MATHEMATICAL TRIPOS: PART IA (Lent 2013)

Probability – Example Sheet 2 (of 4) **FPK**

Exercises

1. A coin with probability p of heads is tossed n times. Let E be the event 'a head is obtained on the first toss' and F_k the event 'exactly k heads are obtained'. For which pairs of integers (n, k) are E and F_k independent?

2. The events A and B are independent. Show that the events A^C and B are independent, and that the events A^C and B^C are independent.

3. Independent trials are performed, each with probability p of success. Let π_n be the probability that n trials result in an even number of successes. Show that

$$\pi_n = \tfrac{1}{2}[1 + (1 - 2p)^n]\,.$$

4. Two darts players A and B throw alternately at a board and the first to score a bull wins the contest. The outcomes of different throws are independent and on each of their throws A has probability p_A and B has probability p_B of scoring a bull. If A has first throw, calculate the probability of A winning the contest.

5. Suppose that X and Y are independent Poisson random variables with parameters λ and μ respectively. Find the distribution of $X + Y$. Prove that the conditional distribution of X, given that $X + Y = n$, is binomial with parameters n and $\lambda/(\lambda+\mu)$.

6. The number of misprints on a page has a Poisson distribution with parameter λ, and the numbers on different pages are independent. What is the probability that the second misprint will occur on page r?

7. $X_1, \dots, X_n$ are independent, identically distributed random variables with mean μ and variance σ^2. Find the mean of

$$S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2, \quad \text{where } \bar{X} = \frac{1}{n}\sum_{1}^{n} X_i.$$

8. In a sequence of n independent trials the probability of a success at the ith trial is p_i. Show that mean and variance of the total number of successes are $n\bar{p}$ and $n\bar{p}(1-\bar{p}) - \Sigma(p_i - \bar{p})^2$ where $\bar{p} = \Sigma p_i/n$. Notice that for a given mean, the variance is greatest when all p_i are equal.

9. Let $(X, Y) = (\cos\theta, \sin\theta)$ where $\theta = \frac{k\pi}{4}$ and k is a random variable such that $P\{k = r\} = \frac{1}{8}$, $r = 0, 1, \dots, 7$. Show that $\mathrm{cov}(X, Y) = 0$, but that X and Y are not independent.

10. Let $a_1, a_2, \dots, a_n$ be a ranking of the yearly rainfalls in Cambridge over the next n years: assume $a_1, a_2, \dots, a_n$ is a random permutation of $1, 2, \dots, n$. Say that

k is a record year if $a_i > a_k$ for all $i < k$ (thus the first year is always a record year). Let $Y_i = 1$ if i is a record year and 0 otherwise. Find the distribution of Y_i and show that $Y_1, Y_2, \ldots, Y_n$ are independent. Calculate the mean and variance of the number of record years in the next n years.

11. Liam's bowl of spaghetti contains n strands. He selects two ends at random and joins them together. He does this until no ends are left. What is the expected number of spaghetti hoops in the bowl?

12. Sarah collects figures from cornflakes packets. Each packet contains one figure, and n distinct figures make a complete set. Show that the expected number of packets Sarah needs to buy to collect a complete set is

$$n \sum_{i=1}^{n} \frac{1}{i}.$$

13. (X_k) is a sequence of independent identically distributed positive random variables where $E(X_k) = a$ and $E(X_k^{-1}) = b$ exist. Let $S_n = \sum_{k=1}^{n} X_k$. Show that $E(S_m/S_n) = m/n$ if $m \leq n$, and $E(S_m/S_n) = 1 + (m-n)aE(S_n^{-1})$ if $m \geq n$.

Problems

Note: These problems should not be tackled at the expense of examples on later sheets.

14. Let X be an integer-valued random variable with distribution

$$P(X = n) = n^{-s}/\zeta(s)$$

where $s > 1$, and $\zeta(s) = \sum_{n \geq 1} n^{-s}$, the Riemann zeta function. Let $p_1 < p_2 < p_3 < \cdots$ be the primes and let A_k be the event $\{X \text{ is divisible by } p_k\}$. Find $P(A_k)$ and show that the events $A_1, A_2, \ldots$ are independent. Deduce that

$$\prod_{k=1}^{\infty} (1 - p_k^{-s}) = 1/\zeta(s).$$

15. You are playing a match against an opponent in which at each point either you or your opponent serves. If you serve you win the point with probability p_1, but if your opponent serves you win the point with probability p_2. There are two possible conventions for serving:

(i) serves alternate;

(ii) the player serving continues to serve until she loses a point.

You serve first and the first player to reach n points wins the match. Show that your probability of winning the match does not depend on the serving convention adopted.
[*Hint:* Under either convention you serve at most n times and your opponent at most $n-1$ times. Recall Pascal and Fermat's 'problem of points', treated in lectures.]