1. Recall the definition of a *Wardrop equilibrium* for the flows in a congested network. Check that if the delay $D_j(y_j)$ at link j is a continuous increasing function of the throughput y_j of link j, then a Wardrop equilibrium exists and solves an optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in J} \int_0^{y_j} D_j(u)du \\ \text{over} \quad & x \geq 0, \quad y, \\ \text{subject to} \quad & Hx = f, Ax = y. \end{aligned}$$

In what sense is the equilibrium unique?

Suppose that, in addition to the delay $D_j(y_j)$, users of link j incur a traffic dependent toll

$$T_j(y_j) = y_j D_j'(y_j).$$

Show that if users select routes in an attempt to minimize the sum of their tolls and their delays, then they will produce a flow pattern that minimizes the average delay in the network.

2. In the definition of a Wardrop equilibrium f_s is the aggregate flow for source-sink pair s, and is assumed fixed. Extend the model to the allow the aggregate flow for source-sink pair s to depend upon the minimal delay over routes serving the source-sink pair s. For the extended model, can an equilibrium be characterized in terms of a solution to an optimization problem?

3. Suppose that user r monitors its rate $x_r(t)$, and chooses $w_r(t)$ to track the optimum to $USER_r(U_r; \lambda_r(t))$, where $\lambda_r(t) = w_r(t)/x_r(t)$ (the charge per unit flow to user r at time t). Determine the consequent relationship between $w_r(t)$ and $x_r(t)$.

The primal algorithm becomes

$$\frac{d}{dt} x_r(t) = \kappa\big(w_r(t) - x_r(t) \sum_{j \in r} \mu_j(t)\big)$$

where

$$\mu_j(t) = p_j\big(\sum_{s: j \in s} x_s(t)\big).$$

Find a Lyapunov function for this system.