STOCHASTIC NETWORKS EXAMPLE SHEET 3 SOLUTIONS

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Exercise 1. Recall the definition of a Wardrop equilibrium for the flows in a congested network. Check that if the delay $D_j(y_j)$ at link j is a continuous increasing function of the throughput y_j of link j, then a Wardrop equilibrium exists and solves an optimization problem of the form

$$\text{minimize} \sum_{j\in J}\int_0^{y_j} D_j(u)du$$
$$\text{over } \mathbf{x}\geq 0,\ \mathbf{y},$$
$$\text{subject to } H\mathbf{x}=\mathbf{f},\ A\mathbf{x}=\mathbf{y}.$$

In what sense is the equilibrium unique?

Suppose that, in addition to the delay $D_j(y_j)$, users of link j incur a traffic-dependent toll

$$T_j(y_j)=y_jD_j'(y_j).$$

Show that if users select routes in an attempt to minimize the sum of their tolls and their delays, then they will produce a flow pattern that minimizes the average delay in the network.

Proof. The first part of the question is done in the lecture notes: the conditions for a Wardrop equilibrium are the same as the KKT (complementary slackness) conditions for the minimization problem. The equilibrium gives a unique set of y values when all D_j are strictly increasing; but in general not a unique set of x values unless the link-route incidence matrix A has full rank.

If we want the system problem to be

$$\text{minimize} \sum_{j\in J} y_jD_j(y_j)$$
$$\text{over } \mathbf{x}\geq 0,\ \mathbf{y},$$
$$\text{subject to } H\mathbf{x}=\mathbf{f},\ A\mathbf{x}=\mathbf{y}$$

(i.e., minimizing the total rate at which traffic accumulates to everyone in the network), then the corresponding Lagrangian is

$$\mathcal{L}(\mathbf{x},\mathbf{y};\boldsymbol{\lambda},\boldsymbol{\mu})=\sum_{j\in J} D_j(y_j)+\boldsymbol{\lambda}\cdot(\mathbf{f}-H\mathbf{x})-\boldsymbol{\mu}\cdot(\mathbf{y}-A\mathbf{x})$$

and differentiating,

$$\frac{\partial\mathcal{L}}{\partial y_j}=D_j(y_j)+y_jD_j'(y_j)-\mu_j,\qquad \frac{\partial\mathcal{L}}{\partial x_r}=-\lambda_{s(r)}+\sum_j \mu_jA_{jr}$$

The derivative with respect to y_j must be zero at the minimum (y_j is unconstrained), while the derivative with respect to x_r must be nonnegative, and zero if $x_r>0$ at the minimum. Thus, we have $\mu_j=D_j(y_j)+y_jD_j'(y_j)$ is the "price" of link j to each customer, and $\lambda_{s(r)}\leq$

$\sum_j \mu_j A_{jr}$ with equality if $x_r > 0$. This corresponds to users comparing the sum of "costs" (delay + congestion charge) over all the links in their chosen route, and travelling only along the route with the smallest total cost $\lambda_{s(r)}$.

Of course, in practice there are various issues with congestion charges, for example privacy (congestion charges are localised, which means someone needs to keep track of who went where when, which seems poor). $\square$

Exercise 2. In the definition of a Wardrop equilibrium f_s is the aggregate flow for source-sink pair s, and is assumed fixed. Extend the model to the allow the aggregate flow for source-sink pair s to depend upon the minimal delay over routes serving the source-sink pair s. For the extended model, can an equilibrium be characterized in terms of a solution to an optimization problem?

Proof. I'll use the letter φ_s to denote the function that gives the actual aggregate flow for the source-sink pair s. We want the Wardrop equilibrium to be a vector of flows $\mathbf{x} = (x_r, r \in R)$ along routes with the property $\sum_{j \in r} x_r = \varphi_s(D_s)$ and

$$x_r > 0 \implies \sum_{j \in r} D_j(y_j) = \min_{r' \in s(r)} \sum_{j \in r'} D_j(y_j) \equiv D_s.$$

where $\mathbf{y} = A\mathbf{x}$ and $D_s = \sum_{j \in r} D_j(y_j)$ for the route(s) with $x_r > 0$.

Let us first of all make some sensible assumptions about the function φ: we expect φ_s to be decreasing, continuous, and probably convex (we certainly want it to be nonnegative!). Let's also assume that it's finite, i.e. $\varphi_s(0) = f_s < \infty$.

In that case, we can still think of a model in which a finite number of people f_s would like to travel along routes serving the source-destination pair s. If there were no delays, they would all travel; as delays build up in the network, more and more people decide that it isn't worth it, and stay at home.

Formally, consider a network with *fixed* flows $f_s = \varphi_s(0)$ for each source-sink pair s, but with an extra one-link route added to each source-sink pair. The route is called "stay at home (and SSH into work)" and has all the people who decided not to travel along the congested roads. We will need to design the "delay" (cost) function on $\tilde{D}_s$ to this link in such a way that the number of people who end up travelling along all the other routes is $\varphi_s(D_s)$.

Assuming the $\tilde{D}_s$ we construct is increasing and continuous, we now have the usual Wardrop equilibrium model. In order for the extra route to be doing anything, we must have traffic on it, which means that it must have the same "cost" as all the other routes. If for a source-destination pair s, the number of people actually travelling is f_s^1 and the number of people staying at home is f_s^0, with $f_s^0 + f_s^1 = f_s$, then we must have

$$\tilde{D}_s(f_s^0) = D_s = \varphi_s^{-1}(f_s^1) = \varphi_s^{-1}(f_s - f_s^0).$$

(If the number of people actually travelling along s is f_s^1, then the delay for that source-destination pair must be $\varphi_s^{-1}(f_s^1)$, and this must be equal to the cost that the people staying at home are incurring.) Observe that $f_s - f_s^0$ is decreasing in f_s^0, and φ_s was decreasing also, so $\tilde{D}_s(f_s^0)$ is increasing in f_s^0; and if φ_s was strictly decreasing and continuous, than D_s is continuous as well.

That is, our model will feature an extra one-link route for each soure-destination pair, with delay function $\tilde{D}_s(y) = \varphi_s^{-1}(f_s - y)$, where $f_s = \varphi_s(0)$. We are interested in the usual

Wardrop equilibrium for this model, which comes from the optimization problem

$$\begin{aligned} &\text{minimize } \sum_{j\in J}\int_0^{y_j} D_j(u)du + \sum_s \int_0^{f_s^0} \tilde{D}_s(u)du \\ &\text{over } \mathbf{x}\geq 0,\ \mathbf{f}^0 \geq 0,\ \mathbf{y}, \\ &\text{subject to } (\sum_{r\in s} x_r) + f_s^0 = f_s\ \forall s,\ A\mathbf{x}=\mathbf{y}. \end{aligned}$$

Since we showed above that $\tilde{D}_s$ is increasing and continuous, this is still a convex optimization problem with unique maximum (in terms of $\mathbf{y}$ and $\mathbf{f}^0$). □

Exercise 3. Suppose that user r monitors its rate $x_r(t)$, and chooses $w_r(t)$ to track the optimum to $USER_r(U_r; r(t))$, where $r(t) = w_r(t)/x_r(t)$ (the charge per unit flow to user r at time t). Determine the consequent relationship between $w_r(t)$ and $x_r(t)$.

The primal algorithm becomes

$$\frac{d}{dt}x_r(t) = \kappa\left(w_r(t) - x_r(t)\sum_{j\in r}\mu_j(t)\right)$$

where

$$\mu_j(t) = p_j\left(\sum_{s:j\in s} x_s(t)\right).$$

Find a Lyapunov function for this system.

Proof. In lecture notes. The relationship between $w_r(t)$ and $x_r(t)$ is $w_r(t) = r(t)x_r(t)$, but more to the point $U_r'(x_r(t)) = r(t)$ (or $U_r'(w_r(t)/r(t)) = r(t)$). This doesn't change because I happen to have added a funny set of symbols (t) at the end of all my variables – this is just an expression for the location of the maximum!

The Lyapunov function is

$$\mathcal{U}(x) = \sum_{r\in R} w_r \log x_r - \sum_{j\in J}\int_0^{\sum_{s:j\in s} x_s} p_j(y)dy$$

with derivative

$$\frac{d}{dt}\mathcal{U}(x(t)) = \sum_{r\in R}\frac{\partial \mathcal{U}}{\partial x_r}|_{x(t)} \times \frac{dx_r(t)}{dt}|_{x(t)} = \sum_{r\in R}\frac{\kappa_r}{x_r(t)}\left(w_r - x_r(t)\sum_{j\in r}p_j(\sum_{s:j\in s}x_s)\right)^2 \geq 0.$$

The derivatives $\partial\mathcal{U}/\partial x_r$ don't depend on time (because $\mathcal{U}$ doesn't, it's a function of the state alone), but I'm then evaluating at the point $x(t)$, which is how I get the time dependence for $\partial\mathcal{U}/\partial x_r|_{x(t)}$. □