MATHEMATICAL TRIPOS: PART III **(Lent 2018)**

Stochastic Networks – Example Sheet 1 **FPK**

1. Determine the stationary distribution, π, of an $M/M/2$ queue. Show that the proportion of time both servers are idle is

$$\pi_0 = \frac{1-\rho}{1+\rho}, \text{ where } \rho = \frac{\nu}{2\mu}.$$

2. Upon an $M/M/1$ queue is imposed the additional constraint that arriving customers who find N customers already present leave and never return. Find the stationary distribution of the queue.

3. A continuous time Markov process has transition rates $(q(j,k), j,k \in S)$, and equilibrium distribution $(\pi(j), j \in S)$. Write down the equations relating $q(.,.)$ and $\pi(.)$. A discrete time Markov chain is formed by observing the jumps of this process: at the successive jump times of the process, the state j just before, and the state k just after, the jump are recorded as an ordered pair $(j,k) \in S^2$. Write down the transition probabilities of the resulting Markov chain, and show that it has equilibrium distribution

$$\pi'(j,k) = G^{-1}\pi(j)q(j,k)$$

provided

$$G = \sum_j \sum_k \pi(j)q(j,k) < \infty.$$

Give an alternative interpretation of $\pi'(j,k)$, in terms of the conditional probability of seeing the original process jump from state j to state k in the interval $(t, t+h)$, given that a jump occurs in this interval.

4. An $M/M/1$ queue has arrival rate ν and service rate μ, where $\rho = \nu/\mu < 1$. Show that the sojourn time (= queuing time + service time) of a typical customer is exponentially distributed with parameter $\mu - \nu$.

5. Consider an $M/M/\infty$ queue with servers numbered $1, 2, \ldots$. On arrival a customer chooses the lowest numbered server which is free. Calculate the equilibrium probability that j out of the first n servers are busy. For what fraction of time is each server busy?

Car parking spaces are labelled $n = 1, 2, \ldots$, where the label indicates the distance (in car lengths) to walk to a shop, and an arriving car parks in the lowest numbered free space. Cars arrive as a Poisson process of rate ν, and parking times are exponentially distributed with unit mean, and are independent of each other and of the arrival process. Show that the distance parked from the shop has mean

$$\sum_{C=0}^{\infty} E(\nu, C)$$

where $E(\nu, C)$ is Erlang's formula.

6. Goldie's Restaurant remains open 24 hours per day, 365 days per year. The total number of customers served in the restaurant during 2017 was 21% greater than the total for 2016. In each year, the number of customers in the restaurant was recorded at a large number of randomly selected times, and the average of those numbers in 2017 was 16% greater than the average in 2016. By how much did the average duration of a customer visit to the restaurant increase or decrease?

7. Does the train station arrangement (one queues for S counters rather than S queues) reduce the expected waiting time? What advantage does it have? (Think about how you would model this so as to make precise the issues.)