it, and a call attempt is lost if the associated direct path includes a vertex already handling a call.

The calling rates correspond to a vertex initiating call attempts at rate λ; a call attempt traverses a distance that is geometrically distributed with parameter q in either the east-west or north-south direction. The rates are clearly very special but serve to focus attention on the question of interest. Using a space-time diagram and a percolation bound, it is possible to establish the existence of, and provide a construction for, the stochastic process representing calls in progress at time t. For small enough values of λ, the construction shows that the process has a unique stationary distribution. But what happens for larger values of λ?

Conjecture. There exist values of λ and q such that the process has more than one stationary distribution.

For certain values of λ and q, there may be a translation invariant stationary distribution under which connected calls lie predominantly in a north-south direction; by symmetry, there would then also exist a stationary distribution favoring east-west calls. The conjecture is related to that of Kelbert and Suhov ([1], [8]) who consider a packet-switched network with queueing. The model described here is simpler, possessing a relatively explicit stationary distribution for any finite truncation, and this may make it easier to study. Marbukh [6] has considered a circuit-switched network based on a complete graph and has shown that if blocked calls are redirected along alternative routes, then instabilities may occur. The intuition behind this result is that alternative routes will be longer, use more of the facilities of the network, and thus that above a certain threshold, alternative routing may lead to greater and greater congestion. The intuition for the conjecture here is geometrical: calls fit together more easily when they are aligned.

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a unique stationary distribution. Some insight into the behavior of the system can be given by describing what is thought to be the unique stationary distribution of $(X(s,t),\ s \in IR)$ for a number of special cases. Suppose, for example, that the distribution of call distance F is exponential with parameter μ. Then it is believed that $(X(s,t),\ s \in IR)$ has the distribution of a certain Markov chain, stationary with respect to its parameter s, on the finite state space $\{0, 1, \ldots, C\}$. The structure of this Markov chain has been considered in detail by Ziedins [9]: roughly speaking, a Markov chain with transition rates $q(n, n+1) = \lambda$, $n = 0, 1, \ldots$, $q(n, n-1) = n\mu$, $n = 1, 2, \ldots$, is conditioned on its sample path lying within the set $\{0, 1, 2, \ldots, C\}$ for $s \in [-L, L]$, and then L is let tend to infinity. For a second example, suppose that F is general and that $C = 1$. Then it is believed that $(X(s,t),\ s \in IR)$ has the distribution of an alternating renewal process, with the lengths of successive intervals in state 1 (corresponding to calls in progress) having distribution function $\lambda\rho^{-1} \int_0^x e^{-\rho z}\, dF(z)$, and with the lengths of the intervening intervals in state 0 (corresponding to unoccupied stretches of cable) having an exponential distribution with parameter ρ; here ρ is the unique solution to the equation

$$\rho = \lambda \int_0^\infty e^{-\rho z}\, dF(z).$$

The acceptance probability for a call of length x is then

$$e^{-\rho x} \left[1 + \lambda \int_0^\infty z\, e^{-\rho z}\, dF(z) \right]^{-1}.$$

The network described in this section can be truncated and discretized so that it becomes a special case of the network of Section 2. From expression (1) the stationary distributions described above can be obtained as limits: further, the limits are not sensitive to the edge conditions imposed on the truncated network.

4. A Tree Network.

In this section, we describe an example which shows that with a countably infinite set of channels, the network of Section 2 may be unstable. Let V be the infinite tree with m (>2) edges emanating from each vertex. Regard the vertices as channels and suppose that each vertex has m circuits. Call attempts centered at vertex i arise as a Poisson process of rate ν. A call centered at vertex i requires m circuits from vertex i and one circuit from each of the m adjacent vertices. Let $X(i,t) = 1$ if a call centered at vertex i is in progress at time t, and let $X(i,t) = 0$ otherwise. Then the stochastic process $\{(X(i,t),\ i \in V),\ t \geq 0\}$ has more than one stationary distribution ([2], [4], [7]). Even when attention is restricted to stationary distributions which are invariant under graph isomorphisms, there may be more than one such distribution. For example, there is certainly more than one such distribution when

$$\nu > \frac{1}{m-1} \left[\frac{m-1}{m-2} \right]^m .$$

Variants can be constructed where the underlying graph is a two-dimensional lattice rather than a tree, the model then resembling the Ising model of an antiferromagnet.

5. A Two-Dimensional Network.

Consider now the two-dimensional lattice Z^2. Vertex $i = (i_1, i_2)$ never attempts to call vertex $j = (j_1, j_2)$ unless either $i_1 = j_1$ or $i_2 = j_2$. Call attempts between vertices (i_1, i_2) and (j_1, j_2) arise at rates

$$\frac{1}{2}\lambda (1-q)\, q^{j_1 - i_1 - 1} \qquad \text{if} \quad i_1 < j_1,\ i_2 = j_2$$

and

$$\frac{1}{2}\lambda (1-q)\, q^{j_2 - i_2 - 1} \qquad \text{if} \quad i_1 = j_1,\ i_2 < j_2.$$

A connected call between two vertices must use the direct (shortest) route between them, passing through each vertex on this route. However, a vertex cannot deal with more than one call terminating at or passing through

2. A Finite Network.

There is a finite set of channels, labeled $i = 1, 2, \ldots, I$. Channel i provides C_i circuits. Call attempts on route $r \in R$ arise as a Poisson process of rate ν_r, and as r varies, it indexes independent Poisson streams. A call attempt on route r requires A_{ir} circuits from channel i for $i = 1, 2, \ldots, I$. If for any $i \in \{1, 2, \ldots, I\}$ the number of free circuits on channel i is less than A_{ir}, then the call is lost. Otherwise, the call is accepted and occupies simultaneously A_{ir} circuits on channel i, for $i = 1, 2, \ldots, I$, for the holding period of the call. The call holding period is randomly distributed with unit mean and is independent of earlier arrival and holding times. Let $n_r(t)$ be the number of calls in progress at time t on route r, and let $n(t) = (n_r(t), r \in R)$. Then the stochastic process $\{n(t), t \geq 0\}$ has a unique stationary distribution and under this distribution $\pi(n) = P\{n(t) = n\}$ is given by

$$\pi(n) = B \prod_r \frac{\nu_r^{n_r}}{n_r!} \quad n \in S, \tag{1}$$

where

$$S = \{n : \sum_r A_{ir}\, n_r \leq C_i,\ i = 1, 2, \ldots, I\}$$

and B is a normalizing constant. Note that π does not depend on the distribution of call holding periods. If call holding periods are exponentially distributed, the stochastic process $\{n(t), t \geq 0\}$ is Markov.

3. A One-Dimensional Network.

Next we introduce some spatial structure. Imagine that users are arranged along an infinitely long cable and that a call between two points on the cable $s_1, s_2 \in IR$ involves just that section of cable between s_1 and s_2. Past any point along its length the cable has the capacity to carry simultaneously up to C calls: a call attempt between $s_1, s_2 \in IR$, $s_1 < s_2$, is lost if, past any point of the interval $[s_1, s_2]$, the cable is already carrying C calls. The statistics of call attempts are most easily defined using a

space-time diagram (Figure 1). A rectangle $\{(s, t) : s_1 \leq s \leq s_2,\ t_1 \leq t \leq t_2\}$ represents a call attempt between points s_1 and s_2 made at time t_1. If accepted, this

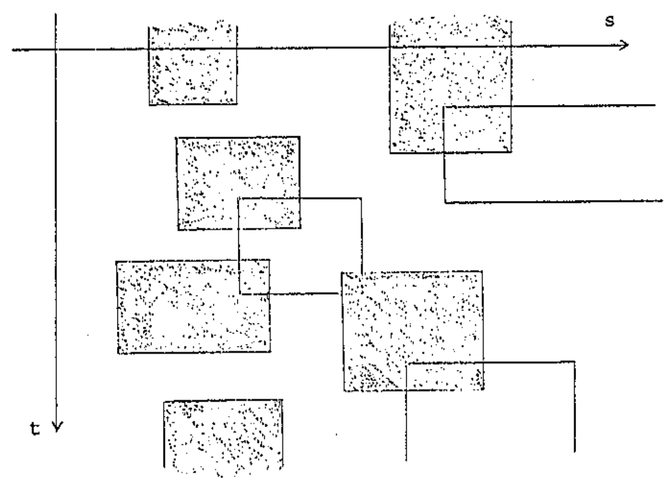

Figure 1. The space-time description of call attempts.

call will last until time t_2. Assume the north-east corners of rectangles are distributed as a Poisson process of rate λ (with respect to Lebesgue measure on IR^2). Assume that heights have unit mean, that widths have a distribution F with finite mean, and that heights and widths are independent of each other and of the positions of north-east corners. Informally, the probability that at time t a call attempt arises connecting a point s to a point $s + z$ is $\lambda\, dt\, ds\, dF(z)$. Let $X(s,t)$ be the number of calls in progress past point s on the cable at time t. It is possible to show that from an initial configuration of calls in progress at time $t = 0$, the space-time diagram defines the stochastic process $\{(X(s,t), s \in IR), t \geq 0\}$ with probability one. It is believed (but has not yet been rigorously proved) that this process has

3.13 INSTABILITY IN A COMMUNICATION NETWORK

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1. Introduction.

The problems described here are concerned with a stochastic model of a communication network. The model represents the interactions between the random demands placed on a network, and the aim is to understand its stationary behavior. In particular, we are interested in any clues that the network may exhibit instabilities, with perhaps various distinct modes of behavior possible.

In Section 2, we describe the model when there is a finite set of channels; it can then be analyzed completely, and a challenge is to extend this analysis to various situations involving an infinite set of channels. In Section 3, we discuss a one-dimensional network which is partially understood and which is believed to be stable. In Section 4, we describe a tree network which is unstable -- it may have more than one stationary distribution. Finally, in Section 5, we describe a two-dimensional network for which there is a conjecture.

The motivation for the problems described here is twofold. First, the model arises naturally in connection with circuit-switching, concurrency control, and some forms of dynamic routing ([2], [3]). Second, the mathematical issues are similar to those that arise in the study of interacting particle systems. There has been enormous progress in this field concerning the relationship between macroscopic phenomena, such as the existence of a phase transition, and the microscopic dynamical description of a system ([4], [5]). This topic is related to the notion of stability in a communication network, and the methods developed may prove useful.