MATHEMATICAL TRIPOS: PART III **(Lent 2018)**

Stochastic Networks – Example Sheet 2 **FPK**

1. Show that the traffic equations for an open migration process have a unique solution, and that this solution is positive. [*Hint:* From the irreducibility of the open migration process deduce the irreducibility of a certain Markov process on $J+1$ states, and then use the fact that the equilibrium distribution for this process is unique.]

Deduce that in an open migration process $\alpha_j \lambda_j$ is the mean arrival rate at colony j, counting arrivals from outside the system and from other colonies.

2. Show that the reversed process obtained from a stationary closed migration process is also a closed migration process, and determine its transition rates.

3. A restaurant has N tables, with a customer seated at each table. Two waiters are serving them.One of the waiters moves from table to table taking orders for food. The time that he spends taking orders at each table is exponentially distributed with parameter μ_1. He is followed by the wine waiter who spends an exponentially distributed time with parameter μ_2 taking orders at each table. Customers always order food first and then wine, and orders cannot be taken concurrently by both waiters from the same customer. All times taken to order are independent of each other. A customer, after having placed her two orders, completes her meal at rate ν, independently of the other customers. As soon as a customer finishes her meal, she departs and a new customer takes her place and waits to order. Model this as a closed migration process. Show that the stationary probability that both waiters are busy can be written in the form

$$\frac{G(N-2)}{G(N)} \cdot \frac{\nu^2}{\mu_1 \mu_2},$$

for a function $G(\cdot)$, which may also depend on ν, μ_1, μ_2, to be determined.

In the above model it is assumed that the restaurant is always full. Develop a model in which this assumption is relaxed: for example, assume that customers enter the restaurant at rate λ while there are tables empty. Again obtain an expression for the probability that both waiters are busy.

4. Show that if the parameters of a stationary open migration process are such that there is no path by which an individual leaving colony k can later reach colony j, then the stream of individuals moving directly from j to k forms a Poisson process.

5. Airline passengers arrive at a passport control desk in accordance with a Poisson process of rate ν. The desk operates as a single-server queue at which service times are independent and exponentially distributed with mean $\mu(< \nu^{-1})$ and are independent of the arrival process. After leaving the passport control desk a passenger must pass through a security check. This also operates as a single-server queue, but one at which service times are all equal to $\tau(< \nu^{-1})$. Show that in equilibrium the probability both queues are empty is

$$(1-\nu\mu)(1-\nu\tau).$$

6. If in the previous question it takes a time σ to walk from the first queue to the second, what is the equilibrium probability that both queues are empty and there is no passenger walking between them?

7. Recall the mathematical model for a loss network with fixed routing.

A network consists of three nodes, with each pair of nodes connected by a link. A call in progress between two nodes may be routed on the direct link between the nodes, or on the two link path through the third node. A call in progress can be rerouted if this will allow an additional arriving call to be accepted. Describing carefully the modelling assumptions you make, obtain an exact expression for the probability an arriving call is lost, and sketch a network with fixed routing which shares the same loss probabilities. Deduce an Erlang fixed point approximation for the loss probabilities in the original network, in terms of blocking probabilities across certain *cuts* of the network.

8. Calls arrive as a Poisson process of rate ν at a link capacity C circuits. A call is blocked and lost if all C circuits are busy; otherwise the call is accepted and occupies a single circuit for an exponentially distributed holding time with mean one. Holding times are independent of each other and of the arrival process. Calculate the mean number of circuits in use, $M(\nu, C)$. Show that, as $n \to \infty$,

$$\frac{M(\nu n, Cn)}{n} \to \nu \wedge C,$$

and the mean number of idle circuits satisfies

$$Cn - M(\nu n, Cn) \to \frac{C}{\nu \vee C - C}.$$

9. Show that solutions of the fixed point equation

$$B = E(\nu[1 + 2B(1-B)], C)$$

locate stationary points of the potential function

$$\nu\left[e^{-y} + e^{-2y}(1 - \frac{2}{3}e^{-y})\right] + \int_0^y U(z, C)dz.$$