STOCHASTIC NETWORKS

(An example from the second example sheet)

Calls arrive as a Poisson process of rate ν at a link of capacity C circuits. A call is blocked and lost if all C circuits are busy; otherwise the call is accepted and occupies a single circuit for an exponentially distributed holding time with mean one. Holding times are independent of each other and of the arrival process. Calculate the mean number of circuits in use, $M(\nu, C)$. Show that, as $n \to \infty$, the proportion of circuits in use satisfies

$$\frac{M(\nu n, Cn)}{n} \to \nu \wedge C,$$

and the mean number of idle circuits satisfies

$$Cn - M(\nu n, Cn) \to \frac{C}{\nu \vee C - C}.$$

The number of circuits in use in a birth and death process whose equilibrium distribution is readily calculated from the detailed balance conditions, and hence we may deduce that

$$\pi(j) \propto \frac{\nu^j}{j!} \qquad j = 0, 1, \ldots, C.$$

Hence we can write Erlang's formula, $E(\nu, 1) = \pi(C)$, in the (unconventional) form

$$E(\nu, C) = \left[1 + \frac{C}{\nu} + \frac{C(C-1)}{\nu^2} + \cdots + \frac{C!}{\nu^C}\right]^{-1}.$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} E(\nu n, Cn) &= \left[1 + \frac{C}{\nu} + \frac{C^2}{\nu^2} + \cdots\right]^{-1} \\
&= 0 \quad \text{if } C \geq \nu \\
&= 1 - \frac{C}{\nu} \quad \text{if } C < \nu.
\end{aligned}$$

Now (from either the distribution π or from Little's formula)

$$M(\nu, C) = \nu(1 - E|\nu, C|)$$

and so

$$\begin{aligned}
\lim_{n\to\infty} \frac{M(\nu n, Cn)}{n} &= \nu\Big(1 - \lim_{n\to\infty} E(\nu n, Cn)\Big) \\
&= \nu \wedge C.
\end{aligned}$$

The mean number of idle circuits is (using the distribution π) monotone decreasing in ν, and may be written

$$\begin{aligned}
C - M(\nu, C) &= \sum_{m=0}^{C} m\pi(C-m) \\
&= \pi(C)\left[0 + 1.\frac{C-1}{\nu} + 2\frac{(C-1)(C-2)}{\nu^2} + \cdots + C\frac{(C-1)!}{\nu^{C-1}}\right].
\end{aligned}$$

If $\nu > C$ then

$$\begin{aligned}
\lim_{n\to\infty}[Cn - M(\nu n, Cn)] &= \left(\lim_{n\to\infty} E(\nu, C)\right). \sum_{m=0}^{\infty} m \left(\frac{C}{\nu}\right)^m \\
&= \left(1 - \frac{C}{\nu}\right). \frac{C}{\nu} \left(1 - \frac{C}{\nu}\right)^{-2} \\
&= \frac{C}{\nu - C}. \qquad (1)
\end{aligned}$$

If $\nu < C$ then, since

$$\lim_{n\to\infty} \frac{M(\nu n, Cn)}{n} = \nu,$$

we have that

$$\lim_{n\to\infty}[Cn - M(\nu n, Cn)] = \infty.$$

Finally, the required result for $\nu = C$ follows from (1) and the monotonicity of the mean number of idle circuits with ν.