**Example Sheet 2 (of 4)**

## Exercises

**1.** A coin with probability $p$ of heads is tossed $n$ times. Let $E$ be the event 'a head is obtained on the first toss' and $F_k$ the event 'exactly $k$ heads are obtained'. For which pairs of integers $(n, k)$ are $E$ and $F_k$ independent?

**2.** The events $A$ and $B$ are independent. Show that the events $A^C$ and $B$ are independent, and that the events $A^C$ and $B^C$ are independent.

**3.** Independent trials are performed, each with probability $p$ of success. Let $P_n$ be the probability that $n$ trials result in an even number of successes. Show that

$$P_n = \tfrac{1}{2}[1 + (1 - 2p)^n].$$

**4.** Two darts players $A$ and $B$ throw alternately at a board and the first to score a bull wins the contest. The outcomes of different throws are independent and on each of their throws $A$ has probability $p_A$ and $B$ has probability $p_B$ of scoring a bull. If $A$ has first throw, calculate the probability of $A$ winning the contest.

**5.** Suppose that $X$ and $Y$ are independent Poisson random variables with parameters $\lambda$ and $\mu$ respectively. Find the distribution of $X + Y$. Prove that the conditional distribution of $X$, given that $X + Y = n$, is binomial with parameters $n$ and $\lambda/(\lambda + \mu)$.

**6.** (i) The number of misprints on a page has a Poisson distribution with parameter $\lambda$, and the numbers on different pages are independent. What is the probability that the second misprint will occur on page $r$?

(ii) A proofreader studies a single page looking for misprints. She catches each misprint (independently of others) with probability $1/2$. Let $X$ be the number of misprints she catches. Find $P(X = k)$. Given that she has found $X = 10$ misprints, what is the distribution of $Y$, the number of misprints she has not caught? How useful is $X$ in predicting $Y$?

**7.** $X_1, \ldots, X_n$ are independent, identically distributed random variables with mean $\mu$ and variance $\sigma^2$. Find the mean of

$$S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2, \quad \text{where } \bar{X} = \frac{1}{n} \sum_{1}^{n} X_i.$$

**8.** In a sequence of $n$ independent trials the probability of a success at the $i$th trial is $p_i$. Show that mean and variance of the total number of successes are $n\bar{p}$ and $n\bar{p}(1 - \bar{p}) - \sum_i (p_i - \bar{p})^2$ where $\bar{p} = \sum_i p_i/n$. Notice that for a given mean, the variance is greatest when all $p_i$ are equal.

**9.** Let $(X, Y) = (\cos\theta, \sin\theta)$ where $\theta = \frac{k\pi}{4}$ and $k$ is a random variable such that $P\{k = r\} = 1/8$, $r = 0, 1, \ldots, 7$. Show that $\text{cov}(X, Y) = 0$, but that $X$ and $Y$ are not independent.

**10.** Let $a_1, a_2, \ldots, a_n$ be a ranking of the yearly rainfalls in Cambridge over the next $n$ years: assume $a_1, a_2, \ldots, a_n$ is a random permutation of $1, 2, \ldots, n$. Say that $k$ is a record year if $a_i > a_k$ for all $i < k$ (thus the first year is always a record year). Let $Y_i = 1$ if $i$ is a record year and 0 otherwise. Find the distribution of $Y_i$ and show that $Y_1, Y_2, \ldots, Y_n$ are independent. Calculate the mean and variance of the number of record years in the next $n$ years.

**11.** Liam's bowl of spaghetti contains $n$ strands. He selects two ends at random and joins them together. He repeats this until no ends are left. What is the expected number of spaghetti hoops in the bowl?

**12.** Sarah collects figures from cornflakes packets. Each packet contains one of $n$ distinct figures. Each type of figure is equally likely. Show that the expected number of packets Sarah needs to buy to collect a complete set of $n$ is

$$n \sum_{i=1}^{n} \frac{1}{i}.$$

[*After doing this, you might like to visit the Wikipedia article about the 'Coupon collector's problem'.*]

**13.** $(X_k)$ is a sequence of independent identically distributed positive random variables where $E(X_k) = a$ and $E(X_k^{-1}) = b$ exist. Let $S_n = \sum_{k=1}^{n} X_k$. Show that $E(S_m/S_n) = m/n$ if $m \leq n$, and $E(S_m/S_n) = 1 + (m-n)aE(S_n^{-1})$ if $m \geq n$.

## Problems

These next questions are more challenging. I hope you will learn and have fun by attempting them.

**14.** You wish to use a fair coin to simulate occurrence or not of an event $A$ that happens with probability $1/3$. One method is to start by tossing the coin twice. If you see HH say that $A$ occurred, if you see HT or TH say that $A$ has not occurred, and if you see TT then repeat the process. Show that this enables you to simulate the event using an expected number of tosses equal to $8/3$.

Can you do better? (i.e. simulate something that happens with probability $1/3$ using a fair coin and with a smaller expected number of tosses.) [Hint. The binary expansion of $1/3$ is $0.0101010101\ldots$.]

**15.** Let $X$ be an integer-valued random variable with distribution

$$P(X = n) = n^{-s}/\zeta(s)$$

where $s > 1$, and $\zeta(s) = \sum_{n \geq 1} n^{-s}$. Let $p_1 < p_2 < p_3 < \cdots$ be the primes and let $A_k$ be the event $\{X$ is divisible by $p_k\}$. Find $P(A_k)$ and show that the events $A_1, A_2, \ldots$ are independent. Deduce that

$$\prod_{k=1}^{\infty}(1 - p_k^{-s}) = 1/\zeta(s).$$

**16.** You are playing a match against an opponent in which at each point either you or your opponent serves. If you serve you win the point with probability $p_1$, but if your opponent serves you win the point with probability $p_2$. There are two possible conventions for serving:

1. serves alternate;

2. the player serving continues to serve until she loses a point.

You serve first and the first player to reach $n$ points wins the match. Show that your probability of winning the match does not depend on the serving convention adopted.

[*Hint:* Under either convention you serve at most $n$ times and your opponent at most $n-1$ times. Recall Pascal and Fermat's 'problem of the points', treated in lectures.]

## Puzzles

This section is for enthusiasts — or for discussion in supervision when you have done everything else. The following puzzles have been communicated to me by Peter Winkler.

**17.** Let $k < n$, $k$ even, $n$ odd. Joey is to play $n$ chess games against his parents, alternating between his father and mother. To receive his allowance he must win $k$ games in a row. Prove that given the choice, he should start against the stronger parent.

[Hint: start by solving the cases $k = 2, n = 3$, and $k = n - 1$.]

**18.** Let $X_1, \dots, X_6$ be i.i.d. $B(1, p)$ with $p = 0.4$. Let $S_n = X_1 + \cdots + X_n$. Argue that

$$P(S_4 \geq 3) = P(S_6 \geq 4)$$

without explicitly computing either the left or right hand sides.

[Hint. Compare $P(S_4 \geq 3 \mid S_5 = i)$ and $P(S_6 \geq 4 \mid S_5 = i)$ for each of $i = 0, \dots, 5$.]