INVARIANT MEASURES AND THE Q-MATRIX

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Abstract

This paper provides a necessary and sufficient condition for a measure to be invariant for a Markov process. The condition is expressed in terms of the q-matrix assumed to generate the process.

1. Introduction

Let $Q = (q_{ij},\ i, j \in S)$ be a stable, conservative, regular and irreducible q-matrix over a countable state space S, and let $P(t) = (p_{ij}(t),\ i,j \in S)$ be the matrix of transition probabilities of the Markov process determined by Q. If (the Markov process determined by) Q is recurrent then the relations

$$\left.\begin{aligned} \sum_i m_i q_{ij} &= 0 \\ m_j &> 0 \end{aligned}\right\} \qquad j \in S \tag{1}$$

have a solution $m = (m_i, i \in S)$, unique up to constant multiples. Call m an invariant measure for $P(t)$ if

$$\sum_i m_i p_{ij}(t) = m_j \qquad t > 0,\ j \in S.$$

When Q is positive recurrent it is known (Doob [5], Kendall and Reuter [13]) that a solution m to (1) is an invariant measure for $P(t)$. This conclusion also holds when Q is null recurrent, but may not when Q is transient. When Q is transient the set of solutions to (1) may be empty or it may contain linearly independent elements: we obtain a necessary and sufficient condition for a given element of the set to be an invariant measure for $P(t)$.

The basic properties of Markov processes which will be needed are taken from Kendall [11] and are briefly stated in Section 2: they can also be found in [3], [6], [10], [12], [13] and [17]. Section 3 contains the main result of the present paper. Here it is shown that a solution to (1) is an invariant measure for $P(t)$ if and only if a time-reversed q-matrix $\tilde{Q}$, defined in terms of m and Q, is regular. It is convenient to obtain the result assuming only that Q is stable and conservative, with $P(t)$ the minimal (Feller) transition matrix determined by Q. The method of proof is a straightforward generalization of an argument used by Kendall [12] in the case where Q is symmetrically reversible. Section 4 is devoted to a cycle criterion relating Q and $\tilde{Q}$, modelled on the criteria discussed by Kolmogorov [14], Reich [16], Kendall [10] and Whittle [19].

My interest in the topic of this paper arose from an observation in an applied probability context which is perhaps worth mentioning here. A technique useful in the study of certain forms of queueing network involves solving relations (1) for each of the q-matrices corresponding to the individual queues of the network operating in isolation and then combining these solutions appropriately to obtain a solution to relations (1) for the q-matrix corresponding to the network [9]. Now the Markov process representing the network may well be positive recurrent even when some or all of the processes representing individual queues fail to be recurrent or even regular. The solution for the network will then have a straightforward interpretation as an invariant measure and it is of interest to ask when the individual solutions from which it has been constructed have interpretations as invariant measures for the individual queues.

2. Preliminaries

Suppose that we are given a stable, conservative q-matrix, that is a collection of real numbers $Q = (q_{ij}, i,j \in S)$ where S is a countable set and

$$\begin{aligned} q_{ij} &\geq 0 && i \neq j \\ \sum_{j\neq i} q_{ij} &= -q_{ii} && i \in S \qquad (2) \\ -q_{ii} &\overset{\Delta}{=} q_i < \infty && i \in S \end{aligned}$$

A Markov process with transition rates Q can be constructed by the standard

method, due to Feller and roughly indicated as follows. Starting from state i allow the process to stay there for a period exponentially distributed with parameter q_i, and then move the process to state j with probability q_{ij}/q_i; let the process remain in state j for a period exponentially distributed with parameter q_j, and so on. This construction defines a Markov process $(X(t), 0 \leq t < T)$ with initial state $X(0) = i$ and with stationary transition rates

$$\begin{aligned} q_{ij} &= \lim_{t\to 0} \frac{1}{t} p_{ij}(t) && i \neq j \\ q_i &= \lim_{t\to 0} \frac{1}{t} [1 - p_{ii}(t)] && \end{aligned} \qquad (3)$$

where

$$p_{ij}(t) = P\{t < T, X(t) = j | X(0) = i\} \ .$$

The terminal time T is the sum of the (random) sequence of exponentially distributed holding times, and may well be finite. The process will then have made infinitely many jumps in a finite time, and will have "run out of instructions". A necessary and sufficient condition for T to be infinite with probability one, whatever the initial state i, is that the equations

$$\sum_j q_{ij} y_j = y_i \qquad i \in S$$

have no non-trivial non-negative bounded solution, and in this case Q is said to be *regular*.

Occasionally it requires some effort to show that a matrix Q is regular, but there are a number of sufficient conditions. If $(q_i, i \in S)$ is bounded above then Q is regular. Call the sequence of states occupied by the process $(X(t), 0 \leq t < T)$ the jump chain, and call the sequence $(X(r\delta), r = 0,1,\ldots,\lfloor T/\delta \rfloor)$ the δ-skeleton. These are both Markov chains and recurrence of either of them implies recurrence of the Markov process, and hence regularity of Q.

From the construction of the process $(X(t), 0 \leq t < T)$ it follows that $p_{ij}(t)$ is the limit of the non-decreasing sequence $(f_{ij}(t,n), n = 0,1,\ldots)$ where $f_{ij}(t,n)$ is the probability that the process is in state j at time t after at most n jumps. Clearly

$$f_{ij}(t,0) = \delta_{ij}\, e^{-q_j t} \tag{4}$$

and the collection $(f_{ij}(t,n))$ can be generated using either the backward integral recurrence

$$f_{ij}(t,n+1) = \delta_{ij}\, e^{-q_i t} + \sum_{k\neq i} \int_0^t q_{ik}\, f_{kj}(u,n) e^{-q_i(t-u)}\, du \tag{5}$$

or the forward integral recurrence

$$f_{ij}(t,n+1) = \delta_{ij}\, e^{-q_j t} + \sum_{k\neq j} \int_0^t f_{ik}(u,n) q_{kj}\, e^{-q_j(t-u)}\, du \tag{6}$$

The transition probabilities $P(t) = (p_{ij}(t),\ i,j \in S)$ thus constructed satisfy

$$\sum_j p_{ij}(t) = P\{t < T \mid X(0) = i\} \leq 1 \ .$$

For each fixed $t > 0$ equality holds in this relation for all $i \in S$ if and only if Q is regular.

The matrix Q is *irreducible* if for each pair (i,j) of distinct states there exists a finite sequence of states $i, k_1, k_2, \ldots, k_r, j$ satisfying

$$q_{ik_1}\, q_{k_1k_2} \ldots q_{k_rj} > 0 \ .$$

This is equivalent to the condition that for each pair (i,j) of distinct states $p_{ij}(t) > 0$ for $t > 0$ ([10], proof of Theorem IV (i); [3], Theorem 18.4).

A collection of positive numbers $m = (m_i,\ i \in S)$ is an invariant measure for the transition probabilities $P(t)$ if

$$\sum_i m_i\, p_{ij}(t) = m_j \qquad t > 0,\ j \in S\ . \tag{7}$$

If $\sum m_j = 1$ and Q is irreducible then the process is positive recurrent (since its δ-skeleton is) and the invariant probability distribution m has an interpretation as either a stationary or a limiting distribution. It also has an ergodic interpretation: the ratio of the time spent in state i to

that spent in state j over the interval $[0,t]$ tends to m_i/m_j with probability one as $t \to \infty$. If $\sum m_j = \infty$ and Q is irreducible and the process is recurrent then the measure m retains the ergodic interpretation. In both these cases recurrence implies that Q is regular and the ergodic interpretation shows that m is essentially unique.

The work of Derman [4] and Brown [2] provides an alternative interpretation of an invariant measure m. This interpretation remains available when Q fails to be recurrent or even regular, and can be described informally as follows. At time $t = 0$ place N_i particles at site i, $i \in S$, where $(N_i,\ i \in S)$ are independent random variables and N_i has a Poisson distribution with mean m_i. From time $t = 0$ onwards allow particles to move independently from site to site, each moving in accordance with a Markov process constructed from the matrix Q. Then at any time $t > 0$ the number of particles at site i, $N_i(t)$, has a Poisson distribution with mean m_i, and $(N_i(t),\ i \in S)$ are independent. Note that if Q is not regular then it is possible for the Markov process describing a particle's motion to have a finite terminal time - in this case the particle disappears from the set of states S at the terminal time. It is quite possible that a q-matrix might admit an invariant measure and yet not be regular: we shall give an example later.

3. The conditions for invariance

This section explores the connection between the relations (1) and the invariance property (7). We begin with our main result.

Theorem. Let Q be a stable, conservative q-matrix and let $m = (m_i,\ i \in S)$ be a collection of positive numbers. Then the following statements are equivalent :

(i) m is an invariant measure for the transition probabilities $P(t)$ constructed from Q;

(ii) m satisfies

$$\sum_i m_i\, q_{ij} = 0 \qquad j \in S \tag{8}$$

and the relations

$$\sum_i z_i q_{ij} = z_j \qquad j \in S \qquad (9)$$
$$z_j \le m_j$$

have no non-trivial non-negative solution.

Proof. Suppose that m is an invariant measure. Then from equation (7)

$$\sum_{i \neq j} m_i \frac{p_{ij}(t)}{t} = m_j \frac{1-p_{jj}(t)}{t} \qquad j \in S$$

Relations (3) and Fatou's lemma thus imply

$$\sum_{i \neq j} m_i q_{ij} \le m_j q_j \qquad j \in S \qquad (10)$$

Define

$$\tilde{q}_{ij} = \frac{m_j}{m_i} q_{ji} \qquad i,j \in S \qquad (11)$$

Then

$$\tilde{q}_i \triangleq -\tilde{q}_{ii} = -q_{ii} < \infty ,$$

and

$$\sum_j \tilde{q}_{ij} \le 0$$

from inequality (10). Define $\tilde{q}_{i\partial}$ by

$$\sum_j \tilde{q}_{ij} + \tilde{q}_{i\partial} = 0$$

and let

$$\tilde{q}_{\partial j} = 0 \qquad j \in S \cup \{\partial\} .$$

Then $\tilde{Q} = (\tilde{q}_{ij},\ i,\ j \in S \cup \{\partial\})$ is a stable, conservative q-matrix on the state space $S \cup \{\partial\}$. Define $(\tilde{f}_{ij}(t,n))$ by the initial condition (4) and the forward integral recurrence (6), and let

$$\tilde{p}_{ij}(t) = \lim_{n \to \infty} \tilde{f}_{ij}(t,n) \qquad i,j \in S \cup \{\partial\} .$$

Clearly

$$m_i f_{ij}(t,0) = m_j \tilde{f}_{ji}(t,0) \qquad i,j \in S .$$

Assume for the moment the inductive hypothesis that

$$m_i f_{ij}(t,n) = m_j \tilde{f}_{ji}(t,n) \qquad i,j \in S \qquad (12)$$

for some $n \ge 0$. Then from the backward integral recurrence (5)

$$m_i f_{ij}(t,n+1) = m_i \delta_{ij} e^{-q_i t} + \sum_{k \neq i} \int_0^t m_i q_{ik} f_{kj}(u,n) e^{-q_i (t-u)} du .$$

But

$$m_i q_{ik} f_{kj}(u,n) = m_k \tilde{q}_{ki} f_{kj}(u,n)$$
$$= m_j \tilde{q}_{ki} \tilde{f}_{jk}(u,n)$$

from equation (11) and the inductive hypothesis (12). The definition of $\tilde{f}_{ji}(t,n+1)$ by means of the forward integral recurrence (6) thus shows that

$$m_i f_{ij}(t,n+1) = m_j \tilde{f}_{ji}(t,n+1) \qquad i,j \in S.$$

The additional state ∂ causes no difficulty, since $q_{\partial i} = 0$. The inductive hypothesis (12) is established, and so, on letting n tend to infinity,

$$m_i p_{ij}(t) = m_j \tilde{p}_{ji}(t) \qquad i,j \in S .$$

The assumed invariance of m thus implies

$$\sum_{i\in S} \tilde{p}_{ji}(t) = 1 \qquad (13)$$

and so

$$\tilde{p}_{j\partial}(t) = 0 \qquad j \in S \quad .$$

But $(\tilde{f}_{j\partial}(t,n),\ n = 0,1,\ldots)$ is a non-decreasing sequence whose limit is $\tilde{p}_{j\partial}(t)$: thus

$$\tilde{f}_{j\partial}(t,n) = 0 \qquad j \in S$$

and so, from the case $n = 1$ of the recurrence (6),

$$\tilde{q}_{j\partial} = 0 \qquad j \in S \quad .$$

Thus

$$\sum_j \tilde{q}_{ij} = 0 \qquad i \in S$$

and so equations (8) follow from the definition (11). The equality in relation (13) implies that the stable conservative q-matrix $(\tilde{q}_{ij},\ i,\ j \in S)$ is regular, and so the equations

$$\sum_{j\in S} \tilde{q}_{ij}\ y_j = y_i \qquad i \in S$$

have no non-trivial non-negative bounded solution. These equations can be rewritten as

$$\sum_j m_j\ y_j\ q_{ji} = m_i\ y_i \qquad i \in S$$

and the substitution $z_j = m_j\ y_j$ then shows that relations (9) can have no non-trivial non-negative solution.

The converse is established similarly. Suppose that statement (ii) holds. Once again define

$$\tilde{q}_{ij} = \frac{m_j}{m_i}\, q_{ji} \qquad i,\ j \in S \qquad (14).$$

Then $\tilde{Q} = (\tilde{q}_{ij},\ i,\ j \in S)$ is stable, and is conservative also, by virtue of the hypothesis (8). The transition probabilities $\tilde{P}(t) = (\tilde{p}_{ij}(t), i,j \in S)$ constructed from $\tilde{Q}$ satisfy

$$m_i\ p_{ij}(t) = m_j\ \tilde{p}_{ji}(t) \qquad i,j \in S \qquad (15)$$

by the inductive argument used earlier. Thus, summing over i, the collection $(m_i,\ i \in S)$ is an invariant measure if

$$\sum_i \tilde{p}_{ji}\ (t) = 1 \qquad t > 0, \qquad j \in S$$

This is equivalent to the regularity of $\tilde{Q}$, which follows from the assumption that relations (9) have no non-trivial non-negative solution.

□

It is perhaps worth noting that the proof of the theorem makes no use of the assumption that Q is conservative: the conclusions of the theorem remain valid when the conservation condition (2) is relaxed to

$$\sum_{j\neq i} q_{ij} \le -q_{ii} \qquad i \in S \quad ,$$

with the appropriately extended definition of the transition probabilities $P(t)$ ([6], Section 5.6). Note also that equations (8) above imply that m is subinvariant,

$$\sum_i m_i\ p_{ij}(t) \le m_j \qquad t > 0,\ j \in S \quad ,$$

from relation (15).

If Q is transient equations (8) may not have a non-trivial solution. A collection of positive numbers $(m_i,\ i \in S)$ satisfies equations (8) if and only if $(m_i\ q_i,\ i \in S)$ is an invariant measure for the jump chain associated with Q, and the work of Harris [7] and Veetch [18] provides a necessary and sufficient condition for the existence of such a measure when Q is irreducible.

Call the matrix $\tilde{Q}$ defined by relation (14) the time-reverse of Q with respect to m. This terminology is suggested by the observation that if $(X(t), -\infty < t < \infty)$ is a stationary Markov process with q-matrix Q and stationary distribution m then $\tilde{Q}$ is the q-matrix of the Markov process $(X(-t), -\infty < t < \infty)$. The particle system interpretation discussed in the previous section provides a further insight. If Q and $\tilde{Q}$ are stable, conservative and regular then m will be an invariant measure for both $P(t)$ and $\tilde{P}(t)$, and a stationary particle system constructed fro· m and Q and then reversed in time will have the same distributional law as a stationary particle system constructed from m and $\tilde{Q}$.

Suppose now that Q is stable and conservative and that the positive collection $m = (m_i, i \in S)$ satisfies equations (8) but is not necessarily invariant. Consider a particle system in which at time $t \geq 0$ the number of particles at site i is $N_i(t)$: as before suppose that $(N_i(0), i \in S)$ are independent random variables, $N_i(0)$ Poisson with mean m_i, and that from time $t = 0$ onwards particles move independently from site to site, each in accordance with a Markov process constructed from the matrix Q. If $\tilde{Q}$ is not regular then $(N_i(t), i \in S)$ will be a collection of independent Poisson random variables but $EN_i(t)$ may well be less than m_i. From the work of Reuter ([17], Section 5.3) it is possible to deduce that relations (9) have a maximal non-negative solution z, and that

$$\int_0^\infty \left[\sum_i m_i\, p_{ij}(t)\right] e^{-t} dt = m_j - z_j \ .$$

For the particle system this implies that

$$EN_j(\theta) = m_j - z_j$$

where θ is an exponential random variable with unit mean. The vector z thus indicates the extent to which the measure m is subinvariant.

We now illustrate the theorem with a simple example.

Example 1 Take the state space S to be the integers $\mathbb{Z}$ and let

$$\begin{aligned} q_{ij} &= q_i && j = i+1 \\ &= -q_i && j = i \\ &= 0 && \text{otherwise} \end{aligned}$$

where $q_i > 0$ for all $i \in \mathbb{Z}$. The (essentially unique) solution to equations (8) is

$$m_i = q_i^{-1} \qquad i \in \mathbb{Z}.$$

A solution to the equations $Qy = y$ (using the usual matrix abbreviation) must have the form

$$\begin{aligned} y_j &= y_0 \prod_{i=0}^{j-1} (1 + q_i^{-1}) && j > 0 \\ &= y_0 \prod_{i=1}^{-j} (1 + q_{-i}^{-1})^{-1} && j < 0 \ , \end{aligned}$$

and a solution to the equations $zQ = z$ must have the form

$$\begin{aligned} z_j &= z_0\, q_0\, q_j^{-1} \prod_{i=1}^{j} (1 + q_i^{-1})^{-1} && j > 0 \\ &= z_0\, q_0\, q_j^{-1} \prod_{i=0}^{-j-1} (1 + q_{-i}^{-1}) && j < 0 \end{aligned}$$

Thus if (and only if)

$$\sum_{i=0}^{\infty} q_i^{-1} < \infty \tag{16}$$

there exists a non-trivial non-negative bounded solution to the equations $Qy = y$. On the other hand if (and only if)

$$\sum_{i=0}^{\infty} q_{-i}^{-1} < \infty \tag{17}$$

there exists a non-trivial non-negative solution to the equations $zQ = z$ which is bounded above by m. When condition (16) fails and condition (17) holds we have an example of a q-matrix which is not regular and yet admits an invariant measure m. When condition (16) holds and condition (17) fails we have an example of a regular q-matrix which does not admit an invariant measure, even though equations (8) admit a positive solution. Note that if conditions (16) and (17) both hold the solution m to equations (8) is summable, although the q-matrix is not regular. This possibility has been pointed out by Miller [15].

The second condition for invariance, that there exist no non-trivial non-negative vector z satisfying (9), should be compared with the condition arising in the investigation by Reuter ([17], Section 6) of the uniqueness of the solution to the forward equations associated with Q. Reuter's condition is that there exist no non-trivial non-negative vector z satisfying

$$\sum_i z_i q_{ij} = z_j \qquad j \in S$$

$$\sum_j z_j < \infty \quad .$$

In Example 1 the two conditions are equivalent, but in general the relationship does not appear to be this straightforward. Note that Reuter's condition is expressed solely in terms of the matrix Q, whereas relations (9) involve both Q and m. When Q is transient there may exist linearly independent positive solutions to equations (8): some of these may be invariant measures while others may not, as the next example illustrates.

Example 2 Let $S = \mathbb{Z}$ and set

$$\begin{aligned} q_{ij} &= \lambda q_i && j = i+1 \\ &= -q_i && j = i \\ &= \mu q_i && j = i-1 \\ &= 0 && \text{otherwise} \end{aligned} \qquad (18)$$

where $\lambda + \mu = 1$, $\lambda > \mu$ and $q_i > 0$ for $i \in \mathbb{Z}$. Consider the Markov process constructed from this q-matrix. From the form of the jump chain it is apparent that with probability one the process will remain in the set $\{-1,-2,\ldots\}$ for just a finite time. If these sections of the sample path are deleted the result is equivalent to reflecting the Markov process at the origin, and so the q-matrix (18) is regular if and only if the q-matrix

corresponding to the reflected process is regular. Thus, from the analysis of Reuter ([17], Section 8.4), the q-matrix (18) is regular if and only if

$$\sum_{i=0}^{\infty} q_i^{-1} = \infty \qquad (19).$$

Now a solution to equations (8) is

$$m_i = \left(\frac{\lambda}{\mu}\right)^i q_i^{-1} \qquad i \in \mathbb{Z} \qquad (20)$$

and with this choice of m the matrix $\tilde{Q}$ defined by (14) is identical to Q. Thus m, given by expression (20), is an invariant measure for Q if and only if condition (19) holds. Another solution to equations (8) is

$$m_i = q_i^{-1} \qquad i \in \mathbb{Z} \qquad (21)$$

and with this choice of m the matrix $\tilde{Q}$ is given by

$$\begin{aligned} \tilde{q}_{ij} &= \mu q_i && j = i+1 \\ &= -q_i && j = i \\ &= \lambda q_i && j = i-1 \\ &= 0 && \text{otherwise} \quad . \end{aligned}$$

This $\tilde{Q}$ is regular if and only if

$$\sum_{i=0}^{\infty} q_{-i}^{-1} = \infty \qquad (22)$$

and so condition (22) is necessary and sufficient for the measure (21) to be invariant.

The methods of this section provide an alternative proof of results, stated in the next Corollary, which are of some use in applied probability contexts. Part (i) is contained in, and part (ii) can be readily deduced from, Theorem 8 of Kendall and Reuter [13]; these statements also follow from the work of Miller [15]. They differ from a number of more widely available results in that the recurrence of Q is not required as a premise. Part (iii) is given for completeness but is hardly surprising: when Q is null recurrent relations (1) have an essentially unique solution, since the jump chain is recurrent, and it is known that $P(t)$ has an essentially unique invariant measure [1].

Corollary Let Q be stable, conservative, regular and irreducible, and suppose that $m = (m_i,\ i \in S)$ is a collection of positive numbers satisfying

$$\sum_i m_i\, q_{ij} = 0 \qquad j \in S$$

(i) If $\sum_i m_i = 1$ then Q is positive recurrent and m is the invariant probability distribution for $P(t)$.

(ii) If $\sum_i m_i = \infty$ then Q is null recurrent or transient.

(iii) If Q is null recurrent then m is the essentially unique invariant measure for $P(t)$.

Proof. Since Q is regular m is an invariant measure for the stable, conservative and irreducible matrix $\tilde{Q}$ defined by relation (14). Thus m is an invariant measure for the δ-skeleton of the Markov process constructed from $\tilde{Q}$.

If $\sum m_i = 1$ then this δ-skeleton is positive recurrent. Thus $\tilde{Q}$ is regular and so m is an invariant measure for Q also. Since $\sum m_i = 1$ the δ-skeleton of the Markov process constructed from Q is positive recurrent, and hence so is $\tilde{Q}$.

If $\sum m_i = \infty$ then the δ-skeleton of the Markov process constructed from $\tilde{Q}$ is null recurrent or transient. In either case

$$\lim_{t\to\infty} \tilde{p}_{ji}(t) = 0 \qquad i, j \in S$$

where $(\tilde{p}_{ji}(t))$ are the transition probabilities of the Markov process constructed from $\tilde{Q}$. But these probabilities satisfy relation (15), and so

$$\lim_{t\to\infty} p_{ij}(t) = 0 \qquad i, j \in S .$$

Thus Q must be either null recurrent or transient.

The null recurrence of Q is equivalent to

$$\int_0^\infty p_{ii}(t)dt = \infty .$$

But, from relation (15),

$$\int_0^\infty p_{ii}(t)dt = \int_0^\infty \tilde{p}_{ii}(t)dt .$$

Thus if Q is null recurrent then so is $\tilde{Q}$, and this in turn implies that $\tilde{Q}$ is regular and m is invariant for $P(t)$.

4. A cycle criterion

Recall that a matrix $\tilde{Q}$ is the time-reverse of the matrix Q with respect to a collection of positive numbers $(m_i,\ i \in S)$ if

$$m_j\, q_{jk} = m_k\, \tilde{q}_{kj} \qquad j, k \in S \qquad (23).$$

Say that $\tilde{Q}$ is a time-reverse of Q if there exists a collection of positive numbers $(m_i,\ i \in S)$ satisfying (23).

Theorem Let Q and $\tilde{Q}$ be stable and conservative, and suppose that Q is irreducible. Then $\tilde{Q}$ is a time-reverse of Q if and only if

$$q_{jk_1} q_{k_1k_2} \cdots q_{k_rj} = \tilde{q}_{jk_r} \tilde{q}_{k_rk_{r-1}} \cdots \tilde{q}_{k_1j} \qquad (24)$$

for each positive integer r and for all $j, k_1, k_2, \ldots, k_r \in S$. When this condition holds the measure m determined up to scalar multiples by

$$m_j\, q_{jk} = m_k\, \tilde{q}_{kj} \qquad j, k \in S$$

is an invariant measure for the transition probabilities $P(t)$ constructed from Q if and only if $\tilde{Q}$ is regular.

Proof. If $\tilde{Q}$ is a time-reverse of Q then (24) follows immediately from (23). Conversely suppose that (24) is satisfied. This condition and the assumed irreducibility of Q imply that $\tilde{Q}$ is irreducible. Choose a base state, labelled 0 say, from the state space S. Set $m_o = 1$ and define

$$m_j = \frac{\tilde{q}_{0k_1}\ \tilde{q}_{k_1k_2} \cdots \tilde{q}_{k_wj}}{q_{jk_w}\ q_{k_wk_{w-1}} \cdots q_{k_10}} \tag{25}$$

where the path $(j, k_w, k_{w-1}, \ldots, k_1, 0)$ is chosen so that the denominator in expression (25) is positive. By virtue of the cycle condition (24) m_j also satisfies

$$m_j = \frac{q_{0h_1}\ q_{h_1h_2} \cdots q_{h_vj}}{\tilde{q}_{jh_v}\ \tilde{q}_{h_vh_{v-1}} \cdots \tilde{q}_{h_10}} \tag{26}$$

for any path $(j, h_v, h_{v-1}, \ldots, h_1, 0)$ chosen so that the denominator in expression (26) is positive. Since $\tilde{Q}$ is irreducible there is at least one such path. The representation (26) together with the cycle condition (24) then imply that every path $(j, k_w, k_{w-1}, \ldots, k_1, 0)$ which produces a positive denominator in expression (25) defines the same value for m_j.

We now show that m_j is positive. Since Q is irreducible there exists a cycle $(j, k_w, k_{w-1}, \ldots, k_1, 0, h_1, h_2, \ldots, h_v)$ such that both the denominator of expression (25) and the numerator of expression (26) are positive: the cycle condition (24) then implies that both the numerator of expression (25) and the denominator of expression (26) are positive.

Now consider two states $i, j \in S$, and suppose $q_{ij} > 0$. Then, using the path $(i, j, k_w, k_{w-1}, \ldots, k_1, 0)$ to define m_i, we have that

$$m_i = \frac{\tilde{q}_{ji}}{q_{ij}}\ m_j \ .$$

Similarly when $\tilde{q}_{ij} > 0$ expression (26) shows that

$$m_i = \frac{q_{ji}}{\tilde{q}_{ij}}\ m_j \ .$$

If $i = j$ then the choice $k_1 = j$ in relation (24) implies $q_{jj} = \tilde{q}_{jj}$. Thus

$$m_j\ q_{jk} = m_k\ \tilde{q}_{kj} \qquad j, k \in S \ , \tag{27}$$

and so $\tilde{Q}$ is the time-reverse of Q with respect to m.

Since $\tilde{Q}$ is assumed conservative equations (27) imply that m satisfies $mQ = 0$, and so the final part of the theorem follows from the results of the previous section. □

The above method can be used to establish a similar cycle criterion for discrete time Markov chains (cf. [8], p.32).

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