varepsilon_j$. The Brownian approximation of the above system, valid when ε_j and η are small, is

$$\begin{aligned} dx_r^{sub}(t) &= \kappa\Big(x_r^{sub}(t)U_r'(x_r^{sub}(t))dt - \sum_{j\in J} A_{jr}\Big(\mu_j(t)dt + (\varepsilon_j x_r^{sub}(t)\mu_j(t))^{1/2}dB_j(t)\Big)\Big) \\ dn_r(t) &= \Big(\alpha_r - \beta_r n_r(t)\Big)dt + \eta^{1/2}\Big(\alpha_r + \beta_r n_r(t)\Big)^{1/2} dB_r(t) \end{aligned}$$

where $(B_j(t), t \geq 0)$, for $j \in J$, and $(B_r(t), t \geq 0)$, for $r \in R$, are collections of independent standard Brownian motions.

The corresponding linearized system is just equation (3.28) where $B(t) = (B_i(t), i \in I)$, $|I| = |J| + |R|$, and M is a $2|R| \times |I|$ matrix given by

$$M = \begin{pmatrix} F & 0 \\ 0 & -G \end{pmatrix} \tag{3.33}$$

where $FF^T = \text{diag}(\kappa^2 \sum_{j\in J} A_{jr}\mu_j\varepsilon_j, r \in R)$, and $GG^T = \text{diag}(2\eta\alpha_r, r \in R)$. The stationary covariance matrix Σ may then be calculated using relation (3.30).

Example: A four-node network

We illustrate the methods and results of this section through a numerical example using a four-node network as shown in Figure 3.1 where $|J| = |R| = 4$. The set-up is as described in this section i.e. Poisson arrivals and departures with individual feedback. Calculations were done using *Mathematica*.

Suppose each resource has unit capacity and marks/charges the packet in service whenever the queue exceeds a threshold N. Then, using a heavy traffic approximation for an M/D/1 queue for the resource, we have

$$\mathbb{P}(Q > N) = e^{-2(1-y)N}$$

where Q, N and y are the queue length, threshold and traffic intensity respectively ([27], Section 7.2.3). Also, the feedback signal is directed at a random sub-user with probability x/y where x is the rate of that particular sub-user. The sub-user

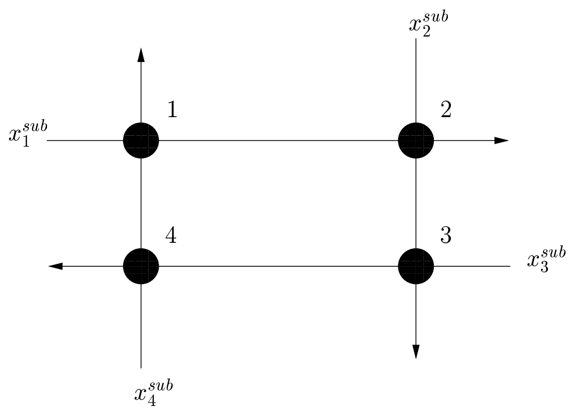


Figure 3.1: A four-node network.

who receives a feedback signal reduces its rate by κ. This model corresponds to equation (3.31) with $\varepsilon_j = 1$ and

$$p_j(y) = \frac{1}{y}e^{-2(1-y)N} \tag{3.34}$$

Let the arrival of sub-users on route r be a Poisson process of constant rate α_r, while departures occur as a time-dependent Poisson process at rate $\beta_r n_r(t)$. So this corresponds to equation (3.32) with $\eta = 1$. Choose the threshold $N = 6$ and let the fixed point be at $y = 0.8$, so that $p_j = 0.11$ and $p_j' = 1.2$ from (3.34), and suppose $-x_r^{sub}U_r'' = 2p_j$. The covariance matrix may then be calculated using relation (3.30). We wish to investigate the effect of varying β (convergence rate of the arrival/departure process), n (number of sub-users per route) and κ on the variances of the rates.

Figure 3.2 plots the coefficient of variation ([57], Section 7.4) of a sub-user's rate, defined as its standard deviation divided by its mean, $\sigma(x)/x$, versus $\log_{10}\beta$. We notice from the graphs that there are broadly 3 system regimes, depending on the relative speeds of convergence of the rate (or x) and arrival/departure (or n) processes. If β is small, corresponding to the n process being slow compared with the x process, then the stochastic effects due to the n process may dominate, especially when n is small. On the other hand, if β is large, the n process converges rapidly, as shown in equation (3.24), so that the stochastic effects due to the x process dominate (tending to the model with a fixed number of users). The third regime is when both processes interact because they have comparable speeds of convergence.

In the first graph (a), the number of sub-users for each route, $n = 20$, and $\kappa = 10^{-4}$. Then we see that when β is of comparable magnitude to κ, the two processes interact. To further illustrate the use of our methods, consider the following specific example. We take a slot to be of length $0.5\,ms$ and the mean rate $x = 0.02$. Let the call holding times be exponentially distributed with mean 200 seconds or 400,000 slots (about the length of a telephone call), so that $\beta = 2.5 \times 10^{-6}$ per unit time. These numbers have been chosen to be broadly comparable with those used in studies of both ATM and IP networks (see [58], Section 8.1.4). Then this corresponds to point (1) on graph (a), from which we can infer that the arrival/departure process of users is slow compared with the rate process so that they barely interact at all. In this case, since $n = 20$ is relatively small, its stochastic effects dominate and we see that $\sigma(x)/x \approx (2n)^{-0.5}$. If the holding times are reduced from 200 seconds to 0.2 seconds (assuming such adaptive fixed duration traffic exists), then the operating point shifts to point (2), so that the stochastic effects due to the x process now dominates.

In the second graph (b), we retain the same parameters as graph (a) but now the number of sub-users per route is $n = 200$. Then, since we we maintain the same traffic intensity, the mean rate is reduced to $x = 0.002$. The graph has broadly the same features as graph (a) except that it has been 'squashed' in the vertical axis. The maximum of $\sigma(x)/x$ has been reduced but its minimum has been increased. The reduction is due basically to the law of large numbers, since the stochastic effects from the n process is now smaller. The increase in the minimum is because the stochastic effects of the x process is larger compared with graph (a), since κ, the quantum of reduction for each congestion signal received, is now bigger relative to the reduced mean rate x.

In the third graph (c), we retain the same parameters as graph (a) but now $\kappa = 10^{-6}$. This has the effect of shifting the curve to the left and reducing the minimum of $\sigma(x)/x$. The left shift of about 2 units is due to the fact that the convergence of the x process is now slower since κ is smaller. The reduction in the minimum of $\sigma(x)/x$ is because the stochastic effects due to the x process has been scaled down. Recall from equation (2.22) that the covariance matrix scales linearly with κ. Hence, we expect the minimum to be reduced by a factor of 10, which is indeed so if we compare graphs (a) and (c).

3.4.2 A mix of users

We know that in practice, there will be a variety of user types and not all of them will stay in the network for just a fixed duration. For example, a typical class of users might be those who wish to perform a file transfer. Hence, in the next model, we consider a single resource with a mix of users, some of which are fixed duration users as in the previous section while others are file transfer users. Our aim is to show that the system exhibits good or bad behaviour depending on whether the number of fixed duration users is large or small.

Consider a single resource with unit capacity comprising a mix of user types, $n_1(t)$ fixed duration users and $n_2(t)$ file transfer users. We first model the system as a deterministic dynamical system using the following equations

$$\frac{d}{dt}n_1(t) = \alpha_1 - \beta_1 n_1(t) \tag{3.35}$$

$$\frac{d}{dt}n_2(t) = \alpha_2 - \frac{\beta_2 n_2(t)}{n_1(t) + n_2(t)} \tag{3.36}$$

The above equations may describe the following system. Fixed duration users arrive at a rate of α_1 and stay in the system for a duration of β_1^{-1}. File transfer users, with a file size of β_2^{-1}, arrive at rate α_2 and receive a service rate or bandwidth allocation of $(n_1(t)+n_2(t))^{-1}$. The latter corresponds to a proportional sharing of the resource capacity among $(n_1(t) + n_2(t))$ users. We note that the system (3.35)–(3.36) has a

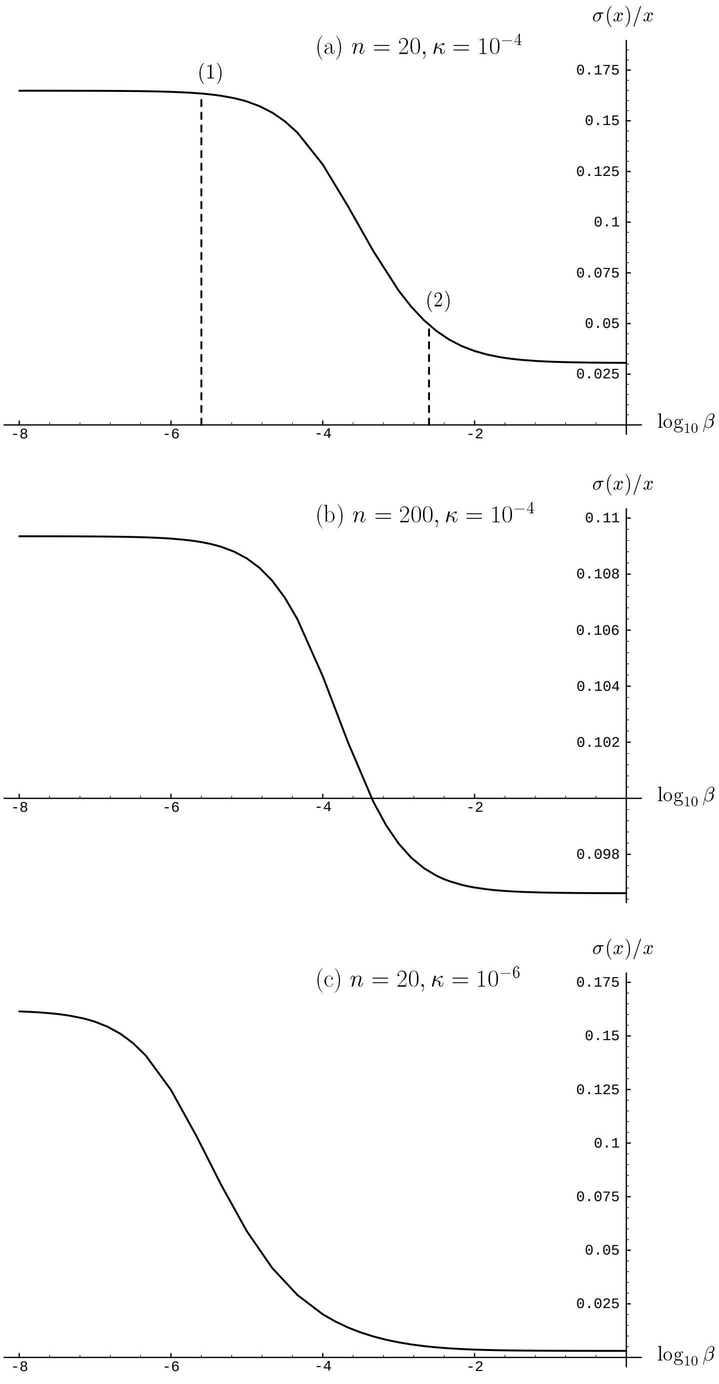


Figure 3.2: Graph of the coefficient of variation of a sub-user's rate, $\sigma(x)/x$, versus $\log_{10}\beta$ where β is the rate of convergence of the arrival/departure process. In (a), the equilibrium number of sub-users per route is $n = 20$ and $\kappa = 10^{-4}$. Points (1) and (2) correspond to traffic with holding times of 200s and 0.2s respectively. In (b), $n = 200$ and $\kappa = 10^{-4}$ while for (c), $n = 20$ and $\kappa = 10^{-6}$.

fixed point for $\alpha_2 \neq \beta_2$ given by

$$(n_1, n_2) = (\frac{\alpha_1}{\beta_1}, \frac{\alpha_1\alpha_2}{\beta_1(\beta_2 - \alpha_2)})$$

provided $\alpha_1, \beta_1 > 0$ and $\beta_2 > \alpha_2$.

The stability of the fixed point may be investigated through the linearization of the system (3.35)–(3.36). Using $n_1(t) = n_1 + m_1(t)$ and $n_2(t) = n_2 + m_2(t)$, after some reduction, we obtain the matrix equation

$$\frac{d}{dt}\begin{pmatrix} m_1(t) \\ m_2(t) \end{pmatrix} = -L \begin{pmatrix} m_1(t) \\ m_2(t) \end{pmatrix}$$

where

$$L = \begin{pmatrix} \beta_1 & 0 \\ -\frac{\beta_2 n_2}{(n_1+n_2)^2} & \frac{\beta_2 n_1}{(n_1+n_2)^2} \end{pmatrix} \tag{3.37}$$

which has positive eigenvalues since $n_1 > 0$. Hence, the fixed point is locally stable. We now consider a stochastic version of the system (3.35)–(3.36).

Poisson arrivals and departures. Suppose the arrival and departure of users are Poisson processes described by the following equations

$$\begin{aligned} dn_1(t) &= \varepsilon\left(dN_1(\varepsilon^{-1}\alpha_1 t) - dN_2(\varepsilon^{-1}\beta_1 \int_0^t n_1(\tau)d\tau)\right) \\ dn_2(t) &= \varepsilon\left(dN_3(\varepsilon^{-1}\alpha_2 t) - dN_4(\varepsilon^{-1}\beta_2 \int_0^t \frac{n_2(\tau)}{n_1(\tau)+n_2(\tau)}d\tau)\right) \end{aligned}$$

where $(N_i(\tau), \tau \geq 0)$ are collections of independent unit rate Poisson processes. Then, using a similar analysis to that of the previous section, the corresponding linearized system is just equation (3.28) where $B(t) = (B_i(t), i \in I)$, $|I| = 2$, with the matrices L given by (3.37) and M given by

$$M = \begin{pmatrix} (2\varepsilon\alpha_1)^{1/2} & 0 \\ 0 & (2\varepsilon\alpha_2)^{1/2} \end{pmatrix} \tag{3.38}$$

and hence the stationary covariance matrix may be calculated from relation (3.30). Without loss of generality, take $\beta_1 = 1$, so that the matrix L is given by

$$L = \begin{pmatrix} 1 & 0 \\ -(1-\rho)k & (1-\rho)^2 k/\rho \end{pmatrix}$$

where $\rho = \alpha_2/\beta_2$ is the traffic intensity of file transfer users and $k = \alpha_2/\alpha_1$ is the ratio of the arrival rates. Using relation (3.30), the covariance matrix, after some reduction, is given by

$$\Sigma = \varepsilon\alpha_1 \begin{pmatrix} 1 & \frac{(1-\rho)\rho k}{\rho+(1-\rho)^2 k} \\ \frac{(1-\rho)\rho k}{\rho+(1-\rho)^2 k} & \frac{\rho}{(1-\rho)^2} + \frac{\rho^2 k}{\rho+(1-\rho)^2 k} \end{pmatrix} \tag{3.39}$$

We note that the variance of $m_1(t)$, fixed duration users, is correctly given by $\varepsilon\alpha_1$ as expected. We now examine the variance of the service rate, $x(t) = (n_1(t) + n_2(t))^{-1}$, since this is the quantity of interest to the user. Using a Taylor series expansion, we have

$$\begin{aligned} \mathrm{Var}(x(t)) &= \mathrm{Var}\left(\frac{1}{(n_1 + n_2 + m_1(t) + m_2(t))}\right) \\ &= \mathrm{Var}\left(\frac{1}{(n_1+n_2)}(1 - \frac{m_1(t)}{(n_1+n_2)} - \frac{m_2(t)}{(n_1+n_2)}\right) \\ &= \frac{1}{(n_1+n_2)^4}\Big(\mathrm{Var}(m_1(t)) + \mathrm{Var}(m_2(t)) + 2\mathrm{Cov}(m_1(t), m_2(t))\Big) \end{aligned}$$

For good behaviour, we expect the coefficient of variation ([57], Section 7.4) of the service rate, $\sqrt{\mathrm{Var}(x(t))}/\mathbb{E}\mathrm{xp}(x(t)) = \sigma/\mu$, to be small. Intuitively, we expect the system to be more variable when the number of fixed duration users is small. This is due to the fact that the system does not have a fixed point as $\alpha_1 \to 0$, and our linearized model is thus expected to break down. Figure 3.3 plots, using *Mathematica*, the coefficient of variation, σ/μ, of the service rate for various values of α_1, the number of fixed duration users, and ρ, the traffic intensity of file transfer users with $\varepsilon = 1, k = 1$ and $\varepsilon = 1, k = 100$. We note that the quantity, σ/μ, is sensitive to α_1, the number of fixed duration users. Surprisingly, it is relatively insensitive to both ρ, the traffic intensity of file transfer users and to $k = \alpha_2/\alpha_1$, the ratio of the arrival rates. In particular, the system is well-behaved when α_1 is large and badly behaved when α_1 is small. We now show why this is expected.

Notice that

$$\frac{\rho^2 k}{\rho + (1-\rho)^2 k} < \frac{\rho^2}{(1-\rho)^2} < \frac{\rho}{(1-\rho)^2}$$

since $\rho < 1$. With $n_2 = \alpha_1\rho/(1-\rho)$, we obtain

$$\frac{\varepsilon n_2^2}{\alpha_1\rho} < \mathrm{Var}(m_2(t)) < \frac{2\varepsilon n_2^2}{\alpha_1\rho} \tag{3.40}$$

Then, using equations (3.39) and (3.40), we have

$$\frac{\sqrt{\mathrm{Var}(x(t))}}{\mathbb{E}\mathrm{xp}(x(t))} > \frac{\varepsilon^{1/2}}{(n_1+n_2)}\sqrt{n_1 + \frac{n_2^2}{\alpha_1\rho} + 2\mathrm{Cov}(m_1(t), m_2(t))} \tag{3.41}$$

The expression given by equation (3.41) blows up if $n_2 \gg n_1$ and $\alpha_1\rho \to 0$. Since $n_1 = \alpha_1$ and $n_2 = \alpha_1\rho/(1-\rho)$, the first condition means

$$\frac{\alpha_1\rho}{1-\rho} \gg \alpha_1 \quad \text{or} \quad \frac{\rho}{1-\rho} \gg 1$$

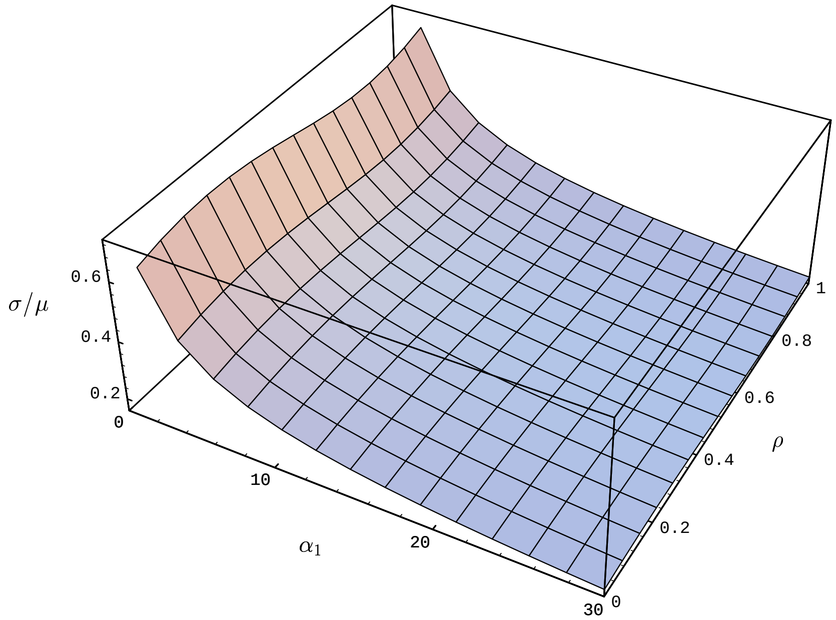
which gives $\rho \to 1$. Then the condition $\alpha_1\rho \to 0$ means that $\alpha_1 \to 0$. Thus, we expect the system to be more unstable when the number of fixed duration users is smaller.

On the other hand, from (3.40), we also have

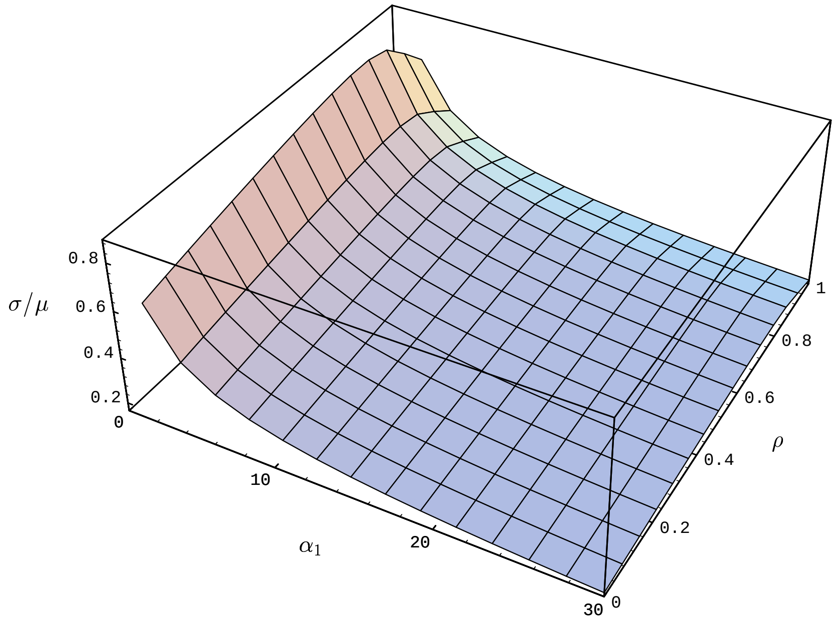
$$\begin{aligned}
\frac{\sqrt{\mathrm{Var}(x(t))}}{\mathbb{E}\mathrm{xp}(x(t))} &< \frac{\varepsilon^{1/2}}{(n_1+n_2)}\sqrt{n_1 + \frac{2n_2^2}{\alpha_1\rho} + 2\mathrm{Cov}(m_1(t), m_2(t))} \\
&< \frac{\varepsilon^{1/2}}{(n_1+n_2)}\sqrt{\frac{2(\rho n_1^2 + n_2^2 + 2n_1n_2)}{\alpha_1\rho} + 2\mathrm{Cov}(m_1(t), m_2(t))} \\
&\to \sqrt{\frac{2\varepsilon}{\alpha_1}}
\end{aligned}$$

as $\rho \to 1$. That is, we have good system behaviour if the number of fixed duration users, α_1, is large.

In summary, the results in this section show that adaptive fixed duration users are, in a sense, 'good traffic'.



(a)



(b)

Figure 3.3: Graphs of σ/μ of the service rate versus α_1, the number of fixed duration users, and ρ, the traffic intensity of file transfer users with (a) $\varepsilon = 1, k = 1$ and (b) $\varepsilon = 1, k = 100$.

Chapter 4

Time lags

In Chapters 2 and 3, we showed the global stability of simple additive increase and multiplicative decrease algorithms using a Lyapunov approach. It was noted that the speed of convergence increases with the gain parameter κ. However, in the presence of time lags, increasing κ may compromise the stability of the system. Recall that if the relevant parameters are known, the stability of the time lagged primal and dual algorithms are determined by the spectral radii of the matrices (2.38) and (2.43) respectively. Our aim in this chapter is to explore this issue of stability in greater depth, studying how the gain parameters relate to the time lags, price/feedback functions and the topology of the network.

The work of Chong et al. [12][13] is closely related to this chapter. In [12][13], Chong et al. analysed the equilibrium and stability of their proposed first-order rate-based flow control (FRFC) algorithm for a single bottleneck resource of multiple users with diverse round-trip delays. In the FRFC algorithm, the rate allocated to each user is the difference between the observed queue length and queue threshold, multiplied by a control gain. The FRFC equations are

$$\begin{aligned} x_r(t) &= \left(-\frac{\kappa_r}{n}(q(t) - q_T)\right)^+ \\ \dot{q}(t) &= \sum_{r=1}^{n} x_r(t - D_r) - C\,, \qquad q(t) > 0 \end{aligned}$$

$$= \ (\sum_{r=1}^{n} x_r(t - D_r) - C)^+ , \qquad q(t) = 0$$

where $x_r(t)$ is the rate allocated to user r, κ_r is the gain parameter, n is the number of users, $q(t)$ is the queue length at the resource, q_T is the queue threshold, D_r is the round-trip delay for user r, C is the capacity of the resource and $(\cdot)^+$ denotes $\max\{\cdot, 0\}$. In the context of this dissertation, the FRFC algorithm is similar to the dual algorithm if we view the observed queue length as a pseudo dual variable or shadow price. Our approach in this chapter is to use the primal algorithm which differs from the dual in various ways; in particular, for a self-clocking (see [33] or [24]) algorithm, each route of single or multiple resources may be characterized by a single round-trip delay. Also, for the dual algorithm, it may be difficult for the resource to know all the diverse round-trip delays of the users using it.

In this chapter, we will first derive the conditions for local stability for a sequence of simple networks using the primal rate control algorithm in discrete time. These include networks with single and multiple resources, and a large network with identical delays. Parallel theorems are then obtained via retarded differential equations. We will show how the rate of convergence depends on the gain parameter, with its maximum being the inverse of the round-trip delay, a similar result to that obtained by Chong et al. [12]. The case of a single route with multiple resources having diverse delays is then analysed. Lastly, we look at the joint feedback model and the impact of adaptive users.

An important issue in the modelling of communication networks is whether the results still hold as the size of the network grows. We shall see that in general, under certain conditions, the results in this chapter scale well with the size of the network as the time delays and number of resources grow. Important insight is obtained on the design of the price/feedback function at each resource. Also, the results indicate why self-clocking rate control algorithms are likely to work well to maintain stability.

4.1 Unresponsive cross-traffic

We first consider the simplest system, that of a single resource as shown in Figure 4.1. The horizontal user employs the following version of the primal algorithm

$$x[t+1] = x[t] + \kappa(w - x[t-D]\mu[t-d_2]) \tag{4.1}$$

where $\mu[t] = p(x[t-d_1]+v)$ and $D = d_1 + d_2$ is a non-negative integer. The vertical user, or cross-traffic, is unresponsive and equal to v for all time t.

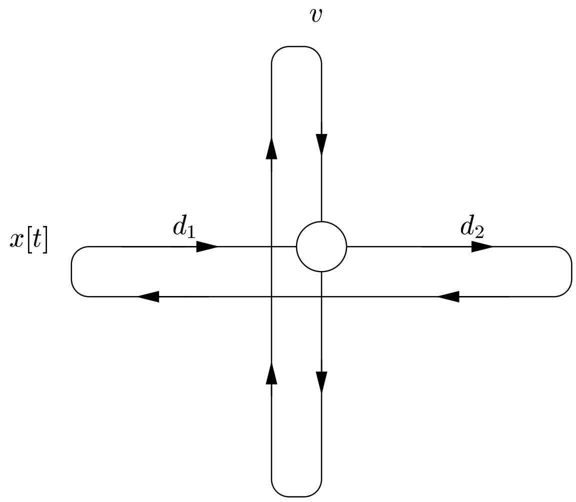


Figure 4.1: Single resource and unresponsive cross-traffic.

We may motivate the above algorithm as follows. Suppose that d_1 and d_2 are the forward and return delays (integer-valued) between the horizontal user and resource respectively. Also, suppose that the resource tags a charge $p(y)$, per unit flow through it, onto packets as they are served when the total load on the resource is y. Then the user is notified of the amount he is charged after a total round-trip delay D and adjusts his rate in order to reach a target value w. Let us term this method of charging or congestion indication as *delayed feedback*.

As $\kappa \to 0$, the system (4.1) may be approximated by equations (2.7)–(2.8), which has been shown to be globally stable. Our aim is to investigate the stability of the equilibrium point for general values of κ via its linearized version. Let the fixed point be (x, p) where $p = p(x + v)$ and $w = xp$, and suppose $p(\cdot)$ is differentiable at this point with derivative p'. Then linearizing with $x[t] = x + y[t]$, we obtain

$$\begin{aligned} y[t+1] &= y[t] + \kappa\left(w - (p + p'y[t-D])(x + y[t-D])\right) \\ &= y[t] - \kappa(p + xp')y[t-D] \end{aligned}$$

neglecting higher order terms. The corresponding characteristic equation is thus

$$\lambda^{D+1} - \lambda^{D} + \kappa(p + xp') = 0 \tag{4.2}$$

The fixed point is locally stable if the above equation has all roots with moduli less than unity ([45], Section 9.4). For each D, we are interested in the maximum value of κ such that the system is locally stable.

The next theorem, Theorem 10, will state the conditions which are essentially necessary and sufficient for the fixed point of the system (4.1) to be locally stable. The strategy of its proof is as follows. We first show that the maximum modulus of the roots of equation (4.2) is continuous in κ. Then we show that equation (4.2) has all roots with moduli less than unity for κ near the origin. Lastly we look for the smallest κ such that equation (4.2) has a root of unit modulus.

Lemma 1 *The maximum modulus of the roots of equation (4.2) is continuous in κ.*

Proof. Define

$$p_a(z) = z^{D+1} - z^{D} + a$$

Hence we need to prove that the maximum modulus of the roots of $p_a(z)$ is continuous in a. Fix ε. Assume first that the roots are distinct. Denote the roots of $p_a(z)$ by $\lambda_i, 1 \le i \le D+1$ and their maximum modulus by $m(a)$. By the Identity Theorem ([55], Section 5.14), the roots of a polynomial are isolated and so we can find $\varepsilon', 0 < \varepsilon' \le \varepsilon$ small enough such that all the $(D+1)$ closed discs, centred at λ_i, radius ε', are disjoint. Let

$$\delta_i = \min\{|p_a(z)| : z \text{ lies on the boundary of the disc centred at } \lambda_i.\}$$

Note that δ_i exists because $p_a(z)$, being a polynomial, is uniformly continuous on the boundary of the disc. Also $\delta_i > 0$ since $p_a(z)$ is non-zero except at λ_i, the centres of the discs. Define $\delta = \min_i\{\delta_i\} > 0$.

We make use of Rouché's theorem ([55], Section 7.7) from complex analysis which states: suppose f and g are analytic on and inside a regular closed curve, with $|f(z)| > |g(z)|$ for all points z lying on the curve. Then $f + g$ and f have the same number of zeros inside the closed curve.

For our case, let $f(z) = p_a(z)$ and $g = \delta$ for all z. Applying Rouché's theorem, we see that for each disc centred at λ_i, $p_a(z) + \delta$ has a zero inside the disc. But $p_a(z) + \delta = p_{a+\delta}(z)$. Thus the maximum modulus of the roots of $p_{a+\delta}(z)$ is clearly

$$m(a+\delta) \quad < \quad m(a) + \varepsilon'$$

The same argument may be used for $g = -\delta$ to obtain

$$m(a-\delta) \quad < \quad m(a) + \varepsilon'$$

Since the above is true for $|g| \leq \delta$, we have

$$|m(b) - m(a)| < \varepsilon \quad \text{if} \quad |b-a| < \delta\,,$$

that is, the maximum modulus of the roots of $p_a(z)$ is continuous in a. The proof is readily extended to the case of repeated roots since both the Identity and Rouché's theorems may still be applied. Thus, we have proven the lemma. □

Lemma 2 *For κ near the origin, equation (4.2) has all roots with moduli less than unity.*

Proof. Note that when $\kappa = 0$, equation (4.2) has roots $\lambda = 0$ of multiplicity D and $\lambda = 1$. By Lemma 1, the D roots initially at the origin remain within the unit circle for small κ. For the root initially at $\lambda = 1$, it moves inside the unit circle since it is approximately given by $\lambda = 1 - a$ where $a = \kappa(p + xp')$. Thus, for κ small enough, equation (4.2) has all roots with moduli less than unity. □

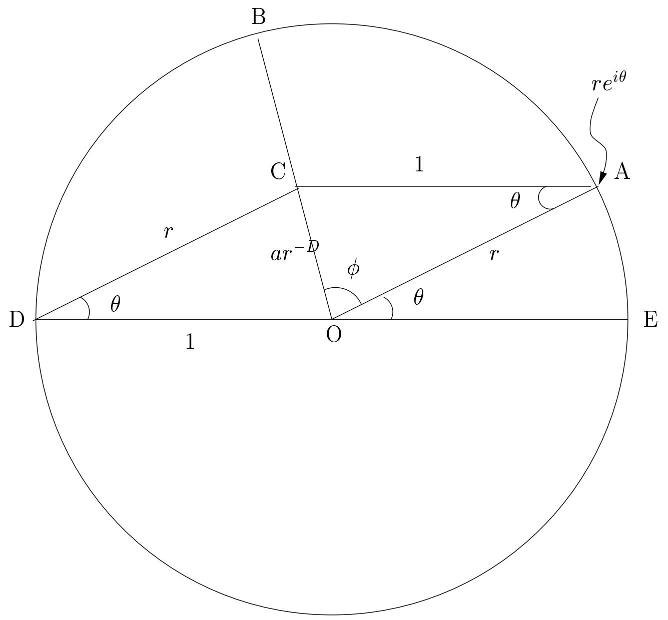


Figure 4.2: The characteristic equation as a map.

Theorem 10 *The system (4.1) is locally stable if*

$$\kappa(p + xp') \quad < \quad 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

and unstable if

$$\kappa(p + xp') \quad > \quad 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

Proof. We can easily check the result for $D = 0$ since the linearized system becomes

$$y[t+1] = (1 - \kappa(p + xp'))y[t]$$

so assume $D \geq 1$ for the rest of the proof. By Lemma 1, the maximum modulus of the roots of equation (4.2) varies continuously with κ. Also we know that the system is locally stable for small κ from Lemma 2. Hence, for the first part of the theorem, it suffices to look for the smallest κ such that equation (4.2) has a root of unit modulus. Let the root be $e^{i\theta}$ where $0 \le \theta \le \pi$ (this is sufficient since $-\pi \le \theta \le 0$ just gives the conjugate root). Considering the following form of equation (4.2)

$$\lambda = 1 - \frac{a}{\lambda^D} \tag{4.3}$$

where $a = \kappa(p + xp')$. This may be thought of as a map in the complex plane of the point $e^{i\theta}$ when a root of unit modulus exists, illustrated in Figure 4.2. Taking $r = 1$, the complex numbers represented by the points A, B and C are $\lambda = e^{i\theta}$, $-\lambda^{-D}$ and $-a\lambda^{-D}$ respectively. Thus, equation (4.3) is the map A→B→C→A. From Figure 4.2, we deduce that θ needs to satisfy the equations

$$a = 2\sin(\frac{\theta}{2}) \tag{4.4}$$

and

$$\begin{aligned} \phi = \quad & \pi - (D+1)\theta + 2n\pi \quad = \frac{\pi - \theta}{2} \\ \text{or} \quad & (2D+1)\theta = (4n+1)\pi \end{aligned} \tag{4.5}$$

where n is an integer. We choose $n = 0$ in (4.5) since we are looking for the smallest positive a such that λ is of unit modulus. Hence, substituting for θ in (4.4), we see that there are no solutions for θ if $a < 2\sin(\pi/(2(2D+1)))$. This gives the first part of the theorem.

For the second part of the theorem, it suffices to prove that there exists a root of modulus $r > 1$ to equation (4.2) when $a > 2\sin(\theta^*/2)$ where $\theta^* = \pi/(2D+1)$. From Figure 4.2, we deduce the following equations to be satisfied by r and θ using the cosine and sine rules.

$$\cos\theta = \frac{1 + r^2 - a^2 r^{-2D}}{2r} \tag{4.6}$$

$$r = \frac{\sin(D\theta)}{\sin(D+1)\theta} \tag{4.7}$$

We know that for $a = 2\sin(\theta^*/2)$, $\cos\theta^* = 1 - \frac{a^2}{2}$ and $r = 1$. Suppose $a > 2\sin(\theta^*/2)$. Then the RHS of (4.6) is less than $\cos\theta^*$ for $r = 1$. Notice that the LHS of (4.6) decreases from $\cos\theta^*$ to $\cos(\pi/(D+1))$ for $\theta \in [\theta^*, \frac{\pi}{D+1}]$. To prove that a root of modulus greater than unity exists, it suffices to show that r increases from 1 to ∞ for $\theta \in [\theta^*, \frac{\pi}{D+1}]$ and that the RHS of (4.6) is increasing in r for $r \geq 1$.

From equation (4.7),

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{D\sin((D+1)\theta)\cos(D\theta) - (D+1)\cos((D+1)\theta)\sin(D\theta)}{\sin^2((D+1)\theta)} \\ &> 0 \end{aligned}$$

since $\sin(D\theta) > \sin((D+1)\theta)$ and $-\cos((D+1)\theta) + \cos(D\theta) > 0$ for $\theta \in [\theta^*, \frac{\pi}{D+1}]$. So r increases from 1 to ∞ for $\theta \in [\theta^*, \frac{\pi}{D+1}]$.

Differentiating the RHS of (4.6) with respect to r gives

$$\frac{d}{dr}(4.6) = \frac{2(r^2-1) + (4D+2)a^2r^{-2D}}{4r^2}$$

which shows that the RHS of (4.6) is increasing in r for $r \geq 1$, tending to $r/2$ when r is large.

Thus a root of modulus greater than unity exists for which $\theta \in [\theta^*, \frac{\pi}{D+1}]$ when $a > 2\sin(\theta^*/2)$. This proves the second part of the theorem. $\square$

Theorem 10 essentially gives the necessary and sufficient conditions for local stability except when there are boundary eigenvalues for which a separate analysis is required ([45], Section 9.4). The result may also be found in the book by Marshall et al. ([48], Section 6.5) but a complete formal proof is not given. We have taken the trouble to prove Theorem 10 separately in the hope that it may provide insight into the more general problem of a network with many resources and diverse delays.

The geometrical proof using Figure 4.2 is certainly interesting. Indeed it is easy from Figure 4.2 to see why there are no roots of unit modulus for all $a < a^* = 2\sin(\theta^*/2)$ where $\theta^* = \pi/(2D+1)$. Suppose $a < a^*$. To look for a solution of unit modulus to the map (4.3), we might try to increase θ. But the length of OC needs to increase as well. Since $a < a^*$, there are no solutions for $\theta \geq \theta^*$. Also, for $0 < \theta < \theta^*$, $\arg(-a\lambda^{-D}) > \pi - D\theta^*$ which means it lies in sector BOD of Figure 4.2. Then, a translation of unit length cannot bring it back to $\lambda = e^{i\theta}$.

We note that since equation (4.1) is linear, it may be used to represent an aggregate number of users on the same route employing the same algorithm, in which case $w = \sum w_i$ and $x[t] = \sum x_i[t]$ where the sum is over all the users on the route.

We now consider an extension of the above result to a system comprising a route using multiple resources and unresponsive cross-traffic $v_j, j \in J$. Then the primal algorithm for the horizontal user is

$$x[t+1] = x[t] + \kappa\Big(w - \sum_{j\in J} x[t-D_j]\mu_j[t-d_2(j)]\Big) \tag{4.8}$$

where $\mu_j[t] = p_j(x[t-d_1(j)] + v_j)$ and $D_j = d_1(j) + d_2(j)$ is a non-negative integer. Similarly, $d_1(j)$ and $d_2(j)$ are the forward and return delays (integer-valued) between the user and resource j respectively. Let the fixed point be (x, p_j) where $p_j = p_j(x + v_j)$ and $w = x\sum_{j\in J} p_j$, and suppose $p_j(\cdot)$ is differentiable at this point with derivative p'_j. Then we may prove the following theorem for the case where the total round-trip delay is the same for all the resources.

Theorem 11 *Suppose $D_j = D$ for all $j \in J$. Then the system (4.8) is locally stable if*

$$\kappa(\sum_{j\in J} p_j)\left(1 + \frac{x\sum_{j\in J} p'_j}{\sum_{j\in J} p_j}\right) \quad < \quad 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

and unstable if

$$\kappa(\sum_{j\in J} p_j)\left(1 + \frac{x\sum_{j\in J} p'_j}{\sum_{j\in J} p_j}\right) \quad > \quad 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

Proof. Linearizing the system (4.8) as before gives the characteristic equation

$$\lambda^{D+1} - \lambda^D + \kappa\sum_{j\in J}(p_j + xp'_j) = 0 \tag{4.9}$$

and thus the result may be obtained using Theorem 10. □

A few remarks should be made on Theorems 10 and 11. Firstly, we note that if D is large, the condition in the theorems becomes approximately $\kappa D < a$ where a is some constant. More importantly, in the case of Theorem 11, we see that if $\sum_{j \in J} p_j$ is bounded and $p'_j / p_j \leq \beta$ for every resource, then the result scales well as the number of resources grow. We shall discuss this again later in the chapter. It is important to note that Theorem 11 is an easy extension of Theorem 10 because the horizontal route may be characterized by a single round-trip delay D (true for self-clocking algorithms) under the primal algorithm. This will not be true in the case of the dual algorithm. Lastly, it is easy to see that Theorem 11 also holds for a collection of disjoint routes.

Intuitively, we might expect that only the maximum delay matters if the network consists of a single route with many resources on its route having different round-trip delays. This is the purpose of the next theorem.

Lemma 3 *The maximum modulus of the roots of the equation*

$$\lambda^{D_{max}+1} - \lambda^{D_{max}} + \sum_{0 \leq i < D_{max}} a_i \lambda^i = 0 \tag{4.10}$$

is continuous in $a_i, 0 \leq i < D_{max}$ where $a_i \geq 0$, $a_0 > 0$ and D_{max} is a non-negative integer.

Proof. It is easy to see that a proof similar to that for Lemma 1 goes through except that we choose discs whose boundaries do not intersect the origin and $g = \delta z^i$ with δ small enough such that Rouché's theorem may be applied. □

Lemma 4 *For $a_i, 0 \leq i < D_{max}$ near the origin, equation (4.10) has all roots with moduli less than unity.*

Proof. Notice that if $a_i = 0, 0 < i < D_{max}$, this reduces to Lemma 2. Then applying Lemma 3 yields the result. □

Lemma 5 *Suppose*

$$\sum_{0\le i<D_{max}} a_i \quad < \quad a^* = 2\sin\left(\frac{\pi}{2(2D_{max}+1)}\right)$$

where $a_i \geq 0$, $a_0 > 0$ *and* D_{max} *is a non-negative integer. Then equation (4.10) has all roots of moduli less than unity.*

Proof. Consider the following form of equation (4.10)

$$\lambda = 1 - \sum_{0\le i<D_{max}} \frac{a_i}{\lambda^{(D_{max}-i)}} \tag{4.11}$$

which may be viewed as a map in the complex plane as before. We are interested in the existence of a solution of unit modulus i.e. $\lambda = e^{i\theta}$. If $a_i = 0$ for all $i > 0$, this reduces to the case in Theorem 10. So assume that at least one other a_i is strictly positive. Consider the second term on the RHS of equation (4.11). Now given the condition of the theorem,

$$\left|\sum_{0\le i<D_{max}} \frac{a_i}{\lambda^{(D_{max}-i)}}\right| \quad < \quad a^*$$

for all θ and so it is clear from Figure 4.2 that there are no solutions to the above map for $\theta \geq \theta^*$ where $\theta^* = \pi/(2D_{max}+1)$. For $\theta < \theta^*$,

$$\arg\left(\sum_{0\le i<D_{max}} \frac{a_i}{\lambda^{(D_{max}-i)}}\right) \quad > \quad (\pi - D_{max}\theta^*)$$

since it is a non-negative linear combination of complex numbers lying in sector BOD. Hence it lies in sector BOD of Figure 4.2. Then, a translation of unit length cannot bring it back to $\lambda = e^{i\theta}$. Thus, there are no roots of unit modulus to the above map. This gives the required result. □

Theorem 12 *Suppose* $D_{max} = \max\{D_j, j \in J\}$. *Then the system (4.8) is locally stable if*

$$\kappa(\sum_{j\in J} p_j)\left(1 + \frac{x\sum_{j\in J} p_j'}{\sum_{j\in J} p_j}\right) \quad < \quad 2\sin\left(\frac{\pi}{2(2D_{max}+1)}\right)$$

Proof. Linearizing the system (4.8) as before with $x[t] = x + y[t]$, we obtain

$$\begin{aligned} y[t+1] &= y[t] + \kappa\Big(w - \sum_{j\in J}(p_j + p_j' y[t-D_j])(x + y[t-D_j])\Big) \\ &= y[t] - \kappa \sum_{j\in J}(p_j + xp_j')y[t-D_j] \end{aligned}$$

where higher order terms are neglected. The corresponding characteristic equation is thus

$$\lambda^{D_{max}+1} - \lambda^{D_{max}} + \kappa \sum_{j\in J}(p_j + xp_j')\lambda^{D_{max}-D_j} = 0.$$

Then applying Lemma 5 yields the required result. □

Theorem 12 gives a fairly crude bound for κ which is sufficient for local stability. We shall derive a tighter bound for a similar network later in Section 4.7.1 using retarded differential equations.

4.2 Responsive cross-traffic

Consider the same system as shown in Figure 4.1 but now with responsive cross traffic. The system equations are then

$$x[t+1] = x[t] + \kappa\left(w - x[t-D]\mu[t-d_2]\right) \tag{4.12}$$

$$v[t] = \frac{w_v}{\mu[t]} \tag{4.13}$$

where $\mu[t] = p(x[t-d_1] + v[t])$ and $D = d_1 + d_2$ is a non-negative integer. In the above system, the horizontal user employs the same algorithm as before while the cross-traffic is responsive, adopting the dual algorithm described in Chapter 2. We assume that there is no time lag for the responsive user. (This may be viewed as an idealization of a system operating in discrete time; however we will derive a parallel result using continuous time retarded differential equations later in this chapter.) Let the fixed point be (x, v, p) where $p = p(x+v)$, $w = xp$ and $w_v = vp$. Then we have the following theorem.

Theorem 13 *The system (4.12)–(4.13) is locally stable if*

$$\kappa(p+xp')\left(1-\frac{(xp')(vp')}{(p+xp')(p+vp')}\right) \quad < \quad 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

and unstable if

$$\kappa(p+xp')\left(1-\frac{(xp')(vp')}{(p+xp')(p+vp')}\right) \quad > \quad 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

Proof. We linearize the system with $x[t] = x + y[t]$ and $v[t] = v + u[t]$. Equation (4.12) gives

$$y[t+1] = y[t] - \kappa((p+xp')y[t-D] + xp'u[t-d_2]) \tag{4.14}$$

From equation (4.13) we get

$$\begin{aligned} v + u[t] &= v(1 - \frac{p'}{p}(y[t-d_1] + u[t])) \\ \text{or} \qquad u[t] &= -\frac{vp'}{p+vp'}y[t-d_1] \end{aligned}$$

Substituting into (4.14) gives

$$y[t+1] = y[t] - \kappa\left((p+xp') - \frac{(xp')(vp')}{(p+vp')}\right)y[t-D] \tag{4.15}$$

Then applying Theorem 10 gives the result. □

We notice that the condition in Theorem 13 is less stringent than that of Theorem 10 since

$$\left(1-\frac{(xp')(vp')}{(p+xp')(p+vp')}\right) < 1$$

This is not surprising since essentially, the responsive cross traffic or vertical user damps the oscillations caused by the horizontal user. In other words, a system is more stable if there are users who back off quickly when the price/congestion goes up.

It is a straight forward extension to the case of a route with multiple resources having identical round-trip delays and responsive cross traffic. Suppose the system equations are

$$x[t+1] = x[t] + \kappa\left(w - \sum_{j\in J} x[t-D]\mu_j[t-d_2(j)]\right) \tag{4.16}$$

$$v_j[t] = \frac{w_{v_j}}{\mu_j[t]} \tag{4.17}$$

where $\mu_j[t] = p_j(x[t-d_1(j)] + v_j[t])$ and $D_j = d_1(j) + d_2(j)$ is a non-negative integer. Let the fixed point be (x, v_j, p_j) where $p_j = p_j(x + v_j)$, $w = x\sum_{j\in J} p_j$ and $w_{v_j} = v_j p_j$. Then the following theorem may be proved.

Theorem 14 *Suppose $D_j = D$ for all $j \in J$. Then the system (4.16)–(4.17) is locally stable if*

$$\kappa\left(\sum_{j\in J} p_j + x\sum_{j\in J} p_j' - \sum_{j\in J}\frac{(xp_j')(v_jp_j')}{(p_j + v_jp_j')}\right) < 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

and unstable if

$$\kappa\left(\sum_{j\in J} p_j + x\sum_{j\in J} p_j' - \sum_{j\in J}\frac{(xp_j')(v_jp_j')}{(p_j + v_jp_j')}\right) > 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

Proof. Use the same approach as Theorem 13. □

4.3 A large network with identical delays

Consider next the lagged, discrete time system

$$x_r[t+1] = x_r[t] + \kappa_r\left(w_r - \sum_{j\in J} A_{jr}x_r[t-D_r]\mu_j[t-d_2(j,r)]\right) \tag{4.18}$$

where

$$\mu_j[t] = p_j(\sum_{s\in R} A_{js}x_s[t-d_1(j,s)]), \tag{4.19}$$

where $d_i(j,r)$, $j \in J, r \in R$, are non-negative integers and $D_r = d_1(j,r) + d_2(j,r)$, for $j \in r$, is the characteristic round-trip delay for route r. This might correspond to an model of congestion indication with delayed feedback, where there is a delay of $d_2(j,r)$ between resource j generating a feedback signal and user r receiving it, and another delay of $d_1(j,r)$ between user r changing its rate and the altered flow reaching resource j. Say that a vector x is an *equilibrium point* of the system (4.18)–(4.19) if $x_r[t] = x_r$, for $t = \ldots, 0, 1, 2, \ldots$, satisfies these equations. In Chapter 2, it was shown in Theorem 3 that the unique x which maximizes $\mathcal{U}(x)$ given by (2.9) is an equilibrium point of the above system.

Let the vector $x = (x_r, r \in R)$ be the equilibrium point and $\mu_j = p_j[\sum_{s\in R} A_{js}x_s]$, and suppose p_j is differentiable at this point, with derivative p'_j. Let $x_r[t] = x_r + \kappa_r^{1/2} x_r^{1/2} y_r[t]$. Then, linearizing the system (4.18)–(4.19) about x, we obtain

$$\begin{aligned} y_r[t+1] &= y_r[t] - \kappa_r^{1/2}\Big(\sum_{j\in J} A_{jr}\mu_j\kappa_r^{1/2} y_r[t-D_r] + \\ &\quad x_r^{1/2}\sum_{j\in J} A_{jr}p'_j \sum_{s\in R} A_{js}x_s^{1/2}\kappa_s^{1/2} y_s[t-d_2(j,r)-d_1(j,s)]\Big) \\ &= y_r[t] - \kappa_r^{1/2}\Big(\kappa_r^{1/2} w_r x_r^{-1} y_r[t-D_r] + \\ &\quad \sum_{j\in J}\sum_{s\in R} p'_j A_{jr}A_{js} x_r^{1/2} x_s^{1/2}\kappa_s^{1/2} y_s[t-d_2(j,r)-d_1(j,s)]\Big) \end{aligned} \tag{4.20}$$

Suppose all the forward delays are equal to d_1 and all the return delays are equal to d_2, and $D = d_1 + d_2$. This might correspond to a network where the resources are all situated very close together. Define the vector $y[t] = (y_r[t], r \in R)$ and rewrite equation (4.20) in the matrix form

$$y[t+1] = y[t] - \kappa^{1/2}(WX^{-1} + X^{1/2}A^T P' A X^{1/2})\kappa^{1/2} y[t-D] \tag{4.21}$$

where $X = \mathrm{diag}(x_r, r \in R)$, $W = \mathrm{diag}(w_r, r \in R)$, $P' = \mathrm{diag}(p'_j, j \in J)$ and $\kappa = (\kappa_r, r \in R)$.

Let

$$\Gamma^T \Phi \Gamma = \kappa^{1/2}(WX^{-1} + X^{1/2}A^T P' A X^{1/2})\kappa^{1/2} \tag{4.22}$$

where Γ is an orthogonal matrix, $\Gamma^T\Gamma = I$, and $\Phi = \mathrm{diag}(\phi_r, r \in R)$ is the matrix of eigenvalues, necessarily positive, of the real, symmetric, positive definite matrix (4.22). Then, by multiplying both sides of equation (4.21) with Γ, we can decouple the system by an orthogonal transformation of coordinates into recursions of the form

$$z_r[t+1] = z_r[t] - \phi_r z[t-D] \tag{4.23}$$

where $z_r[t] = (z[t])_r = (\Gamma y[t])_r$. This leads to the following theorem.

Theorem 15 *Suppose all the forward delays are equal to d_1 and all the return delays are equal to d_2, and $D = d_1 + d_2$. Then the system (4.18)–(4.19) is locally stable if*

$$\left\|\kappa^{1/2} M \kappa^{1/2}\right\| \quad < \quad 2\sin\left(\frac{\pi}{2(2D+1)}\right) \tag{4.24}$$

and unstable if

$$\left\|\kappa^{1/2} M \kappa^{1/2}\right\| \quad > \quad 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

where $M = WX^{-1} + X^{1/2}A^T P' A X^{1/2}$ and $\|\cdot\|$ denotes spectral radius.

Proof. Applying Theorem 10 to equations (4.23), we get the condition

$$\phi_{max} < 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

where $\phi_{max} = \max\{\phi_r, r \in R\}$ is the maximum eigenvalue of the matrix (4.22). Since the spectral radius $||\kappa^{1/2} M \kappa^{1/2}|| = \phi_{max}$, we have the required result. □

Note that by similarity transformations, the condition (4.24) in Theorem 15 may be rewritten in various ways. For example, an equivalent condition is

$$\left\|\kappa(WX^{-1} + A^T P' A X)\right\| \quad < \quad 2\sin\left(\frac{\pi}{2(2D+1)}\right)$$

since

$$\begin{aligned}
\left\|\kappa^{1/2} M \kappa^{1/2}\right\| &= \left\|\kappa^{1/2}\kappa^{1/2} M \kappa^{1/2}\kappa^{-1/2}\right\| \\
&= \left\|X^{-1/2}\kappa M X^{1/2}\right\| \\
&= \left\|\kappa X^{-1/2} M X^{1/2}\right\|
\end{aligned}$$

This gives rise to the following theorem.

Theorem 16 *Suppose all the forward delays are equal to d_1 and all the return delays are equal to d_2, and $D = d_1 + d_2$. Then the system (4.18)–(4.19) is locally stable if for each $r \in R$*

$$\kappa_r(\sum_{j \in J} A_{jr} p_j) \left(1 + \frac{\sum_{j \in J} A_{jr} y_j p'_j}{\sum_{j \in J} A_{jr} p_j}\right) \quad < \quad 2 \sin\left(\frac{\pi}{2(2D+1)}\right) \tag{4.25}$$

where $y_j = \sum_{s \in R} A_{js} x_s$ is the equilibrium load on resource j.

Proof. Now, since the spectral radius of a matrix is bounded by the maximum absolute row sum ([32], Chapter 5), we obtain the required condition for each row r from the equivalent condition. □

4.4 Discussion

We now briefly discuss the significance of Theorems 10–16. Notice that if the relevant system parameters are known, stability bounds for the gain parameter κ may be calculated. Also, if the routes and system operating point are fixed, the condition for stability becomes $\kappa D < constant$. In a typical network where the round-trip delays may not be known, these results mean that *self-clocking* rate control algorithms will work well to maintain network stability. In a self-clocking algorithm, the sender uses an acknowledgement from the receiver to prompt a step forward. This gives the inverse dependence of the gain κ on the round-trip delay D. (For more details on self-clocking algorithms, please see [24] or [33].)

Notice that Theorem 16 allows for a decentralized implementation by route. More importantly, assuming $\sum_j p_j \leq \alpha$ and $y_j p'_j / p_j \leq \beta$ for all j, then the LHS of equation (4.25) is $\leq \kappa_r \alpha(1 + \beta)$ no matter how many resources lie on each route r i.e. the result is independent of routeing and scales well as the number of resources in the network grow.

Consider the uniform bound $y_j p'_j / p_j \leq \beta$ for every resource in the network. Integrating, this means

$$p(y) \leq b y^{\beta}$$

where b is some constant of integration. In other words, although we can have a decentralised implementation, there is a requirement for a kind of design parameter for the price/feedback function at every resource. Recall from equation (2.5) that p_j is the derivative of the cost function c_j, so that $y_j p_j'/p_j$ is just the elasticity of the cost function at y_j.

As an example, consider a resource j which marks the served packet when the queue exceeds a threshold N. Suppose we model this as an M/D/1 queue in heavy traffic ([27], Section 7.2.3). Then the function p_j is given by

$$p_j(y_j) = \frac{e^{-2N(1-y_j)}}{y_j}$$

and we want

$$\frac{y_j p_j'}{p_j} = 2Ny_j - 1 \quad < \quad \beta$$

Suppose $\sum_{j \in J} p_j < 1$. Then if the equilibrium traffic intensity y_j is known, the condition imposes a bound on N, the threshold used in the resource charging strategy.

4.5 Parallel results from retarded differential equations

In this section, we obtain parallel results of the preceding theorems by formulating the problem using retarded differential or differential-difference equations, the theory of which may be found in the books by Bellman and Cooke [2], Hale [29] and Stépán [61]. For convenience, we restate here a result due to Hayes [30] taken from Bellman and Cooke ([2], Theorem 13.8).

Theorem 17 (Hayes) *All the roots of $be^{\lambda} + c - \lambda e^{\lambda} = 0$, where b and c are real, have negative real parts if and only if*

(a) $b < 1$, and

(b) $b < -c < \sqrt{a_1^2 + b^2}$,

where a_1 is the root of $a = b \tan a$ such that $0 < a < \pi$. If $b = 0$, we take $a_1 = \pi/2$.

We start again with the simplest system, that of a single resource and unresponsive cross-traffic as shown in Figure 4.1. The horizontal user adopts the primal algorithm

$$\frac{d}{dt}x[t] = \kappa\left(w - x[t-D]\mu[t-d_2]\right) \tag{4.26}$$

where $\mu[t] = p(x[t-d_1]+v)$ and $D = d_1 + d_2$. The vertical user, or cross-traffic, is unresponsive and equal to v for all time t. Notice the similarity with equation (4.1).

We study the stability of this simple system via its linearized version ([2], Theorem 11.2). Let the fixed point be (x, p) where $p = p(x+v)$ and $w = xp$. Then linearizing with $x[t] = x + y[t]$, we obtain

$$\begin{aligned} \frac{d}{dt}y[t] &= \kappa\big(w - (x + y[t-D])(p + p'y[t-D])\big) \\ &= -\kappa(p + xp')y[t-D] \end{aligned}$$

neglecting higher order terms. The corresponding characteristic equation, obtained by substituting $y = e^{st}$, is thus

$$s = -\kappa(p + xp')e^{-sD}$$

which after substituting $\lambda = sD$, reduces to

$$-\kappa D(p + xp') - \lambda e^{\lambda} = 0 \tag{4.27}$$

The fixed point is locally stable if the above equation have all roots with negative real parts ([45], Section 9.4). For each D, we are interested in the maximum value of κ such that the system is locally stable.

Theorem 18 *The system (4.26) is locally stable if*

$$\kappa(p + xp') \quad < \quad \frac{\pi}{2D}$$

and unstable if

$$\kappa(p + xp') \quad > \quad \frac{\pi}{2D}$$

Proof. A direct application of Theorem 17 to equation (4.27) with $b = 0$ and $c = -\kappa D(p + xp')$. □

We can now obtain the parallel results of Theorems 11 and 13–16. For a system with multiple resources and unresponsive cross-traffic v_j, we have the primal algorithm for the horizontal user

$$\frac{d}{dt}x[t] = \kappa\Big(w - \sum_{j\in J} x[t - D_j]\mu_j[t - d_2(j)]\Big) \tag{4.28}$$

where $\mu_j[t] = p_j(x[t - d_1(j)] + v_j)$ and $D_j = d_1(j) + d_2(j)$. Let the fixed point be (x, p_j) where $p_j = p_j(x + v_j)$ and $w = x\sum_{j\in J} p_j$.

Theorem 19 *Suppose $D_j = D$ for all $j \in J$. Then the system (4.28) is locally stable if*

$$\kappa(\sum_{j\in J} p_j)\left(1 + \frac{x\sum_{j\in J} p'_j}{\sum_{j\in J} p_j}\right) \quad < \quad \frac{\pi}{2D}$$

and unstable if

$$\kappa(\sum_{j\in J} p_j)\left(1 + \frac{x\sum_{j\in J} p'_j}{\sum_{j\in J} p_j}\right) \quad > \quad \frac{\pi}{2D}$$

Proof. Linearize the system (4.28) to obtain the characteristic equation

$$-\kappa D\Big(\sum_{j\in J} p_j + x\sum_{j\in J} p'_j\Big) - \lambda e^{\lambda} = 0$$

and apply Theorem 17 to obtain the result. □

For a system with one resource and responsive cross-traffic, the equations for the horizontal and vertical users are

$$\frac{d}{dt}x[t] \quad = \quad \kappa\,(w - x[t - D]\mu[t - d_2]) \tag{4.29}$$

$$v[t] \quad = \quad \frac{w_v}{\mu[t]} \tag{4.30}$$

where $\mu[t] = p(x[t - d_1] + v[t])$ and $D = d_1 + d_2$. Let the fixed point be (x, v, p) where $p = p(x + v)$, $w = xp$ and $w_v = vp$. Then we have the following result.

Theorem 20 *The system (4.29)–(4.30) is locally stable if*

$$\kappa(p + xp')\left(1 - \frac{(xp')(vp')}{(p + xp')(p + vp')}\right) \;<\; \frac{\pi}{2D}$$

and unstable if

$$\kappa(p + xp')\left(1 - \frac{(xp')(vp')}{(p + xp')(p + vp')}\right) \;>\; \frac{\pi}{2D}$$

Proof. Using the approach of Theorem 13, linearize the system (4.29)–(4.30) and apply Theorem 17. □

For the case of a system with multiple resources and responsive cross-traffic, the equations for the horizontal and vertical users are

$$\begin{aligned} \frac{d}{dt}x[t] &= \kappa\Big(w - \sum_{j \in J} x[t-D]\mu_j[t - d_2(j)]\Big) && (4.31) \\ v_j[t] &= \frac{w_{v_j}}{\mu_j[t]} && (4.32) \end{aligned}$$

where $\mu_j[t] = p_j(x[t - d_1(j)] + v_j[t])$ and $D_j = d_1(j) + d_2(j)$ is a non-negative integer. Let the fixed point be (x, v_j, p_j) where $p_j = p_j(x + v_j)$, $w = x\sum_{j \in J} p_j$ and $w_{v_j} = v_j p_j$. Then the following theorem may be proved.

Theorem 21 *Suppose $D_j = D$ for all $j \in J$. The system (4.31)–(4.32) is locally stable if*

$$\kappa\left(\sum_{j \in J} p_j + x\sum_{j \in J} p'_j - \sum_{j \in J}\frac{(xp'_j)(v_j p'_j)}{(p_j + v_j p'_j)}\right) \;<\; \frac{\pi}{2D}$$

and unstable if

$$\kappa\left(\sum_{j \in J} p_j + x\sum_{j \in J} p'_j - \sum_{j \in J}\frac{(xp'_j)(v_j p'_j)}{(p_j + v_j p'_j)}\right) \;>\; \frac{\pi}{2D}$$

Proof. Using the approach of Theorem 13, linearize the system (4.31)–(4.32) and apply Theorem 17. □

In the case of a large network with forward and return delays, we have the system equations

$$\frac{d}{dt}x_r[t] = \kappa_r\left(w_r - \sum_{j\in J} A_{jr}x_r[t-D_r]\mu_j[t-d_2(j,r)]\right) \tag{4.33}$$

where

$$\mu_j[t] = p_j(\sum_{s\in R} A_{js}x_s[t-d_1(j,s)]), \tag{4.34}$$

and $D_r = d_1(j,r) + d_2(j,r), j \in r$.

Suppose all the forward delays are equal to d_1 and all the return delays are equal to d_2, and $D = d_1 + d_2$. Linearizing with $x_r[t] = x_r + \kappa_r^{1/2}x_r^{1/2}y_r[t]$, after some reduction, we obtain the matrix equation

$$\frac{d}{dt}y[t] = -\kappa^{1/2}(WX^{-1} + X^{1/2}A^TP'AX^{1/2})\kappa^{1/2}y[t-D] \tag{4.35}$$

where $y[t] = (y_r[t], r \in R)$ and $\kappa = (\kappa_r, r \in R)$.

As before, using the relation (4.22), we can decouple the system by an orthogonal transformation of coordinates into equations of the form

$$\frac{d}{dt}z_r[t] = -\phi_r z[t-D] \tag{4.36}$$

where $z_r[t] = (z[t])_r = (\Gamma y[t])_r$. This leads to the following theorem.

Theorem 22 *Suppose all the forward delays are equal to d_1 and all the return delays are equal to d_2, and $D = d_1 + d_2$. Then the system (4.33)–(4.34) is locally stable if*

$$\left\|\kappa^{1/2}M\kappa^{1/2}\right\| \quad < \quad \frac{\pi}{2D}$$

and unstable if

$$\left\|\kappa^{1/2}M\kappa^{1/2}\right\| \quad > \quad \frac{\pi}{2D}$$

where $M = WX^{-1} + X^{1/2}A^TP'AX^{1/2}$ and $\|\cdot\|$ denotes spectral radius.

Proof. Applying Theorem 18 to equations (4.36), we get the condition

$$\phi_{max} < \frac{\pi}{2D}$$

where $\phi_{max} = \max\{\phi_r, r \in R\}$ is the maximum eigenvalue of the matrix (4.22). Since the spectral radius $\|\kappa^{1/2} M \kappa^{1/2}\| = \phi_{max}$, we have the required result. □

Then, using a similarity transformation as before, an equivalent condition is

$$\left\| \kappa(WX^{-1} + A^T P' A X) \right\| \quad < \quad \frac{\pi}{2D} \tag{4.37}$$

and hence the following theorem.

Theorem 23 *Suppose all the forward delays are equal to d_1 and all the return delays are equal to d_2, and $D = d_1 + d_2$. Then the system (4.33)–(4.34) is locally stable if for each $r \in R$*

$$\kappa_r (\sum_{j \in J} A_{jr} p_j) \left(1 + \frac{\sum_{j \in J} A_{jr} y_j p'_j}{\sum_j A_{jr} p_j} \right) \quad < \quad \frac{\pi}{2D}$$

where $y_j = \sum_{s \in R} A_{js} x_s$ is the equilibrium load on resource j.

Proof. Now, since the spectral radius of a matrix is bounded by the maximum absolute row sum ([32], Chapter 5), we obtain the required condition for each row r from equation (4.37). □

Notice that the results from the retarded differential equation approach are just the corresponding discrete time theorems with $2\sin(\pi/(4D+2))$ replaced by $\pi/2D$. This is not surprising since the retarded differential equations may be obtained by taking an appropriate limit of the discrete time system. Suppose we rewrite equation (4.1) in the form

$$x[t + \frac{1}{2}] = x[t - \frac{1}{2}] + \kappa \left(w - x[(t - \frac{1}{2}) - (D - \frac{1}{2})] \mu[t - d_2] \right)$$

and let t and D take values in $\{n/2 : n \text{ odd}\}$. Then, setting $t' = t - \frac{1}{2}$ and replacing D by $(D - \frac{1}{2})$, the bound for Theorem 10 becomes $2\sin(\pi/4D)$, which tends to $\pi/2D$ as $D \to \infty$.

4.6 Critical damping and optimal rate of convergence

Formulated as retarded differential equations, the primal algorithm gives rise to a *damped* system. Using the analogy from linear ordinary differential equations (ODE) with constant real coefficients, we may discuss the concepts of under-, over- and critical damping ([8], Chapter 3). For a first order ODE, the characteristic equation has real roots only, while for a second order ODE, the possibility of complex roots arises. In the latter, under-damping/over-damping corresponds to the case when the eigenvalues are complex/real and critical damping occurs when the eigenvalues are real and equal. In contrast, we shall see that the possibility of complex roots arises even in a first order retarded differential equation. Consider the equation

$$\frac{d}{dt}y[t] = -ky[t-D] \tag{4.38}$$

where $k > 0$ and its characteristic equation, obtained by substituting $y = e^{st}$, is

$$se^{st} = -ke^{s(t-D)}$$

which reduces to

$$kD = -\lambda e^{\lambda} \tag{4.39}$$

where we require all its roots $\lambda = sD$ to have negative real parts for stability. Suppose the system (4.38) is stable. Let λ^* be the root of equation (4.39) with the smallest (in modulus) negative real part. Then the rate of convergence to the stable point is equal to $|\mathrm{Re}\lambda^*|D^{-1}$. We are interested in how the rate of convergence depends on k in the region of stability $(0, \pi(2D)^{-1})$.

Lemma 6 *The rate of convergence of the system (4.38) is monotonic decreasing from D^{-1} to 0 for $k \in [(eD)^{-1}, \pi(2D)^{-1})$.*

Proof. Let the root be $\lambda = -\gamma + \delta i$ where $\gamma > 0$ and $i = \sqrt{-1}$. Then from equation (4.39)

$$\begin{aligned} \gamma e^{-\gamma}\cos\delta + \delta e^{-\gamma}\sin\delta &= kD \\ \gamma e^{-\gamma}\sin\delta - \delta e^{-\gamma}\cos\delta &= 0 \end{aligned}$$

Notice that if (γ, δ) satisfies the above equations, then so does $(\gamma, -\delta)$. Hence, without loss of generality, we may assume that $\delta \geq 0$. The second equation gives

$$\gamma = \frac{\delta}{\tan \delta} \tag{4.40}$$

which on substituting into the other equation yields

$$\frac{\delta}{\sin \delta} e^{-\frac{\delta}{\tan \delta}} = kD \tag{4.41}$$

Observe that $\delta \in [2n\pi, (2n + \frac{1}{2})\pi]$ where $n \in \mathbb{Z}_+$ since $\tan \delta \geq 0$ and $\sin \delta \geq 0$ from equations (4.40)–(4.41). Also, from (4.40) γ is monotonic decreasing in δ from 1 to 0 for $\delta \in [0, \pi/2]$. Then the LHS of (4.41) is monotonic increasing in δ from e^{-1} to $\pi/2$ for $\delta \in [0, \pi/2]$. Hence, a root of equations (4.40)–(4.41) exists for $\delta \in [0, \pi/2]$ which is decreasing in γ as k increases from $(eD)^{-1}$ to $\pi(2D)^{-1}$. It remains to show that this is the root with the smallest γ. Notice that if $\delta / \sin \delta$ is larger, then γ has to be larger for equation (4.41) to be satisfied. But $\delta / \sin \delta \leq \pi/2$ for $n = 0$ and $\delta / \sin \delta > \pi/2$ for $n > 0$, so taking $n = 0$ yields the root with the smallest γ. Then, since the rate of convergence is γD^{-1}, we have the required result. □

Lemma 7 *The rate of convergence of the system (4.38) is monotonic increasing and convex from 0 to D^{-1} for $k \in (0, (eD)^{-1}]$.*

Proof. Consider the function on the RHS of equation (4.39) when λ is real and negative. Suppose we maximize the function with respect to λ. Now

$$\frac{d}{d\lambda}(-\lambda e^{\lambda}) = -e^{\lambda} - \lambda e^{\lambda},$$

which equals zero when $\lambda = -1$, giving a maximum of e^{-1}. Moreover, the RHS of (4.39) is increasing in λ for the interval $(-\infty, -1]$ and concave decreasing for $[-1, 0]$. The function is shown in Figure 4.3, from which it is clear that equation (4.39) has 2 negative real roots when $k < (eD)^{-1}$ and 1 negative root when $k = (eD)^{-1}$. Moreover, the modulus of the smaller root (in modulus) increases from 0 to 1 as k increases from 0 to $(eD)^{-1}$, and is clearly a convex function of k.

It remains to show that taking λ to be real yields the root with the smallest (in modulus) negative real part. First note that the modulus of the smaller real root is not greater than 1. Suppose there are complex roots for $k \leq (eD)^{-1}$. Then using the same argument in Lemma 6, we see that there are no roots for $n = 0$, $\delta \neq 0$, since the LHS of equation (4.41) is greater than e^{-1}. Also, for $n > 0$, $\delta / \sin \delta > \pi/2$, so the complex root, if any, will have $\gamma > 1$. This completes the proof. □

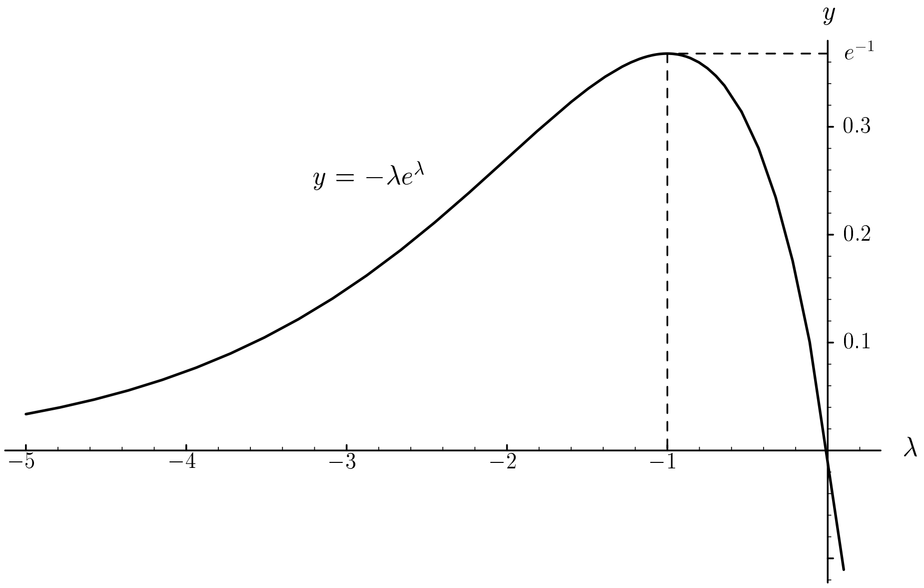


Figure 4.3: Plot of the function $y = -\lambda e^{\lambda}$.

We combine the above lemmas into the following theorem.

Theorem 24 *The maximum rate of convergence for the system (4.38) is D^{-1} when $k = (eD)^{-1}$. Moreover, the rate of convergence is monotonic increasing and convex from 0 to D^{-1} for $k \in (0, (eD)^{-1}]$ and monotonic decreasing from D^{-1} to 0 for $k \in [(eD)^{-1}, \pi(2D)^{-1})$.*

Proof. A straight deduction from Lemmas 6 and 7. □

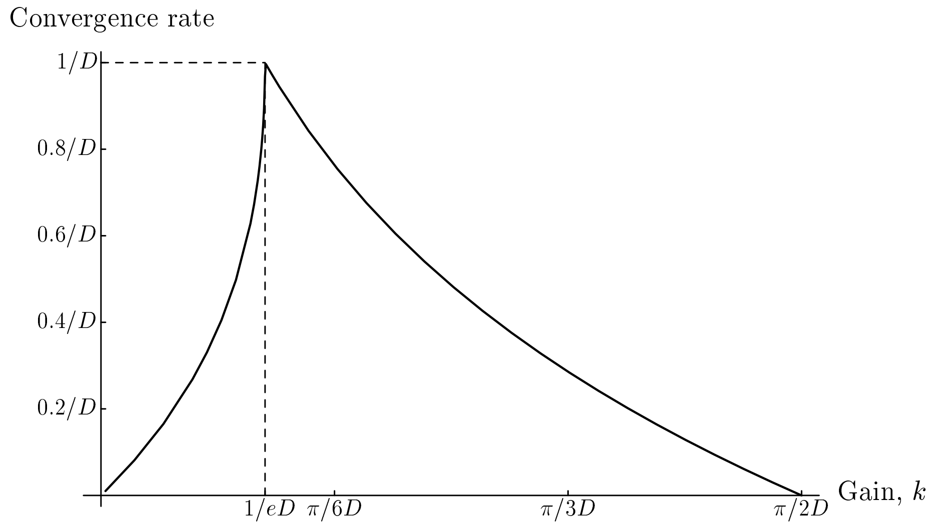


Figure 4.4: Rate of convergence versus gain.

Figure 4.4 illustrates the result of Theorem 24, showing how the rate of convergence for the system (4.38) varies with $k \in (0, \pi(2D)^{-1})$ in the region of stability. Using the analogy from ordinary differential equations, we see that the system is over-damped for $0 < k < (eD)^{-1}$ and under-damped for $(eD)^{-1} < k < \pi(2D)^{-1}$. The system is critically damped when $k = (eD)^{-1}$, and this also gives the optimal rate of convergence of D^{-1} or time constant of D. Intuitively, this makes sense since D is the round-trip delay.

The result of Theorem 24 has also been shown by Chong et al. [12] using essentially Theorem 17. However, their claim that the rate of convergence is a concave function of gain differs from what is depicted in Figure 4.4. In fact, by Lemma 7, the rate of convergence is convex for $k \in (0, (eD)^{-1}]$. We briefly sketch a second proof along similar lines to [12] for the result that $k = (eD)^{-1}$ gives the optimal rate of convergence of D^{-1}. Let $z = \lambda + \gamma$ where γ is a positive real number. Then rewrite the characteristic equation (4.39) as

$$\gamma e^z - kDe^\gamma - ze^z = 0 \tag{4.42}$$

We wish to find the supremum γ^* of positive real numbers γ such that the roots of equation (4.42), z, have negative real parts. Then the maximum rate of convergence for the system (4.38) is just $\gamma^* D^{-1}$. From Theorem 17 we have the conditions

$$\begin{aligned} \gamma &< 1 \qquad (4.43) \\ \gamma < kDe^{\gamma} &< \frac{a_1}{\sin a_1} \qquad (4.44) \end{aligned}$$

where a_1 is the root of $a = \gamma \tan a$ such that $0 < a < \pi$. The key observation from equation (4.44) is that as γ increases from 0 to 1, $a/\sin a$ decreases from $\pi/2$ to 1, giving equalities when $\gamma = 1$. Hence, the supremum is $\gamma^* = 1$, that is, the maximum rate of convergence is D^{-1} when $k = (eD)^{-1}$.

Applying Theorem 24 to our previous results in this chapter gives the following theorem.

Theorem 25 *The maximum rate of convergence of D^{-1} is achieved in systems (4.26), (4.28),(4.29)–(4.30) and (4.31)–(4.32) if and only if*

$$\begin{aligned} \kappa(p + xp') &= \frac{1}{eD} \\ \kappa(\sum_{j\in J} p_j)\left(1 + \frac{x\sum_{j\in J} p'_j}{\sum_{j\in J} p_j}\right) &= \frac{1}{eD} \\ \kappa(p + xp')\left(1 - \frac{(xp')(vp')}{(p + xp')(p + vp')}\right) &= \frac{1}{eD} \\ \kappa\left(\sum_{j\in J} p_j + x\sum_{j\in J} p'_j - \sum_{j\in J}\frac{(xp'_j)(v_j p'_j)}{(p_j + v_j p'_j)}\right) &= \frac{1}{eD} \end{aligned}$$

hold for each of them respectively.

Proof. A straight application of Theorem 24. □

In the case of a network, if $\|\kappa^{1/2} M \kappa^{1/2}\| = (eD)^{-1}$ in Theorem 22, then in the rotated frame, the coordinates with the maximum eigenvalue will be critically damped while the rest will be over-damped i.e. this may or may not be optimal. For the discrete time systems in Theorems 10, 11, 13 and 14, we would expect the bound $(eD)^{-1}$ to be approximately optimal for large D.

4.7 Diverse delays

4.7.1 A single route with diverse delays

Consider a single route with multiple resources where the round-trip delays from the user to each resource are different e.g this may arise in a multicasting context. Assume also that the cross-traffic at each resource is unresponsive and equal to $v_j, j \in J$. The horizontal user adopts the primal algorithm

$$\frac{d}{dt}x[t] = \sum_{j \in J} \kappa_j \left(w_j - x[t - D_j]\mu_j[t - d_2(j)]\right) \tag{4.45}$$

where $\mu_j[t] = p_j(x[t - d_1(j)] + v_j)$ and $D_j = d_1(j) + d_2(j)$. Let the fixed point be (x, p_j) where $p_j = p_j(x + v_j)$ and $w_j = xp_j$. Linearizing the system as before with $x[t] = x + y[t]$, we obtain the equation

$$\frac{d}{dt}y[t] = -\sum_{j \in J} \kappa_j(p_j + xp_j')y[t - D_j]$$

which has the characteristic function

$$H(\lambda) = \lambda + \sum_{j \in J} \kappa_j(p_j + xp_j')e^{-\lambda D_j} \tag{4.46}$$

For local stability, the equation $H(\lambda) = 0$ must have all roots with negative real parts. We will make use of the following theorem by Stépán ([61], Theorem 2.19), which we restate here for convenience of reference.

Theorem 26 (Stépán) *Consider the characteristic function given by*

$$H(\lambda) = \sum_{k=0}^{n} (-1)^k a_k(\lambda)\lambda^{n-k}$$

Let $\rho_1 \geq \ldots \geq \rho_r \geq 0$ *and* $\sigma_1 \geq \ldots \geq \sigma_s \geq 0$ *denote the non-negative real zeros of* R *and* S *respectively, where*

$$R(\omega) = \operatorname{Re} H(i\omega), \qquad S(\omega) = \operatorname{Im} H(i\omega).$$

The trivial solution $y = 0$ of the system is stable if and only if

$$n = 2m$$

$$S(\rho_k) \neq 0, \qquad k = 1, \ldots, r, \tag{4.47}$$

$$\textstyle\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = (-1)^m m; \tag{4.48}$$

or

$$n = 2m + 1$$

$$R(\sigma_k) \neq 0, \qquad k = 1, \ldots, s-1, \tag{4.49}$$

$$R(0) > 0 \tag{4.50}$$

$$\textstyle\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) + \frac{1}{2}((-1)^s + (-1)^m) + (-1)^m m = 0 \tag{4.51}$$

where m is integer and sgn denotes the sign function.

Then, using an approach similar to that by Chong et al. [13], we may prove the following result.

Theorem 27 *The system (4.45) is locally stable if*

$$\sum_{j \in J} \kappa_j D_j (p_j + x p_j') \quad < \quad 1 \tag{4.52}$$

In particular, this is true if for each resource j on the route

$$\kappa_j D_j \quad < \quad \kappa^*$$

where $\kappa^* \sum_{j \in J} (p_j + x p_j') = 1$.

Proof. From equation (4.46), we have $n = 1$ and $m = 0$. Also

$$\begin{aligned} R(\omega) &= \sum_{j \in J} \kappa_j (p_j + x p_j') \cos \omega D_j \\ S(\omega) &= \omega - \sum_{j \in J} \kappa_j (p_j + x p_j') \sin \omega D_j \end{aligned}$$

Obviously, the condition $R(0) > 0$, is satisfied. Also, we can estimate $S(\omega)$ for $\omega \in (0, \infty)$ as follows

$$S(\omega) > (1 - \sum_{j \in J} \kappa_j D_j (p_j + x p_j'))\omega$$

since $\sin \omega D_j < \omega D_j$ for $\omega \in (0, \infty)$. If the condition (4.52) in the theorem holds, then $S(\omega)$ has no real positive zeros since it is positive for $\omega \in (0, \infty)$. Hence, there is only one non-negative real zero at $\omega = 0$ so that conditions (4.49) and (4.51) are satisfied as well. Therefore, by Theorem 26, the system (4.45) is locally stable, proving the first part of the theorem. In particular, if $\kappa_j D_j < \kappa^*$, then

$$\begin{aligned} \sum_{j \in J} \kappa_j D_j (p_j + x p_j') &< \kappa^* \sum_{j \in J} (p_j + x p_j') \\ &= 1 \end{aligned}$$

This completes the proof. □

If we compare the bound of Theorem 27 with that of Theorem 19 (which is essentially necessary and sufficient), we see that they differ only by a factor of $\pi/2$ i.e. the bound is fairly tight.

4.7.2 Network with diverse delays - conjecture

What about a network with diverse delays? We notice from the preceding results, e.g. Theorems 16, 23 and 27, that κ scales linearly with the round-trip time. Thus, this gives rise to the following conjecture of sufficient conditions for the stability of a large-scale network with diverse delays.

Conjecture 1 *Suppose the round-trip delays, $D_r, r \in R$, are large. Then the system (4.18)–(4.19) is locally stable if for each $r \in R$*

$$\kappa_r \Big(\sum_{j \in J} A_{jr} p_j \Big) \left(1 + \frac{\sum_{j \in J} A_{jr} y_j p_j'}{\sum_{j \in J} A_{jr} p_j} \right) < \frac{\pi}{2 D_r} \qquad (4.53)$$

where $y_j = \sum_{s \in R} A_{js} x_s$ is the equilibrium load on resource j.

Notice the similarity of condition (4.53) with that of Theorems 16 and 23. Conjecture 1, if true, would allow a decentralised implementation for an arbitrarily large network. Assuming that $\sum_{j\in J} A_{jr}p_j$ is bounded and $y_j p'_j/p_j < \beta$ for every resource, then the condition scales well as the number of resources grow. Then, each route only needs to keep the product $\kappa_r D_r$ constant for network stability e.g. using a self-clocking rate control algorithm. We shall investigate this conjecture further through computer simulations in the next chapter.

4.8 Joint feedback

We now consider the following version of the primal algorithm

$$x[t+1] = x[t] + \kappa\,(w - x[t]\mu[t-d_2]) \tag{4.54}$$

where $\mu[t] = p(x[t-d_1])$ and $D = d_1 + d_2$ is a non-negative integer. Here, d_1 and d_2 are again the forward and return delays between the user and resource respectively. The above equation may be motivated by a system where the price/congestion signals are fedback to the source where a multiplicative decrease is applied to its current rate. Let us call this method of congestion indication *joint feedback*. Let the fixed point be (x,p). Then, proceeding as before, we linearize with $x[t] = x + y[t]$ and obtain

$$\begin{aligned} y[t+1] &= y[t] + \kappa\left(w - (p + p'y[t-D])(x + y[t])\right) \\ &= (1-\kappa p)y[t] - \kappa x p' y[t-D] \end{aligned}$$

which has the characteristic equation

$$\lambda^{D+1} - (1-\kappa p)\lambda^D + \kappa x p' = 0 \tag{4.55}$$

The fixed point is locally stable if the above equation have all roots with moduli less than unity ([45], Section 9.4). For each D, we are interested in the maximum value of κ such that the system is stable. Notice the similarity between equations (4.55) and (4.2). Intuitively, we would expect this system to be more stable than the system with delayed feedback. This is the purpose of the next theorem.

Theorem 28 *Suppose $\kappa < \kappa^*$ where κ^* is the bound given by Theorem 10. Then the system (4.54) is locally stable.*

Proof. Since Lemmas 3 and 4 applies to the characteristic equation (4.55), it suffices to prove that equation (4.55) has no roots with unit modulus for $\kappa < \kappa^*$. We may represent equation (4.55) by the following map in the complex plane, illustrated in Figure 4.5.

$$\lambda = (1 - \kappa p) - \frac{\kappa x p'}{\lambda^D} \tag{4.56}$$

Let $\theta^* = \pi/(2D+1)$ and $a^* = 2\sin(\theta^*/2)$. We wish to show that the above map has no solutions of the form $e^{i\theta}$ for all $\kappa < \kappa^*$. Using Figures 4.2 and 4.5, consider what happens when OC and CA are both reduced in length by $\kappa p < 1$. This requires a reduction in angle CAO as the cosine rule gives

$$\begin{aligned} \cos\theta &= \frac{1 + (1-\kappa p)^2 - (a^* - \kappa p)^2}{2(1-\kappa p)} \\ &= 1 - \frac{(a^*)^2}{2} + \frac{a^*\kappa p(2 - a^*)}{2(1-\kappa p)} \\ &> \cos\theta^* \end{aligned}$$

since $a^* < 2$. Thus, there are no solutions for $\theta \geq \theta^*$. As for $\theta < \theta^*$,

$$\arg(\frac{\kappa x p'}{\lambda^D}) \quad > \quad (\pi - D_{max}\theta^*)$$

which means that it lies in sector BOD of Figure 4.5. Then, a translation of length $(1 - \kappa p)$ cannot bring it back to $\lambda = e^{i\theta}$. Thus, there are no roots of unit modulus to equation (4.55) for $\kappa < \kappa^*$. This gives the required result. □

The next theorem gives a better bound for κ which is sufficient for local stability.

Theorem 29 *The system (4.54) is locally stable if $\kappa < \hat{\kappa}$ where $\hat{\kappa}$ is the unique positive solution to the set of equations*

$$\cos\theta = \frac{1 + (1-\kappa p)^2 - (\kappa x p')^2}{2(1-\kappa p)} \tag{4.57}$$

$$\cos\phi = \frac{1 + (\kappa x p')^2 - (1-\kappa p)^2}{2\kappa x p'} \tag{4.58}$$

$$\phi + \theta = \pi - D\theta \tag{4.59}$$

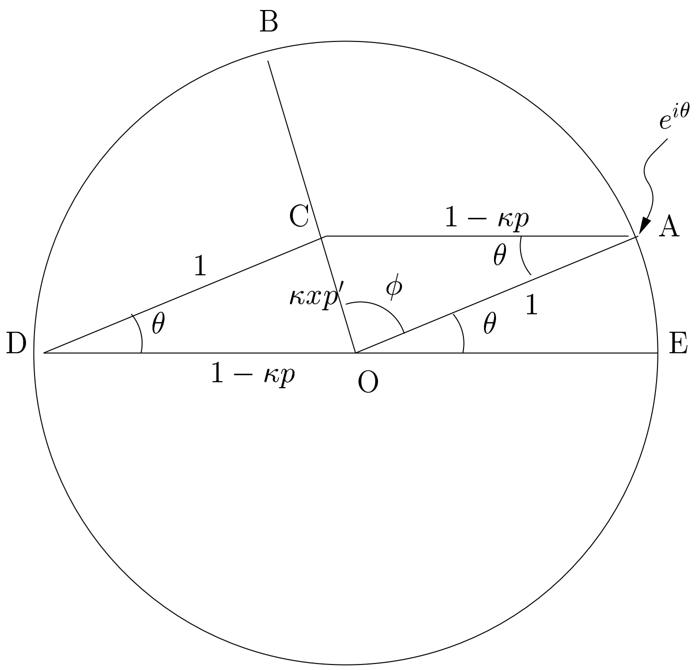


Figure 4.5: A modified map.

Proof. Using Figure 4.5, we obtain equations (4.57) and (4.58) straight away from the cosine rule. Equation (4.59) may be obtained by substituting $\lambda = e^{i\theta}$ into equation (4.56). The result follows. $\square$

If $xp' = \alpha p$ where $\alpha > 1$, then equations (4.57) and (4.58) become

$$\cos\theta = 1 - \frac{(\alpha^2-1)\kappa^2p^2}{2(1-\kappa p)}$$
$$\cos\phi = \frac{2+(\alpha^2-1)\kappa p}{2\alpha}$$

Substituting into equation (4.59) gives

$$(D+1)\cos^{-1}\left(1-\frac{(\alpha^2-1)\kappa^2p^2}{2(1-\kappa p)}\right) = \pi - \cos^{-1}\left(\frac{2+(\alpha^2-1)\kappa p}{2\alpha}\right)$$

which may be easily solved numerically for κ.

For a numerical example, suppose $x = 0.6$, $p = 0.05$, $p' = 0.2$ and $D = 10$. Then, for the system with delayed feedback (4.1), the critical gain calculated from

Theorem 10 is $\kappa_d = 0.88$. However, for the system with joint feedback (4.54), Theorem 29 gives the critical gain of $\kappa_j = 1.67$. This illustrates that joint feedback is more stable than delayed feedback.

4.9 Adaptive user

Thus far in this chapter, we have assumed that the w_r's are fixed. In this section, we discuss the situation when the users are adaptive and adopt accurate tracking as defined in Chapter 3, i.e. $w = xU'(x)$. We again consider the simplest system, that of a single route with a single resource, but now the primal algorithm is

$$x[t+1] = x[t] + \kappa \left(x[t]U'(x[t]) - x[t-D]\mu[t-d_2] \right) \tag{4.60}$$

where $\mu[t] = p(x[t-d_1])$ and $D = d_1 + d_2$ is a non-negative integer. Here, $U(\cdot)$ is the strictly concave utility function of the user.

We again study the stability of this simple system via its linearized version. Let the fixed point be (x, p) where $U'(x) = p$. Then linearizing with $x[t] = x + y[t]$, we obtain

$$\begin{aligned} y[t+1] &= \left(1 + \kappa(U'(x) + xU''(x))\right) y[t] - \kappa(p + xp')y[t-D] \\ &= \left(1 + \kappa p(1+\epsilon)\right) y[t] - \kappa(p + xp')y[t-D] \end{aligned} \tag{4.61}$$

where ϵ is defined as the elasticity of marginal utility with respect to consumption ([5], Chapter 2) at x. The corresponding characteristic equation is thus

$$\lambda^{D+1} - \left(1 + \kappa p(1+\epsilon)\right) \lambda^D + \kappa(p + xp') = 0 \tag{4.62}$$

The fixed point is locally stable if the above equation have all roots with moduli less than unity ([45], Section 9.4). For each D, we are interested in the maximum value of κ such that the system is locally stable. It is straight forward to obtain the following theorem using the approach of Section 4.8.

Theorem 30 *The system (4.60) is locally stable if* $\kappa < \hat{\kappa}$ *where* $\hat{\kappa}$ *is the unique positive solution to the set of equations*

$$\cos\theta = \frac{1 + (1 + \kappa p(1+\epsilon))^2 - (\kappa(p + xp'))^2}{2\,(1 + \kappa p(1+\epsilon))} \tag{4.63}$$

$$\cos\phi = \frac{1 + (\kappa(p + xp'))^2 - (1 + \kappa p(1+\epsilon))^2}{2\kappa(p + xp')} \tag{4.64}$$

$$\phi + \theta = \pi - D\theta \tag{4.65}$$

Proof. Use the approach of Theorem 29. □

From equation (4.62), we can see how the stability of the system is affected by the shape of the utility function $U(\cdot)$. Since $U(\cdot)$ is concave, the elasticity ϵ is non-positive. The case $\epsilon = -1$ corresponds to the logarithmic utility function. We note that the system is more stable than that for logarithmic utilities, i.e. the critical κ is larger, if $\epsilon < -1$ and less stable if $\epsilon > -1$. Recall that the elasticity is a measure of the consumer's sensitivity to price changes, the absolute value of ϵ large corresponding to price insensitivity.

More importantly, we note that the shape of the utility function is not a major factor for stability. Consider the worst case when $\varepsilon \to 0$, that is, when $U(\cdot)$ is almost linear. Now, suppose the logarithmic utility was used as an approximation, and we obtain a bound from Theorem 10, for example. Assuming the round-trip delay D is large, then $\kappa p \ll 1$, and hence we see from equations (4.62)–(4.65) that the logarithmic approximation is good.

Chapter 5

Examples

In the preceding chapters, we have developed our model of a communication network in order to study the issues of stability, rates of convergence, stochastic effects and time lags. An obvious question is how well our model of a network approximates that of a real system. Firstly, our model uses a *fluid flow approximation* instead of discrete data packets. Also, queues at the resources are not modelled at all. In this chapter, we will illustrate the use of the methods discussed earlier and compare the theoretical results with that obtained through simulations.

We will model the resource as an M/D/1 queue. We begin with some discussion on *sample path shadow prices*, using an approach similar to Gibbens and Kelly [24]. Simulation 1 shows the convergence to proportionally fair rates for small κ. Simulation 2 illustrates the result of Theorems 10 and 18. We also discuss the concept of *lag instability* versus *stochastic instability*. Simulation 3 shows the link between lag instability and the phenomenon of *phase-locking*, discussed by Floyd and Jacobson [20]. Simulation 4 investigates Conjecture 1 for a linear network, yielding results which support its claim.

5.1 An M/D/1 queue with delay cost marking

Consider the following model of a resource. Suppose on a fine time-scale, each user generates Poisson traffic independent of one another so that the arrival of packets at the resource is Poisson, and service times are all deterministic and equal to 1.

Consider next the charging/marking scheme. Suppose the network costs incurred comes mainly from the delays experienced by the data packets. Then using the Pollaczek-Khintchine formula ([27], Section 5.1.1) applied to an M/D/1 queue, the cost function is given by

$$\begin{aligned} c(y) &= L_q \\ &= \frac{y^2}{2(1-y)} \end{aligned}$$

where L_q is the mean queue size and y is the traffic intensity. Then from equation (2.5),

$$\begin{aligned} p(y) &= \frac{d}{dy}c(y) \\ &= \frac{1}{2}\left(\frac{1}{(1-y)^2} - 1\right) \qquad (5.1) \end{aligned}$$

How should the resource charge in order to achieve the correct expected charge given by equation (5.1)? Consider the *sample path shadow price* [24] of a packet in any busy period. If there are no other packets in the queue when a packet is being served, then the shadow price is zero since it does not cause any delay to other users. In general, if there are m more packets which will be served before the end of the busy period, the removal of the packet being served will improve the total delay by m, and hence m is precisely the shadow price for the served packet. Since it is not possible to know the number of packets to the end of the busy period, we adopt the following scheme. Suppose the resource charges each packet T where $T+1$ is the time from the start of the busy period. Then, since the distribution of a random packet is uniform on the busy period, the expected charge is equal to the sample path shadow price.

We now verify that the expected charge is equal to that given by equation (5.1). For an M/D/1 queue, the length of the busy period is given by the Borel-Tanner distribution ([34], Section 10.7). For this distribution

$$\begin{aligned} \mathbb{E}(x) &= (1-y)^{-1} \\ \mathrm{Var}(x) &= y(1-y)^{-3} \end{aligned}$$

where y is the traffic intensity. Let π_x be the Borel-Tanner distribution. Then the expected charge is

$$\begin{aligned} \mathbb{E}(T) &= \frac{\sum_{x=1}^{\infty} x\pi_x \mathbb{E}(T|x)}{\sum_{x=1}^{\infty} x\pi_x} \\ &= \frac{\sum_{x=1}^{\infty} \pi_x x(\frac{x-1}{2})}{\sum_{x=1}^{\infty} x\pi_x} \\ &= \frac{\mathbb{E}(x^2) - \mathbb{E}(x)}{2\mathbb{E}(x)} \\ &= \frac{y(1-y)^{-3} + (1-y)^{-2} - (1-y)^{-1}}{2(1-y)^{-1}} \\ &= \frac{1}{2}(\frac{1}{(1-y)^2} - 1) \end{aligned}$$

as required. Note that we have taken into account the fact that it is more likely to find a packet in a longer busy period.

For a single resource, each user employs the following primal algorithm

$$x_r[t+1] = x_r[t] + \kappa_r\left(w_r - \varepsilon x_r[t-D_r]\mu[t-d_2(r)]\right) \tag{5.2}$$

where $\mu[t] = p(\sum_{r\in R} x_r[t-d_1(r)])$ with $p(y)$ given by equation (5.1). Also, $D_r = d_1(r) + d_2(r)$ with $d_1(r)$ and $d_2(r)$ being the forward and return delays between user r and the resource.

We now describe the essential details of the computer simulation of the M/D/1 queue with delay cost marking. At each iteration t, service, marking and arrivals happen at the resource according to the algorithm

```
if (qlength[t]>0)
  {
```

```
      feedback[queue[1]][t] = marks;    %charge the served packet
      marks = marks+1;                  %increase charge
      qlength[t+1] = qlength[t]-1;      %decrease queue
      if (qlength[t+1]==0) marks=0;     %reset charge if queue empty
    }
  for (r=0;r<R;++r) total_arrival+=arrival[r][t-d[r][1]];
  qlength[t+1]=qlength[t+1]+total_arrival                %arrivals
```

Also, for each iteration t, each user transmits a Poisson number of packets and updates his rate according to the algorithm

```
  if (x[r]>0.0) arrival[r][t]=poisson(x[r]);    %Poisson traffic
  else  arrival[r][t]=0;
  x[r] = x[r] + (kappa[r]*                      %primal algorithm
         (W[r]-EPS*feedback[r][t-D[r]]));
```

5.1.1 Simulation 1: Convergence to proportionally fair rates

The purpose of this simulation is to show that if the gain parameter κ is small enough, the primal algorithm (5.2) converges to rates which are proportionally fair as shown in the preceding chapters. We simulate a single route with a single resource used by 10 users, with 5 users having $w_r = 0.005$ and 5 users having $w_r = 0.015$. The other parameters are $\varepsilon = 0.025$, $D_r = 1000$ and $\kappa = 0.0001$. Using the equation $\sum_r w_r = \varepsilon y p(y)$, the fixed point is calculated to be $(0.036, 0.106)$. The rates of the users are all initialized at 0.071. The result of the simulation is plotted in Figure 5.1.

The 5 users with $w_r = 0.015$ increases their rate, tending towards the predicted equilibrium allocation of 0.106 while the other 5 users with the smaller $w's$ move towards the lower allocation of 0.036. This illustrates the convergence to proportionally fair rates for small κ. Another point worth noting is that the convergence time is of the order of the round-trip delay. This supports the result of Theorem 24 in Section 4.6 where we showed that the maximum rate of convergence is the inverse of the round-trip delay.

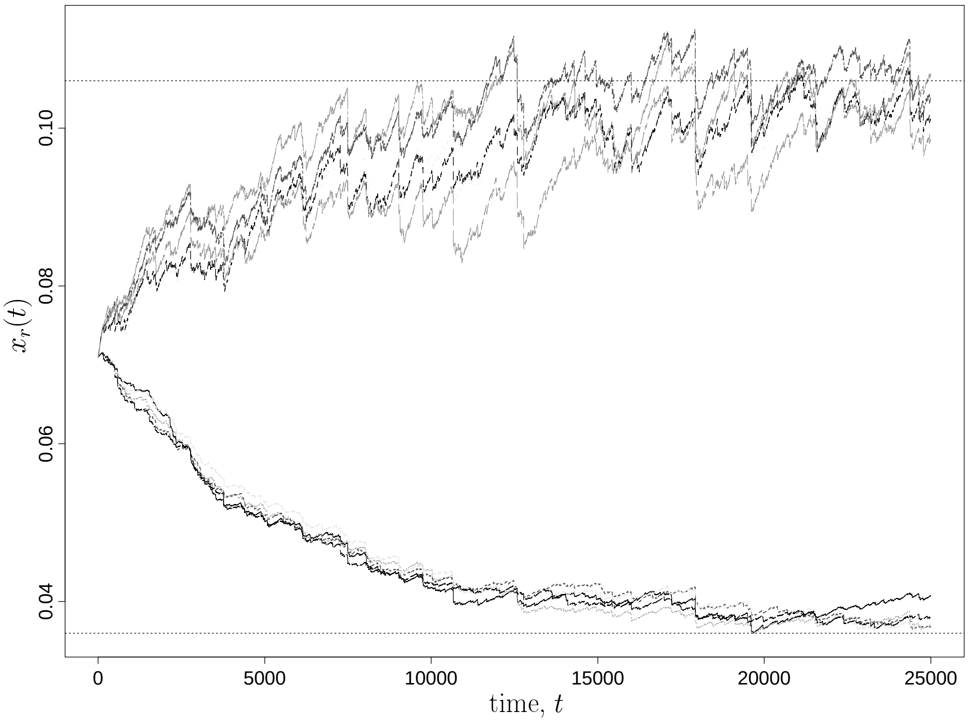


Figure 5.1: Convergence to proportionally fair rates for small κ.

5.1.2 Simulation 2: Relationship between critical gain and delay

We wish to illustrate the result obtained in Theorems 10 and 18. Note that if the other network parameters are fixed, then from Theorem 10, $\kappa^* D$ tends to a constant as $D \to \infty$ where κ^* denote the critical gain for local stability. In a real system i.e. non-linearized, the rates do not diverge for $\kappa > \kappa^*$ as the function $p(y)$ given by equation (5.1) tends to infinity as y tends to unity. Hence, we need to define stability of the fixed point in another way. For our purposes, we shall define κ^* as the κ when $\sigma = 0.1x$, that is, when the coefficient of variation ([57], Section 7.4) of a rate is 0.1.

Using the above definition, instability may also be caused by stochastic effects

RTT	κ_t^* (theory)	κ_s^* (simulation)
100	0.017	0.001
1000	0.0017	0.0006
5000	0.00035	0.00026
10000	0.00017	0.00015
100000	0.000017	0.000017

Table 5.1: Critical κ - theory vs simulation.

in the network. Let us term this as *stochastic instability* as opposed to *lag instability* which is due to the delays in the network. Recall that for small κ, equation (2.22) tells us that the variance of the rates scale linearly with κ. We shall see that stochastic effects is the dominant factor causing instability for short round-trip delays when the critical gain due to lag instability, predicted by Theorems 10 and 18, is larger. For long round-trip delays, the critical gain (based on Theorems 10 and 18) required for local stability is much smaller and hence this becomes the dominant factor causing instability since stochastic effects are negligible in this regime.

We simulate a single route with a single resource used by 10 users. The parameters are $w_r = 0.01$, $\varepsilon = 0.025$ and round-trip times (RTT) of 10^2 to 10^5. The gain κ is varied for each RTT and the variance of the rates at equilibrium is obtained from the simulation. The results are summarised in Table 5.1 and Figure 5.2.

Table 5.1 compares the theoretical value of the critical gain predicted by Theorem 10, κ_t^*, with κ_s^*, the critical gain obtained by simulation. It is clear from the table that the theoretical predictions are very good for long RTT but not for short RTT due to reasons discussed earlier. For RTT $= 100$, κ_t^* and κ_s^* differ by an order of magnitude. This is due to the dominance of stochastic effects. Then for RTT $= 1000$, the theoretical prediction is more accurate, differing from simulation by less than a factor of 3. In this regime, both stochastic effects and time lags are contributing to the system's instability. For RTT $= 10^4$ and 10^5, κ_t^* and κ_s^* are

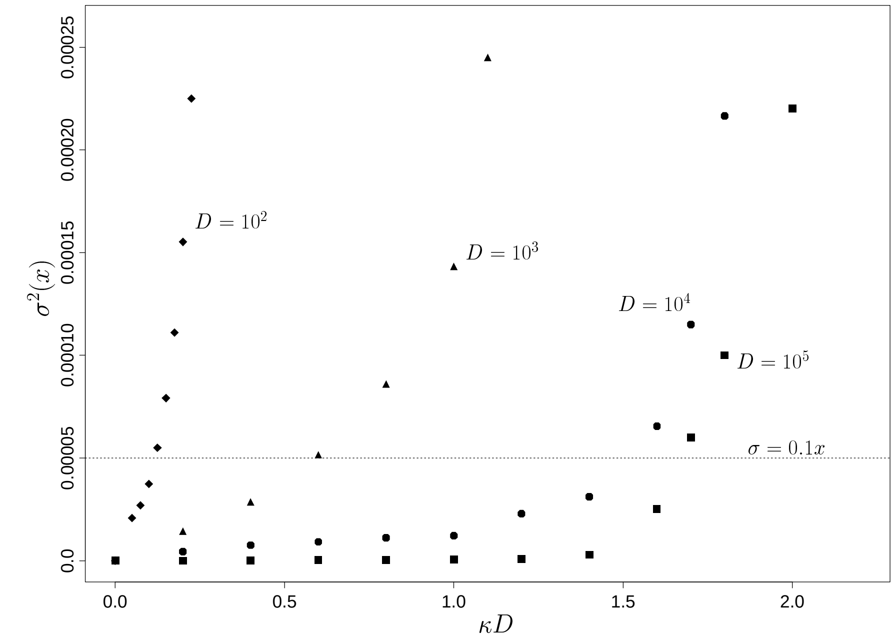


Figure 5.2: Relationship between variance and κD.

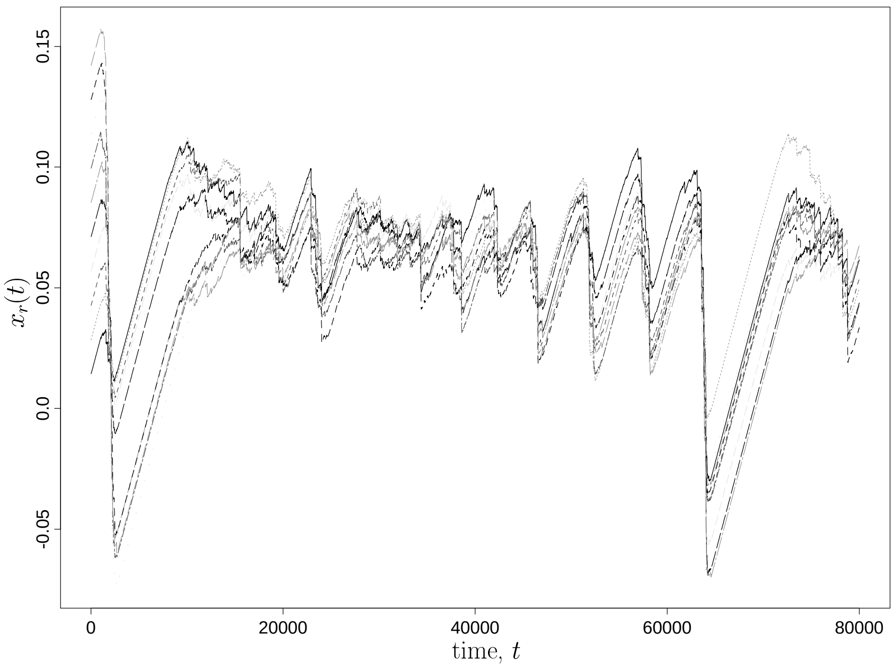


Figure 5.3: Phase locking of rates due to lag instability.

about the same, showing that lag instability dominate in this regime.

Figure 5.2 plots the variance of the rates σ^2 versus κD at equilibrium for round-trip times of 10^2, 10^3, 10^4 and 10^5. From our theory in Chapter 4, we expect the system to become unstable when $\kappa D = 1.7$ for these round-trip times i.e. the variance of the rates should diverge at $\kappa D = 1.7$. Here, we see again that for short RTT, this is not true. In this regime, the stochastic effects is the main cause of instability due to the larger gain parameter κ, the effect of which is clear from equation (2.22). For longer RTT, as lag instability becomes the dominant factor, we see that the coefficient of variation exceeds 0.1 (or $\sigma = 0.1x$) when κD is about 1.7 as predicted.

5.1.3 Simulation 3: Lag instability and phase-locking.

We wish to have a better understanding about what happens when the system becomes unstable due to time lags. For this simulation, we again have a single route with a single resource with 10 users. The parameters are $w_r = 0.01$, $\varepsilon = 0.025$, $D = 1000$ and $\kappa = 0.0015$, so that the system is almost *lag unstable*. The equilibrium point is calculated to be $x_r = 0.071$ for all the users. We initialize the rates of the users at $x_r = 0.0142r, 1 \leq r \leq 10$. The result of the simulation is plotted in Figure 5.3. We see that although the rates of the users are different initially, with some higher and some lower than the equilibrium allocation, they eventually become *phase locked* [20] in this regime, producing large oscillations in the system.

We can understand the influence of time delays in a system easily if we consider the deterministic system (this is a good approximation when the RTT is large and the gain parameter κ is small). As the gain parameter increases, the users' rates are changing faster so that the charging/congestion information from the resource received by the users becomes more and more outdated due to the delays. Instability results when the users are changing their rates wrongly, either increasing their rates when the resource is congested or decreasing when the resource is non-congested. This manifests itself as large oscillations in the rates as shown in Figure 5.3.

5.1.4 Simulation 4: A linear network

In this simulation, we wish to investigate Conjecture 1. We simulate a linear network which has n resources as shown in Figure 5.4 for $n = 4$.

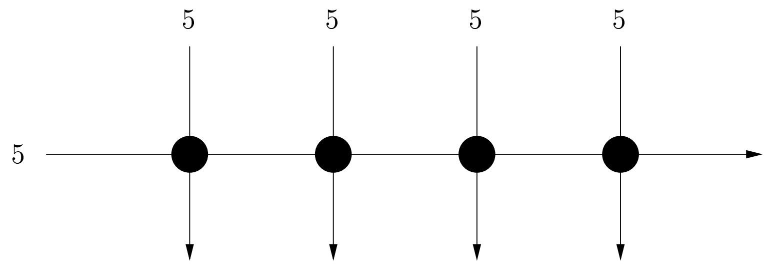


Figure 5.4: A linear network ($n = 4$).

Each resource has a vertical route with 5 users which uses that particular resource only. In addition, there is a horizontal route with 5 users which uses all the resources. The one-way delay is 2500 per hop so that the RTT is 5000 for the vertical routes and $5000n$ for the horizontal route. We scale w appropriately for the horizontal route to keep the fixed point at $x = 0.071$ for all users. The gains for the vertical routes are kept constant at $\kappa_v = 0.00026$, which is the critical gain for $D = 5000$ obtained from Simulation 2. The gain for the horizontal route is set at $\kappa_h = 0.0035/n^2$ using Conjecture 1. The other parameters for the vertical routes are $w_r = 0.01$, $\varepsilon = 0.025$ and $d_1(j,r) = d_2(j,r) = 2500$. For the horizontal route we have $w_r = 0.01n$, $\varepsilon = 0.025$, $d_1(j,r) = 2500j$ and $d_2(j,r) = 5000n - d_1(j,r)$ where $j \in [1,n]$. The variance of the rates for different n are obtained from simulations and these are shown in Table 5.2.

Recall that we defined the system as unstable when the variance of the rates exceeds $0.01x^2 = 0.00005$. From Simulation 2, the critical gain is $\kappa^* = 0.00026$ for $n = 1$. This is less than that predicted by Theorem 10 due to stochastic effects as discussed earlier. Hence, for $n = 1, \kappa_h = 0.00035$, the variance is greater than 0.00005 as expected. For $n = 2, 4, 10$, the variances are all less than 0.00005, supporting the claim of Conjecture 1. Notice that the variances are of the same

n	κ_h	κ_v	Variance
1	0.00026	0.00026	0.000049
1	0.00035	0.00026	0.00011
2	0.0000875	0.00026	0.000027
4	0.00002187	0.00026	0.000022
10	0.0000035	0.00026	0.000022

Table 5.2: Variance of rates as n varies.

order of magnitude, which indicate that the bound given by the conjecture, if true, is fairly tight. Unfortunately, due to computer memory and time constraints, it was not possible to simulate larger values of n.

Chapter 6

Conclusion

In this dissertation we have addressed the issue of how available bandwidth within a large-scale broadband network should be shared among competing streams of elastic traffic. An optimization framework leads to a decomposition of the overall system problem into a separate problem for each user, in which the user chooses a charge per unit time that he is willing to pay, and one for the network; we have shown that two classes of rate control algorithm are naturally associated with the objective functions appearing in, respectively, the primal and dual formulation of the network's problem.

The first main result obtained is the the stability and fairness of two classes of rate control algorithms, corresponding to the primal and dual problems. The algorithms, which operate on a decentralized basis, provide natural generalizations to large-scale networks of simple linear increase/multiplicative decrease schemes, and are shown to be stable about a system optimum characterized by a proportional fairness criterion using Lyapunov methods. The model is then extended to include two important aspects of decentralized systems, namely stochastic perturbations and time lags. These results are a significant extension of previous work which involved only a single bottleneck resource.

The above results assumes a constant number of users in the network, each user fixing the charge per unit time that he is willing to pay while the network performs the rate allocation. With increasing intelligence embedded in end-systems, users may well be able to vary their choice of charges even within the short time scales of the network optimization process. The model has been shown to extend to this situation, and under the assumption that each user tracks the optimum to his own optimization problem, the stability and convergence of the system may again be established. We also extended the basic model to allow routeing choices to be made.

Two models were considered where the user population changes due to arrivals and departures. For the first model, each route r is a superposition of adaptive fixed duration sub-users. We established stability and convergence for the system and then showed how our methods may be used to study the stochastic effects due to the rate and arrival/departure processes, the relative importance of which is dependent on their speeds of convergence. For the second model, we considered a single resource with a mix of two user types, some are fixed duration users (as in the first case) while others are file transfer users leaving the network. Here, the general conclusion is that fixed duration users are, in a sense, 'good traffic'.

The relationship between time lags and stability was further analysed. We derived the conditions for local stability for a series of simple networks using the primal rate control algorithm. These included networks with single and multiple resources, and a large network with identical delays. Parallel theorems were also obtained via retarded differential equations. The case of a single route with multiple resources having diverse delays was also analysed. In general, the results scale well with the size of the network as the time delays and number of resources grow. Important insight is obtained on the design of the price/feedback function at the resource. Also, they indicate why self-clocking algorithms are likely to work well to maintain stability.

In summary, the results obtained provide important insights into the behaviour of the system, and allow us to quantify the various relationships and trade-offs between the rate of convergence of the algorithms, the magnitude of fluctuations about the equilibrium point, and the stability of the system.

Further work is needed on a number of issues. Firstly, is Conjecture 1 true? As discussed in Chapter 4, Conjecture 1, if true, would allow a decentralised implementation for an arbitrarily large network with diverse delays. The second issue is more practical and longer term, and relates to the implementation of our results on an actual network e.g. the Internet. Preliminary work in this direction includes that of Crowcroft and Oechslin [17], Gibbens and Kelly [24], Key and McAuley [41] and Key et al. [42]. Also, there is the enabling work of Ramakrishnan and Floyd [56] which proposes to add Explicit Congestion Notification (ECN) to the Internet protocol.

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