

Heavy traffic on a controlled motorway

F. P. Kelly^a and R. J. Williams^b**Abstract**

Unlimited access to a motorway network can, in overloaded conditions, cause a loss of capacity. Ramp metering (signals on slip roads to control access to the motorway) can help avoid this loss of capacity. The design of ramp metering strategies has several features in common with the design of access control mechanisms in communication networks.

Inspired by models and rate control mechanisms developed for Internet congestion control, we propose a Brownian network model as an approximate model for a controlled motorway and consider it operating under a proportionally fair ramp metering policy. We present an analysis of the performance of this model.

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1 Introduction

The study of heavy traffic in queueing systems began in the 1960s, with three pioneering papers by Kingman [26, 27, 28]. These papers, and the early work of Prohorov [35], Borovkov [5, 6] and Iglehart [20], concerned a single resource. Since then there has been significant interest in networks of resources, with major advances by Harrison and Reiman [19], Reiman [37], Williams [43] and Bramson [7]. For discussions, further ref-

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erences and overviews of the very extensive literature on heavy traffic for networks, Williams [42], Bramson and Dai [8], Harrison [17, 18] and Whitt [41] are recommended.

Research in this area is motivated in part by the need to understand and control the behaviour of communications, manufacturing and service networks, and thus to improve their design and performance. But researchers are also attracted by the elegance of some of the mathematical constructs: in particular, the multi-dimensional reflecting Brownian motions that often arise as limits.

A question that arises in a wide variety of application areas concerns how flows through a network should be controlled, so that the network responds sensibly to varying conditions. Road traffic was an area of interest to early researchers [33], and more recently the question has been studied in work on modelling the Internet. In each of these cases the network studied is part of a larger system: for example, drivers generate demand and select their routes in ways that are responsive to the delays incurred or expected, which depend on the controls implemented in the road network. It is important to address such interactions between the network and the larger system, and in particular to understand the signals, such as delay, provided to the larger system.

Work on Internet congestion control generally addresses the issue of fairness, since there exist situations where a given scheme might maximise network throughput, for example, while denying access to some users. In this area it has been possible to integrate ideas of fairness of a control scheme with overall system optimization: indeed fairness of the control scheme is often the means by which the right information and incentives are provided to the larger system [24, 38].

Might some of these ideas transfer to help our understanding of the control of road traffic? In this paper we present a preliminary exploration of a particular topic: ramp metering. Unlimited access to a motorway network can, in overloaded conditions, cause a loss of capacity. Ramp metering (signals on slip roads to control access to the motorway) can help avoid this loss of capacity. The problem is one of access control, a common issue for communication networks, and in this paper we describe a ramp metering policy, *proportionally fair metering*, inspired by rate control mechanisms developed for the Internet.

The organisation of this paper is as follows. In Section 2 we review early heavy traffic results for a single queue. In Section 3 we describe a model of Internet congestion control, which we use to illustrate the simplifications and insights heavy traffic allows. In Section 4 we describe

a Brownian network model, which both generalizes a model of Section 2 and arises as a heavy traffic limit of the networks considered in Section 3. Sections 3 and 4 are based on the recent results of [21, 25]. These heavy traffic models help us to understand the behaviour of networks operating under policies for sharing capacity fairly.

In Section 5 we develop an approach to the design of ramp metering flow rates informed by the earlier Sections. For each of three examples, we present a Brownian network model operating under a proportionally fair metering policy. Our first example is a linear network representing a road into a city centre with several entry points; we then discuss a tree network, and, in Section 6, a simple network where drivers have routing choices. Within the Brownian network models we show that in each case the delay suffered by a driver at an entry point to the network can be expressed as a sum of dual variables, one for each of the resources to be used, and that under their stationary distribution these dual variables are independent exponential random variables. For the final example we show that the interaction of proportionally fair metering with choices available to arriving traffic has beneficial consequences for the performance of the system.

John Kingman's initial insight, that heavy traffic reveals the essential properties of queues, generalises to networks, where heavy traffic allows sufficient simplification to make clear the most important consequences of resource allocation policies.

2 A single queue

In this Section we review heavy traffic results for the M/G/1 queue, to introduce ideas that will be important later when we look at networks.

Consider a queue with a single server of unit capacity at which customers arrive as a Poisson process of rate ν . Customers bring amounts of work for the server which are independent and identically distributed with distribution G , and are independent of the arrival process. Assume the distribution G has mean $1/\mu$ and finite second moment, and that the load on the queue, $\rho = \nu/\mu$, satisfies $\rho < 1$.

Let $W(t)$ be the *workload* in the queue at time t ; for a server of unit capacity this is the time it would take for the server to empty the queue if no more arrivals were to occur after time t .

Kingman [26] showed that the stationary distribution of $(1 - \rho)W$ is asymptotically exponentially distributed as $\rho \rightarrow 1$. Current approaches

to heavy traffic generally proceed via a weaker assumption that the cumulative arrival process of work satisfies a functional central limit theorem, and use this to show that as $\rho \rightarrow 1$, the appropriately normalized workload process

$$\hat{W}(t) = (1 - \rho) W\left(\frac{t}{(1 - \rho)^2}\right), \quad t \geq 0 \quad (2.1)$$

can be approximated by a reflecting Brownian motion \tilde{W} on \mathbb{R}_+ . In the interior $(0, \infty)$ of \mathbb{R}_+ , \tilde{W} behaves as a Brownian motion with drift -1 and variance determined by the variance of the cumulative arrival process of work. When \tilde{W} hits zero, then the server may become idle; this is where delicacy is needed. The stationary distribution of the reflecting Brownian motion \tilde{W} is exponential, corresponding to Kingman's early result.

We note an important consequence of the scalings appearing in the definition (2.1), the *snapshot principle*. Because of the different scalings applied to space and time, the workload is of order $(1 - \rho)^{-1}$ while the workload can change significantly only over time intervals of order $(1 - \rho)^{-2}$. Hence the time taken to serve the amount of work in the queue is asymptotically negligible compared to the time taken for the workload to change significantly [36, 41].

Note that the workload $(W(t), t \geq 0)$ does not depend on the queue discipline (provided the discipline does not allow idling when there is work to be done), although the waiting time for an arriving customer certainly does. Kingman [29] makes elegant use of the snapshot principle to compare stationary waiting time distributions under a range of queue disciplines.

It will be helpful to develop in detail a simple example. Consider a Markov process in continuous time $(N(t), t \geq 0)$ with state space \mathbb{Z}_+ and non-diagonal infinitesimal transition rates

$$q(n, n') = \begin{cases} \nu & \text{if } n' = n + 1, \\ \mu & \text{if } n' = n - 1 \text{ and } n > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

Let $\rho = \nu/\mu$. If $\rho < 1$ then the Markov process $(N(t), t \geq 0)$ has stationary distribution

$$\mathbb{P}\{N^s = n\} = (1 - \rho)\rho^n, \quad n = 0, 1, 2, \dots \quad (2.3)$$

(here, the superscript s signals that the random variable is associated

with the stationary distribution). The Markov process corresponds to an M/M/1 queue, at which customers arrive as a Poisson process of rate ν , and where customers bring an amount of work for the server which is exponentially distributed with parameter μ .

Next consider an M/G/1 queue with the processor-sharing discipline (under the processor-sharing discipline, while there are n customers in the queue each receives a proportion $1/n$ of the capacity of the server). The process $(N(t), t \geq 0)$ is no longer Markov, but it nonetheless has the same stationary distribution as in (2.3). Moreover in the stationary regime, given $N^s = n$, the amounts of work left to be completed on each of the n customers in the queue form a collection of n independent random variables, each with distribution function

$$G^*(x) = \mu \int_0^x (1 - G(z)) dz, \quad x \geq 0,$$

a distribution recognisable as that of the forward recurrence time in a stationary renewal process whose inter-event time distribution is G . Thus the stationary distribution of W is just that of the sum of N^s independent random variables each with distribution G^* , where N^s has the distribution (2.3) [2, 23]. Let S be a random variable with distribution G . Then we can deduce that the stationary distribution of (W, N) has the property that in probability $W^s/N^s \rightarrow \mathbb{E}(S^2)/2\mathbb{E}(S)$, the mean of the distribution G^* , as $\rho \rightarrow 1$. For fixed x , under the stationary distribution for the queue, let N_x^s be the number of customers in the queue with a remaining work requirement of not more than x . Then, $N_x^s/N^s \rightarrow G^*(x)$ in probability as $\rho \rightarrow 1$. At the level of stationary distributions, this is an example of a property called state-space collapse: in heavy traffic the stochastic behaviour of the system is essentially given by W , with more detailed information about the system (in this case, the numbers of customers with various remaining work requirements) not being necessary.

The amount of work arriving at the queue over a period of time, τ , has a compound Poisson distribution, with a straightforwardly calculated mean and variance of $\rho\tau$ and $\rho\sigma^2\tau$ respectively, where $\sigma^2 = \mathbb{E}(S^2)/\mathbb{E}(S)$. An alternative approach [15] is to directly model the cumulative arrival process of work as a Brownian motion $\check{E} = (\check{E}(t), t \geq 0)$ with matching mean and variance parameters: thus

$$\check{E}(t) = \rho t + \rho^{1/2}\sigma\check{Z}(t), \quad t \geq 0,$$

where $(\check{Z}(t), t \geq 0)$ is a standard Brownian motion. Let

$$\check{X}(t) = \check{E}(t) - t, \quad t \geq 0,$$

a Brownian motion starting from the origin with drift $-(1 - \rho)$ and variance $\rho\sigma^2$. In this approach we define the queue's workload $\check{W}(t)$ at time t by the system of equations

$$\check{W}(t) = \check{W}(0) + \check{X}(t) + \check{U}(t), \quad t \geq 0, \quad (2.4)$$

$$\check{U}(t) = - \inf_{0 \leq s \leq t} \check{X}(s), \quad t \geq 0. \quad (2.5)$$

The interpretation of the model is as follows. While \check{W} is positive, it is driven by the Brownian fluctuations caused by arrival of work less the work served. But when \check{W} hits zero, the resource may not be fully utilized. The process \check{U} defined by equation (2.5) is continuous and non-decreasing, and is the minimal such process that permits \check{W} , given by equation (2.4), to remain non-negative. We interpret $\check{U}(t)$ as the cumulative unused capacity up to time t . Note that \check{U} can increase only at times when \check{W} is at zero.

The stationary distribution of \check{W} is exponential with mean $\rho\sigma^2/2(1 - \rho)$ [15]. This is the same as the distribution of $(1 - \rho)^{-1}\tilde{W}^s$ where \tilde{W}^s has the stationary distribution of the reflecting Brownian motion \tilde{W} that approximates the scaled process \hat{W} given by (2.1). Furthermore, the mean of the stationary distribution of \check{W} is the same as the mean of the exact stationary distribution of the workload W , calculated from its representation as the geometric sum (2.3) of independent random variables each with distribution G^* and hence mean $\mathbb{E}(S^2)/2\mathbb{E}(S)$.

In other words, for the M/G/1 queue, we obtain the same exponential stationary distribution either by (a) approximating the workload arrival process directly by a Brownian motion without any space or time scaling, or by (b) approximating the scaled workload process in (2.1) by a reflecting Brownian motion, finding the stationary distribution of the latter, and then formally unwinding the spatial scaling to obtain a distribution in the original spatial units. Furthermore, this exponential distribution has the same mean as the exact stationary distribution for the workload in the M/G/1 queue and provides a rather good approximation, being of the same order of accuracy as the exponential approximation of the geometric distribution with the same mean.

The main point of the above discussion is that, in the context of this example, we observe that for the purposes of computing approximations to the stationary workload, using a direct Brownian model for the workload arrival process (by matching mean and variance parameters) provides the same results as use of the heavy traffic diffusion approximation coupled with formal unwinding of the spatial scaling, and the approxi-

mate stationary distribution that this yields compares remarkably well with exact results. We shall give another example of this kind of fortuitously good approximation in Section 4. Chen and Yao [9] have also noted remarkably good results from using such ‘strong approximations’ without any scaling.

3 A model of Internet congestion

In this Section we describe a network generalization of processor sharing that has been useful in modelling flows through the Internet, and outline a recent heavy traffic approach [21, 25] to its analysis.

3.1 Fair sharing in a network

Consider a network with a finite set \mathcal{J} of *resources*. Let a *route* i be a non-empty subset of \mathcal{J} , and write $j \in i$ to indicate that resource j is used by route i . Let \mathcal{I} be the set of possible routes. Assume that both \mathcal{J} and \mathcal{I} are non-empty and finite, and let J and I denote the cardinality of the respective sets. Set $A_{ji} = 1$ if $j \in i$, and $A_{ji} = 0$ otherwise. This defines a $J \times I$ matrix $A = (A_{ji}, j \in \mathcal{J}, i \in \mathcal{I})$ of zeroes and ones, the *resource-route incidence matrix*. Assume that A has rank J , so that it has full row rank.

Suppose that resource j has capacity $C_j > 0$, and that there are n_i connections using route i . How might the capacities $C = (C_j, j \in \mathcal{J})$ be shared over the routes \mathcal{I} , given the numbers of connections $n = (n_i, i \in \mathcal{I})$? This is a question which has attracted attention in a variety of fields, ranging from game theory, through economics to political philosophy. Here we describe a concept of fairness which is a natural extension of Nash’s bargaining solution and, as such, satisfies certain natural axioms of fairness [32]; the concept has been used extensively in the modelling of rate control algorithms in the Internet [24, 38].

Let $\mathcal{I}_+(n) = \{i \in \mathcal{I} : n_i > 0\}$. A *capacity allocation policy* $\Lambda = (\Lambda(n), n \in \mathbb{R}_+^I)$, where $\Lambda(n) = (\Lambda_i(n), i \in \mathcal{I})$, is called *proportionally fair* if for each $n \in \mathbb{R}_+^I$, $\Lambda(n)$ solves

$$\text{maximise} \quad \sum_{i \in \mathcal{I}_+(n)} n_i \log \Lambda_i \quad (3.1)$$

$$\text{subject to} \quad \sum_{i \in \mathcal{I}} A_{ji} \Lambda_i \leq C_j, \quad j \in \mathcal{J}, \quad (3.2)$$

$$\text{over} \quad \Lambda_i \geq 0, \quad i \in \mathcal{I}_+(n), \quad (3.3)$$

$$\Lambda_i = 0, \quad i \in \mathcal{I} \setminus \mathcal{I}_+(n). \quad (3.4)$$

Note that the constraint (3.2) captures the limited capacity of resource j , while constraint (3.4) requires that no capacity be allocated to a route which has no connections.

The problem (3.1)–(3.4) is a straightforward convex optimization problem, with optimal solution

$$\Lambda_i(n) = \frac{n_i}{\sum_{j \in \mathcal{J}} q_j A_{ji}}, \quad i \in \mathcal{I}_+(n), \quad (3.5)$$

where the variables $q = (q_j, j \in \mathcal{J}) \geq 0$ are Lagrange multipliers (or *dual variables*) for the constraints (3.2). The solution to the optimization problem is unique and satisfies $\Lambda_i(n) > 0$ for $i \in \mathcal{I}_+(n)$ by the strict concavity on $(\Lambda_i > 0, i \in \mathcal{I}_+(n))$ and boundary behaviour of the objective function in (1.6) [25].

The dual variables q are unique if $n_i > 0$ for all $i \in \mathcal{I}$, but may not be unique otherwise. In any event they satisfy the *complementary slackness* conditions

$$q_j \left(C_j - \sum_{i \in \mathcal{I}} A_{ji} \Lambda_i(n) \right) = 0, \quad j \in \mathcal{J}. \quad (3.6)$$

3.2 Connection level model

The allocation $\Lambda(n)$ describes how capacities are shared for a given number of connections n_i on each route $i \in \mathcal{I}$. Next we describe a stochastic model [31] for how the number of connections within the network varies.

A connection on route i corresponds to continuous transmission of a document through the resources used by route i . Transmission is assumed to occur simultaneously through all the resources used by route i . Let the number of connections on route i at time t be denoted by $N_i(t)$, and let $N(t) = (N_i(t), i \in \mathcal{I})$. We consider a Markov process in continuous time $(N(t), t \geq 0)$ with state space \mathbb{Z}_+^I and non-diagonal infinitesimal transition rates

$$q(n, n') = \begin{cases} \nu_i & \text{if } n' = n + e_i, \\ \mu_i \Lambda_i(n) & \text{if } n' = n - e_i \text{ and } n_i > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (3.7)$$

where e_i is the i -th unit vector in \mathbb{Z}_+^I , and $\nu_i, \mu_i > 0, i \in \mathcal{I}$.

The Markov process $(N(t), t \geq 0)$ corresponds to a model where new

connections arrive on route i as a Poisson process of rate ν_i , and a connection on route i transfers a document whose size is exponentially distributed with parameter μ_i . In the case where $I = J = 1$ and $C_1 = 1$, the transition rates (3.7) reduce to the rates (2.2) of the M/M/1 queue.

Define the *load* on route i to be $\rho_i = \nu_i/\mu_i$ for $i \in \mathcal{I}$. It is known [4, 11] that the Markov process is positive recurrent provided

$$\sum_{i \in \mathcal{I}} A_{ji} \rho_i < C_j, \quad j \in \mathcal{J}. \quad (3.8)$$

These are natural constraints: the load arriving at the network for resource j must be less than the capacity of resource j , for each $j \in \mathcal{J}$. Let $[\rho]$ be the $I \times I$ diagonal matrix with the entries of $\rho = (\rho_i, i \in \mathcal{I})$ on its diagonal, and define ν , $[\nu]$, μ , $[\mu]$ similarly.

Each connection on route i brings with it an amount of work for resource j which is exponentially distributed with mean $1/\mu_i$, for $j \in i$. The Markov process N allows us to estimate the workload for each resource: define the *workload process* by

$$W(t) = A[\mu]^{-1}N(t), \quad t \geq 0. \quad (3.9)$$

3.3 Heavy traffic

To approximate the workload in a heavily loaded connection-level model by that in a Brownian network model, we view a given connection-level model as a member of a sequence of such models approaching the heavy traffic limit. More precisely, we consider a sequence of connection-level models indexed by r where the network structure, defined by A and C , does not vary with r . Each member of the sequence is a stochastic system as described in the previous section. We append a superscript of r to any process or parameter associated with the r^{th} system that depends on r . Thus, we have processes N^r , W^r , and parameters ν^r . We suppose $\mu^r = \mu$ for all r , so that $\rho_i^r = \nu_i^r/\mu_i$, for each $i \in \mathcal{I}$. We shall assume henceforth that the following *heavy traffic* condition holds: as $r \rightarrow \infty$,

$$\nu^r \rightarrow \nu \quad \text{and} \quad r(A\rho^r - C) \rightarrow -\theta \quad (3.10)$$

where $\nu_j > 0$ and $\theta_j > 0$ for all $j \in \mathcal{J}$. Note that (3.10) implies that $\rho^r \rightarrow \rho$ as $r \rightarrow \infty$ and that $A\rho = C$.

We define fluid scaled processes \bar{N}^r , \bar{W}^r as follows. For each r and $t \geq 0$, let

$$\bar{N}^r(t) = N^r(rt)/r, \quad \bar{W}^r(t) = W^r(rt)/r. \quad (3.11)$$

What might be the limit of the sequence $\{\bar{N}^r\}$ as $r \rightarrow \infty$? From the transition rates (3.7) and the observation that $\Lambda(rn) = \Lambda(n)$ for $r > 0$, we would certainly expect that the limit satisfies

$$\frac{d}{dt} n_i(t) = \nu_i - \mu_i \Lambda_i(n(t)), \quad i \in \mathcal{I}, \quad (3.12)$$

whenever n is differentiable at t and $n_i(t) > 0$ for all $i \in \mathcal{I}$. Indeed, this forms part of the fluid model developed in [25] as a functional-law-of-large-numbers approximation. Extra care is needed in defining the fluid model at any time t when $n_i(t) = 0$, for any $i \in \mathcal{I}$: the function $\Lambda(n)$ may not be continuous on the boundary of the region \mathbb{R}_+^I , and so when any component $\bar{N}_i^r(t)$ is hitting zero, $\Lambda(\bar{N}^r(t))$ may jitter.

It is shown in [25] that the set of invariant states for the fluid model is

$$\mathcal{N} = \left\{ n \in \mathbb{R}_+^I : n_i = \rho_i \sum_{j \in \mathcal{J}} q_j A_{ji}, i \in \mathcal{I} \text{ for some } q \in \mathbb{R}_+^J \right\}$$

as we would expect from formally setting the derivatives in (3.12) to zero and using relation (3.5). Call \mathcal{N} the *invariant manifold*. If $n \in \mathcal{N}$, then since A has full row rank the representation of n in terms of q is unique; furthermore, $\Lambda_i(n) = \rho_i$ for $i \in \mathcal{I}_+(n)$ and then since $A\rho = C$, the vector q satisfies equation (3.5) and the complementary slackness conditions (3.6), and hence gives dual variables for the optimization problem (3.1)–(3.4).

For each $n \in \mathbb{R}_+^I$, define $w(n) = (w_j(n), j \in \mathcal{J})$, the workload associated with n , by $w(n) = A[\mu]^{-1}n$. For each $w \in \mathbb{R}_+^J$, define $\Delta(w)$ to be the unique value of $n \in \mathbb{R}_+^I$ that solves the following optimization problem:

$$\begin{aligned} & \text{minimize} && F(n) \\ & \text{subject to} && \sum_{i \in \mathcal{I}} A_{ji} \frac{n_i}{\mu_i} \geq w_j, \quad j \in \mathcal{J}, \\ & \text{over} && n_i \geq 0, \quad i \in \mathcal{I}, \end{aligned}$$

where

$$F(n) = \sum_{i \in \mathcal{I}} \frac{n_i^2}{\nu_i}, \quad n \in \mathbb{R}_+^I.$$

The function $F(n)$ was introduced in [4] and can be used to show positive recurrence of N under conditions (3.8). In [25] the difference $F(n) - F(\Delta(w(n)))$ is used as a Lyapunov function to show that any fluid model solution $(n(t), t \geq 0)$ converges towards the invariant manifold \mathcal{N} . It is

straightforward to check that $n \in \mathcal{N}$ if and only if $n = \Delta(w(n))$ and it turns out that

$$\Delta(w) = [\rho]A'(A[\mu]^{-1}[\nu][\mu]^{-1}A')^{-1}w.$$

Note that if \bar{N}^r lives in the space \mathcal{N} then \bar{W}^r , given by equations (3.9) and (3.11) as $A[\mu]^{-1}\bar{N}^r$, lives in the space $\mathcal{W} = A[\mu]^{-1}\mathcal{N}$, which we can write as

$$\mathcal{W} = \left\{ w \in \mathbb{R}_+^J : w = A[\mu]^{-1}[\nu][\mu]^{-1}A'q \text{ for some } q \in \mathbb{R}_+^J \right\}, \quad (3.13)$$

generally a space of lower dimension. Call \mathcal{W} the *workload cone*. Let

$$\begin{aligned} \mathcal{W}^j = \left\{ w \in \mathbb{R}_+^J : w = A[\mu]^{-1}[\nu][\mu]^{-1}A'q \right. \\ \left. \text{for some } q \in \mathbb{R}_+^J \text{ satisfying } q_j = 0 \right\}, \end{aligned} \quad (3.14)$$

which we refer to as the j^{th} face of the workload cone \mathcal{W} .

We define diffusion scaled processes \hat{N}^r, \hat{W}^r as follows. For each r and $t \geq 0$, let

$$\hat{N}^r(t) = \frac{N^r(r^2t)}{r}, \quad \hat{W}^r(t) = \frac{W^r(r^2t)}{r}.$$

In the next sub-section we outline the convergence in distribution of the sequence $\{(\hat{N}^r, \hat{W}^r)\}$ as $r \rightarrow \infty$. As preparation, note that if $N(t) \in \mathcal{N}$ and $N_i(t) > 0$ for all $i \in \mathcal{I}$, then $\Lambda_i(N(t)) = \rho_i$ for all $i \in \mathcal{I}$. Suppose, as a thought experiment, that for each $i \in \mathcal{I}$ the component \hat{N}_i^r behaves as the queue-length process in an independent M/M/1 queue, with a server of capacity ρ_i . Then a Brownian approximation to \hat{N}_i^r would have variance $\nu_i + \mu_i \rho_i = 2\nu_i$. Next observe that if the covariance matrix of \hat{N}^r is $2[\nu]$ then the covariance matrix of $\hat{W}^r = A[\mu]^{-1}\hat{N}^r$ is

$$\Gamma = 2A[\mu]^{-1}[\nu][\mu]^{-1}A'. \quad (3.15)$$

4 A Brownian network model

Let (A, ν, μ) be as in Section 3: thus A is a matrix of zeroes and ones of dimension $J \times I$ and of full row rank, and ν, μ are vectors of positive entries of dimension I . Let $\rho_i = \nu_i/\mu_i$, $i \in \mathcal{I}$, and $\rho = (\rho_i, i \in \mathcal{I})$. Let \mathcal{W} and \mathcal{W}^j be defined by expressions (3.13) and (3.14) respectively. Let $\theta \in \mathbb{R}^J$ and Γ be given by (3.15).

In the following, all processes are assumed to be defined on a fixed

filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and to be adapted to the filtration $\{\mathcal{F}_t\}$. Let η be a probability distribution on \mathcal{W} . Define a Brownian network model by the following relationships:

- (i) $\tilde{W}(t) = \tilde{W}(0) + \tilde{X}(t) + \tilde{U}(t)$ for all $t \geq 0$,
- (ii) \tilde{W} has continuous paths, $\tilde{W}(t) \in \mathcal{W}$ for all $t \geq 0$, and $\tilde{W}(0)$ has distribution η ,
- (iii) \tilde{X} is a J -dimensional Brownian motion starting from the origin with drift $-\theta$ and covariance matrix Γ such that $\{\tilde{X}(t) + \theta t, \mathcal{F}_t, t \geq 0\}$ is a martingale under \mathbb{P} ,
- (iv) for each $j \in \mathcal{J}$, \tilde{U}_j is a one-dimensional process such that
 - (a) \tilde{U}_j is continuous and non-decreasing, with $\tilde{U}_j(0) = 0$,
 - (b) $\tilde{U}_j(t) = \int_0^t 1_{\{\tilde{W}(s) \in \mathcal{W}^j\}} d\tilde{U}_j(s)$ for all $t \geq 0$.

The interpretation of the above Brownian network model is as follows. In the interior of the workload cone \mathcal{W} each of the J resources are fully utilized, route i is receiving a capacity allocation ρ_i for each $i \in \mathcal{I}$, and the workloads \tilde{W} are driven by the Brownian fluctuations caused by arrivals and departures of connections. But when \tilde{W} hits the j^{th} face of the workload cone \mathcal{W} , resource j may not be fully utilized. The cumulative unused capacity \tilde{U}_j at resource j is non-decreasing, and can increase only on the j^{th} face of the workload cone \mathcal{W} .

The work of Dai and Williams [10] establishes the existence and uniqueness in law of the above diffusion $\tilde{W} = (\tilde{W}(t), t \geq 0)$. In [21] it is shown that, if $\theta_j > 0$ for all $j \in \mathcal{J}$, then \tilde{W} has a unique stationary distribution; furthermore, if \tilde{W}^s denotes a random variable with this stationary distribution, then the components of $\tilde{Q}^s = 2\Gamma^{-1}\tilde{W}^s$ are independent and \tilde{Q}_j^s is exponentially distributed with parameter θ_j for each $j \in \mathcal{J}$.

Now let C be a vector of positive entries of dimension J , define a sequence of networks as in Section 3.3, and suppose θ and C are related by the heavy traffic condition (3.10). In [21] it is shown that, subject to a certain local traffic condition on the matrix A and suitable convergence of initial variables $(\hat{W}^r(0), \hat{N}^r(0))$, the pair (\hat{W}^r, \hat{N}^r) converges in distribution as $r \rightarrow \infty$ to a continuous process (\tilde{W}, \tilde{N}) where \tilde{W} is the above diffusion and $\tilde{N} = \Delta(\tilde{W})$. The proof in [21] relies on both the existence and uniqueness results of [10] and an associated invariance principle developed by Kang and Williams [22]. (The local traffic condition under which convergence is established requires that the matrix A contains amongst its columns the columns of the $J \times J$ identity matrix:

this corresponds to each resource serving at least one route which uses only that resource. The local traffic condition is not needed to show that \tilde{W} has the aforementioned stationary distribution; that requires only the weaker condition that A have full row rank.)

It is convenient to define $\tilde{Q} = 2\Gamma^{-1}\tilde{W}$, a process of *dual variables*. From this, the form of Δ , and the relation $\tilde{N} = \Delta(\tilde{W})$, it follows that $\tilde{N} = [\rho]A'\tilde{Q}$. The dimension of the space in which \tilde{Q} lives is J , and so this is an example of state-space collapse, with the I -dimensional process \tilde{N} living on a J -dimensional manifold where $J \leq I$ is often considerably less than I .

Using the stationary distribution for \tilde{W} , we see that $\tilde{N}^s = [\rho]A'\tilde{Q}^s$ has the stationary distribution of \tilde{N} . Then, after formally unwinding the spatial scaling used to obtain our Brownian approximation, we obtain the following simple approximation for the stationary distribution of the number-of-connections process in the original model described in Section 3.2:

$$N_i^s \approx \rho_i \sum_{j \in \mathcal{J}} Q_j^s A_{ji}, \quad i \in \mathcal{I}, \quad (4.1)$$

where Q_j^s , $j \in \mathcal{J}$, are independent and Q_j^s is exponentially distributed with parameter $C_j - \sum_{i \in \mathcal{I}} A_{ji}\rho_i$.

As mentioned in Section 2, an alternative approach is to directly model the cumulative arrival process of work for each route i as a Brownian motion:

$$\check{E}_i(t) = \rho_i t + \left(\frac{2\rho_i}{\mu_i} \right)^{1/2} \check{Z}_i(t), \quad t \geq 0,$$

where $(\check{Z}_i(t), t \geq 0)$, $i \in \mathcal{I}$, are independent standard Brownian motions; here the form of the variance parameter takes account of the fact that the document sizes are exponentially distributed. Under this model, the potential netflow (inflow minus potential outflow, ignoring underutilization of resources) process of work for resource j is

$$\check{X}_j(t) = \sum_{i \in \mathcal{I}} A_{ji} \check{E}_i(t) - C_j t, \quad t \geq 0,$$

a J -dimensional Brownian motion starting from the origin with drift $A\rho - C$ and covariance matrix $2A[\rho][\mu]^{-1}A' = \Gamma$. Then the workload is modelled by a J -dimensional process \check{W} that satisfies properties (i)–(iv) above, but with \check{W} in place of \tilde{W} and $A\rho - C$ in place of the drift $-\theta$; the covariance matrix remains the same. By the results of [21], if $A\rho < C$, there is a unique stationary distribution for the process \check{W}

such that if \check{W}^s has this stationary distribution then the components of $\check{Q}^s = 2\Gamma^{-1}\check{W}^s$ are independent and \check{Q}_j^s is exponentially distributed with parameter $C_j - \sum_{i \in \mathcal{I}} A_{ji}\rho_i$ for each $j \in \mathcal{J}$. The random variable

$$\check{N}^s = [\rho]A'\check{Q}^s, \quad (4.2)$$

has the stationary distribution of $\check{N} = [\rho]A'\check{Q}$, which is the same as the distribution of the right member of (4.1). Thus, just as in the simple case considered in Section 2, in this connection-level model, using the direct Brownian model yields the same approximation for the stationary distribution of the number-of-connections process as that obtained using the heavy traffic diffusion approximation and formally unwinding the spatial scaling in its stationary distribution.

If we specialize the direct Brownian network model to the case where $I = J = 1$ and $C = 1$, then we obtain the Brownian model of Section 2, with $\Gamma = 2\nu/\mu^2 = \rho\sigma^2$ and where the stationary distribution for \check{W} is exponentially distributed with mean $\rho\sigma^2/2(1 - \rho)$, yielding the same approximation as in Section 2.

A more interesting example is obtained when $I = J + 1$ and A is the $J \times (J + 1)$ matrix:

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \end{pmatrix},$$

so that J routes each use a single resource in such a way that there is exactly one such route for each resource, and one route uses all J resources. In this case, the stationary distribution given by (4.2) accords remarkably well with the exact stationary distribution described by Massoulié and Roberts [31]; it is again of the order of accuracy of the exponential approximation of the geometric distribution with the same mean. (We refer the interested reader to [21] for the details of this good approximation.)

In this Section and in Section 2 we have seen intriguing examples of remarkably good approximations that the direct Brownian modelling approach can yield. Inspired by this, in the next two Sections we explore the use of the direct Brownian network model as a representation of workload for a controlled motorway. Rigorous justification for use of this modelling framework in the motorway context has yet to be investigated. See the last section of the paper for further comments on this issue.

5 A model of a controlled motorway

Once motorway traffic exceeds a certain threshold level (measured in terms of density—the number of vehicles per mile) both vehicle speed and vehicle throughput drop precipitously [13, 12, 39]. The smooth pattern of flow that existed at lower densities breaks down, and the driver experiences stop-go traffic. Maximum vehicle throughput (measured in terms of the number of vehicles per minute) occurs at quite high speeds—about 60 miles per hour on Californian freeways and on London’s orbital motorway, the M25 [13, 12, 39]—while after flow breakdown the average speed may drop to 20–30 miles per hour. Particularly problematic is that flow breakdown may persist long after the conditions that provoked its onset have disappeared.

Variable speed limits lessen the number and severity of accidents on congested roads and are in use, for example, on the south-west quadrant of the M25. But variable speed limits do not avoid the loss of throughput caused by too high a density of vehicles [1, 14]. Ramp metering (signals on slip roads to control access to the motorway) can limit the density of vehicles, and thus can avoid the loss of throughput [30, 34, 40, 44]. But a cost of this is queueing delay on the approaches to the motorway. How should ramp metering flow rates be chosen to control these queues, and to distribute queueing delay fairly over the various users of the motorway? In this Section we introduce a modelling approach to address this question, based on several of the simplifications that we have seen arise in heavy traffic.

5.1 A linear network

Consider the linear¹ road network illustrated in Figure 5.1. Traffic can enter the main carriageway from lines at entry points, and then travels from left to right, with all traffic destined for the exit at the right hand end (think of this as a model of a road collecting traffic all bound for a city). Let $M_1(t)$, $M_2(t)$, \dots , $M_J(t)$ taking values in \mathbb{R}_+ be the line sizes² at the entry points at time t , and let C_1 , C_2 , \dots , C_J be the respective capacities of sections of the road. We assume the road starts at the left hand end, with line J feeding an initial section of capacity C_J , and that

¹ We caution the reader that here we use the descriptive term ‘linear network’ in a manner that differs from its use in [21].

² The term line size is used here to mean a quantity measuring the amount of work in the queue, rather than the more restrictive number of jobs that is often associated with the term queue size.

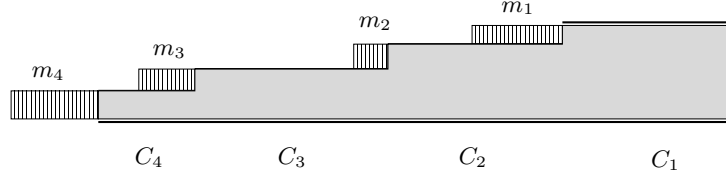


Figure 5.1 Lines of size m_1, m_2, m_3, m_4 are held on the slip roads leading to the main carriageway. Traffic on the main carriageway is free-flowing. Access to the main carriageway from the slip roads is metered, so that the capacities C_1, C_2, C_3, C_4 of successive sections are not overloaded.

$C_1 > C_2 > \dots > C_J > 0$. The corresponding resource-route incidence matrix is the square matrix

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \quad (5.1)$$

We model the traffic, or work, arriving at line i , $i \in \mathcal{I}$, as follows: let $E_i(t)$ be the cumulative inflow to line i over the time interval $(0, t]$, and assume $(E_i(t), t \geq 0)$ is an ergodic process with non-negative, stationary increments, with $\mathbb{E}[E_i(t)] = \rho_i t$, where $\rho_i > 0$, and suppose these processes are independent over $i \in \mathcal{I}$. Suppose the metering rates for lines $1, 2, \dots, J$ at time t can be chosen to be any measurable vector-valued function $\Lambda = \Lambda(M(t))$ satisfying constraints (3.2)–(3.4) with $n = M(t)$, and such that

$$M_i(t) = M_i(0) + E_i(t) - \int_0^t \Lambda_i(M(s)) ds \geq 0, \quad t \geq 0 \quad (5.2)$$

for $i \in \mathcal{I}$. Observe that we do not take into account travel time along the road: motivated by the snapshot principle, we suppose that $M(\cdot)$ varies relatively slowly compared with the time taken to travel through the system.³

³ The time taken for a vehicle to travel through the system comprises both the queueing time at the entry point and the travel time along the motorway. If the motorway is free-flowing, the aim of ramp metering, then the travel time along the motorway may be reasonably modelled by a constant not dependent on $M(t)$, say τ_i from entry point i . A more refined treatment might insist that the rates $\Lambda(M_i(t - \tau_i), i \in \mathcal{I})$ satisfy the capacity constraints (3.2). We adopt the simpler approach, since we expect that in heavy traffic travel times along the motorway will be small compared with the time taken for $M(t)$ to change significantly.

How might the rate function $\Lambda(\cdot)$ be chosen? We begin by a discussion of two extreme strategies. First we consider a strategy that prioritises the upstream entry points. Suppose the metered rate from line J , $(\Lambda_J(M(t)), t \geq 0)$, is chosen so that for each $t \geq 0$ the cumulative outflow from line J , $\int_0^t \Lambda_J(M(s)) ds$, is maximal, subject to the constraint (5.2) and $\Lambda_J(M(t)) \leq C_J$ for all $t \geq 0$: thus there is equality in the latter constraint whenever $M_J(t)$ is positive. For each of $j = J - 1, J - 2, \dots, 1$ in turn define $\int_0^t \Lambda_j(M(s)) ds$ to be maximal, subject to the constraint (5.2) and

$$\Lambda_j(M(t)) \leq C_j - \sum_{i=j+1}^J \Lambda_i(M(t)), \quad t \geq 0. \quad (5.3)$$

In consequence there is equality in constraint (5.3) at time t if $M_j(t) > 0$, and by induction for each $t \geq 0$ the cumulative flow along link j , $\int_0^t \sum_{i=j}^J \Lambda_i(M(s)) ds$, is maximal, for $j = J, J - 1, \dots, 1$. Thus this strategy minimizes, for all times t , the sum of the line sizes at time t , $\sum_j M_j(t)$.

The above optimality property is compelling if the arrival patterns of traffic are exogenously determined. The strategy will, however, concentrate delay upon the flows entering the system at the more downstream entry points. This seems intuitively unfair, since these flows use fewer of the system's resources, and it may well have perverse and suboptimal consequences if it encourages growth in the load ρ_i arriving at the upstream entry points. For example, growth in ρ_J may cause the natural constraint (3.8) to be violated, even while traffic arriving at line J suffers only a small amount of additional delay.

Next we consider a strategy that prioritises the downstream entry points. To present the argument most straightforwardly, let us suppose that the cumulative inflow to line i is discrete, i.e., $(E_i(t), t \geq 0)$ is constant except at an increasing, countable sequence of times $t \in (0, \infty)$, for each $i \in \mathcal{I}$. Suppose the inflow from line 1 is chosen to be $\Lambda_1(M(t)) = C_1$ whenever $M_1(t)$ is positive, and zero otherwise. Then link 1 will be fully utilized by the inflow from line 1 a proportion ρ_1/C_1 of the time. Let $\Lambda_j(M(t)) = C_j$ whenever both $M_j(t)$ is positive and $\Lambda_i(M(t)) = 0$ for $i < j$, and let $\Lambda_j(M(t)) = 0$ otherwise. This strategy minimizes lexicographically the vector $(M_1(t), M_2(t), \dots, M_J(t))$ at all times t . Provided the system is stable, link 1 will be utilized solely by the inflow from line

j a proportion ρ_j/C_j of the time. Hence the system will be unstable if

$$\sum_{j=1}^J \frac{\rho_j}{C_j} > 1,$$

and thus may well be unstable even when the condition (3.8) is satisfied. Essentially the strategy starves the downstream links, preventing them from working at their full capacity. Our assumption that the cumulative inflow to line i is discrete is not essential for this argument: the stability region will be reduced from (3.8) under fairly general conditions.

The two extreme strategies we have described each have their own interest: the first has a certain optimality property but distributes delay unfairly, while the second can destabilise a network even when all the natural capacity constraints (3.8) are satisfied.

5.2 Fair sharing of the linear network

In this sub-section we describe our preferred ramp metering policy for the linear network, and our Brownian network model for its performance.

Given the line sizes $M(t) = m$, we suppose the metered rates $\Lambda(m)$ are chosen to be proportionally fair: that is, the capacity allocation policy $\Lambda(\cdot)$ solves the optimization problem (3.1)–(3.4). Hence for the linear network we have from relations (3.5)–(3.6) that

$$\Lambda_i(m) = \frac{m_i}{\sum_{j=1}^i q_j}, \quad i \in \mathcal{I}_+(m),$$

where the q_j are Lagrange multipliers satisfying

$$q_j \geq 0, \quad q_j \left(C_j - \sum_{i=j}^J \Lambda_i(m) \right) = 0, \quad j \in \mathcal{J}. \quad (5.4)$$

Under this policy the total flow along section j will be its capacity C_j whenever $q_j > 0$.

Given line sizes $M(t) = m$, the ratio $m_i/\Lambda_i(m)$ is the time it would take to process the work currently in line i at the current metered rate for line i . Thus

$$d_i = \sum_{j=1}^i q_j, \quad i \in \mathcal{I}, \quad (5.5)$$

give estimates, based on current line sizes, of queueing delay in each of the I lines. Note that these estimates do not take into account any

change in the line sizes over the time taken for work to move through the line.

Next we describe our direct Brownian network model for the linear network operating under the above policy. We make the assumption that the inflow to line i is a Brownian motion $\check{E}_i = (\check{E}_i(t), t \geq 0)$ starting from the origin with drift ρ_i and variance parameter $\rho_i \sigma^2$, and so can be written in the form

$$\check{E}_i(t) = \rho_i t + \rho_i^{1/2} \sigma \check{Z}_i(t), \quad t \geq 0 \quad (5.6)$$

for $i \in \mathcal{I}$, where $(\check{Z}_i(t), t \geq 0)$, $i \in \mathcal{I}$, are independent standard Brownian motions. For example, if the inflow to each line were a Poisson process, then this would be the central limit approximation, with $\sigma = 1$. More general choices of σ could arise from either a compound Poisson process, or the central limit approximation to a large class of inflow processes.

Our Brownian network model will be a generalization of the model (2.4)–(2.5) of a single queue, and a specialization of the model of Section 4 to the case where $\mu_i = 2/\sigma^2$, $i \in \mathcal{I}$, and the matrix A is of the form (5.1).

Let

$$\check{X}_j(t) = \sum_{i \in \mathcal{I}} A_{ji} \check{E}_i(t) - C_j t, \quad t \geq 0;$$

note that the first term is the cumulative workload entering the system for resource j over the interval $(0, t]$. Write $\check{X}(t) = (\check{X}_j(t), j \in \mathcal{J})$ and $\check{X} = (\check{X}(t), t \geq 0)$. Then \check{X} is a J -dimensional Brownian motion starting from the origin with drift $A\rho - C$ and covariance matrix $\Gamma = \sigma^2 A[\rho]A'$. We assume the stability condition (3.8) is satisfied, so that $A\rho < C$.

Write

$$\mathcal{W} = A[\rho]A' \mathbb{R}_+^J \quad (5.7)$$

for the workload cone, and

$$\mathcal{W}^j = \{A[\rho]A'q : q \in \mathbb{R}_+^J, q_j = 0\}, \quad (5.8)$$

for the j^{th} face of \mathcal{W} . Our Brownian network model for the resource level workload AM is then the process \check{W} defined by properties (i)–(iv) of Section 4 with \check{W} in place of \tilde{W} , $C - A\rho$ in place of θ and $\Gamma = \sigma^2 A[\rho]A'$.

The form (5.1) of the matrix A allows us to rewrite the workload cone (5.7) as

$$\mathcal{W} = \left\{ w \in \mathbb{R}^J : \frac{w_{j-1} - w_j}{\rho_{j-1}} \leq \frac{w_j - w_{j+1}}{\rho_j}, j = 1, 2, \dots, J \right\}$$

where $w_{J+1} = 0$ and we interpret the left hand side of the inequality as 0 when $j = 1$. Under this model, at any time t when the workloads $\check{W}(t)$ are in the interior of the workload cone \mathcal{W} , each resource is fully utilized. But when \check{W} hits the j^{th} face of the workload cone \mathcal{W} , resource j may not be fully utilized. Our model corresponds to the assumption that there is no more loss of utilization than is necessary to prevent \check{W} from leaving \mathcal{W} . This assumption is made for our Brownian network model by analogy with the results reviewed in Sections 3 and 4, where it emerged as a property of the heavy traffic diffusion approximation.

In a similar manner to that in Section 4, we define a process of dual variables: $\check{Q} = (A[\rho]A')^{-1}\check{W}$. Since \check{W} is our model for AM , our Brownian model for the line sizes is given by

$$\check{M} = A^{-1}\check{W} = [\rho]A'\check{Q}. \quad (5.9)$$

Within our Brownian model we represent (nominal) delays $\check{D} = (\check{D}_i, i \in \mathcal{I})$ at each line as given by

$$\check{D} = A'\check{Q}, \quad (5.10)$$

since these would be the delays if line sizes remained constant over the time taken for a unit of traffic to move through the line, with ρ_i both the arrival rate and metered rate at line i .⁴ Relation (5.10) becomes, for the linear network,

$$\check{D}_i = \sum_{j=1}^i \check{Q}_j, \quad i = 1, 2, \dots, J,$$

parallelling relation (5.5). Note that when \check{W} hits the j^{th} face of the workload cone \mathcal{W} , then $\check{Q}_j = 0$ and $\check{D}_{j-1} = \check{D}_j$; thus the loss of utilization at resource j when \check{W} hits the j^{th} face of the workload cone \mathcal{W} is just sufficient to prevent the delay at line j becoming smaller than the delay at the downstream line $j - 1$.

If \check{W}^s has the stationary distribution of \check{W} , then the components of $\check{Q}^s = (A[\rho]A')^{-1}\check{W}^s$ are independent and \check{Q}_j^s is exponentially distributed

⁴ The nominal delay $\check{D}_i(t)$ for line i at time t will not in general be the realized delay (the time taken for the amount of work $\check{M}_i(t)$ found in line i at time t to be metered from line i). Since $A\rho < C$ the metered rate $\Lambda_i(m)$ will in general differ from ρ_i even when $m = A^{-1}\check{W}$ and $\check{W} \in \mathcal{W}$. Our definition of nominal delay is informed by our earlier heavy traffic results: as $A\rho$ approaches C we expect scaled realized delay to converge to scaled nominal delay. Metered rates do fluctuate as a unit of traffic moves through the line, but we expect less and less so as the system moves into heavy traffic.

with parameter

$$\frac{2}{\sigma^2} \left(C_j - \sum_{i=j}^J \rho_i \right), \quad j = 1, 2, \dots, J.$$

The stationary distributions of \check{M} and \check{D} are then given by the distributions of \check{M}^s and \check{D}^s , respectively, where

$$\check{M}_i^s = \rho_i \sum_{j=1}^i \check{Q}_j^s, \quad \check{D}_i^s = \sum_{j=1}^i \check{Q}_j^s \quad i = 1, 2, \dots, J.$$

In the above example the matrix A is invertible. As an example of a network with a non-invertible A matrix, suppose that in the linear network illustrated in Figure 5.1 one section of road is unconstrained, say $C_3 = \infty$. Then, removing the corresponding row from the resource-route incidence matrix we have

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The workload cone is the collapse of \mathcal{W} obtained by setting $w_2 = w_3$, and in consequence the construction of \check{W} and $\check{M} = [\rho]A'(A[\rho]A')^{-1}\check{W}$ enforces the relationship $\check{M}_2/\check{M}_3 = \rho_2/\rho_3$. Since the matrix A is not invertible, this is no longer a necessary consequence of the network topology, but is a natural modelling assumption, motivated by the forms of state-space collapse we have seen earlier. Essentially lines 2 and 3 use the same network resources and face the same queueing delays.

A Brownian network model of the first strategy from Section 5.1 could also be constructed, but the workload cone and its faces would not be of the required form (5.7) and (5.8), but instead would be defined by

$$\mathcal{W} = \{w \in \mathbb{R}^J : 0 \leq w_J \leq \dots \leq w_2 \leq w_1\}, \quad (5.11)$$

and the requirement that if $w \in \mathcal{W}^j$ then $w_j = w_{j+1}$, with the interpretation $w_{J+1} = 0$. Thus face j represents the requirement that the workload for resource j comprises at least the workload for resource $j+1$, for $j = 1, 2, \dots, J$. Under this model, resource j is fully utilized except when \check{W} hits the j^{th} face of the workload cone (5.11): it is not possible for \check{W} to leave \mathcal{W} , since the constraints expressed in the form (5.11) follow necessarily from the topology of the network embodied in A . The model corresponds to the assumption that there is no more loss of utilization than is a necessary consequence of the network topology. Note

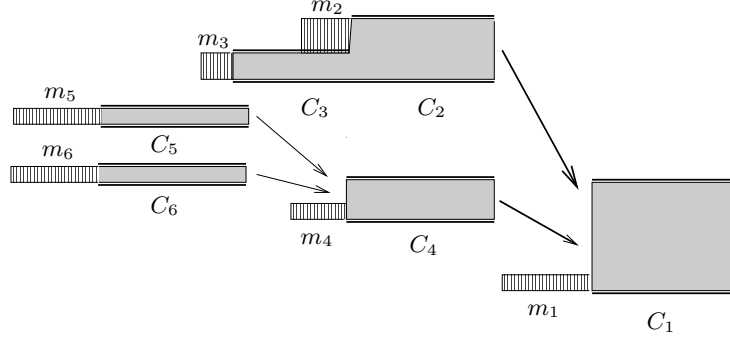


Figure 5.2 There are six sources of traffic, starting in various lines and all destined to eventually traverse section 1 of the road. Once traffic has passed through the queue at its entry point, it does not queue again.

that the proportionally fair policy may fail to fully utilize a resource not only when this is a necessary consequence of the network topology, but also when this would cause an upstream entry point to obtain more than what the policy considers a fair share of a scarce downstream resource.

5.3 A tree network

Next consider the tree network illustrated in Figure 5.2. Access is metered at the six entry points so that the capacities C_1, C_2, \dots, C_6 are not overloaded. There is no queueing after the entry point, and the capacities satisfy the conditions $C_3 < C_2$, $C_5 + C_6 < C_4$, $C_2 + C_4 < C_1$.

Given the line sizes $M(t) = m$, we suppose the metered rates $\Lambda(m)$ are chosen to be proportionally fair: that is, the capacity allocation policy $\Lambda(\cdot)$ solves the optimization problem (3.1)–(3.4) where for this network

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We assume, as in the last Section, that the cumulative inflow of work to line i is given by equation (5.6) for $i \in \mathcal{I}$, where $(\check{Z}_i(t), t \geq 0)$, $i \in \mathcal{I}$, are independent standard Brownian motions. Our Brownian network model is again the process \check{W} defined by properties (i)–(iv) of Section 4

with \check{W} in place of \tilde{W} , $C - A\rho$ in place of θ , $\Gamma = \sigma^2 A[\rho]A'$, and with the workload cone and its faces defined by equations (5.7)–(5.8) for the above choice of A . We assume the stability condition (3.8) is satisfied, so that all components of $C - A\rho$ are positive.

If \check{W}^s denotes a random variable with the stationary distribution of \check{W} , then the components of $\check{Q}^s = (A[\rho]A')^{-1}\check{W}^s$ are independent and \check{Q}_j^s is exponentially distributed with parameter $2\sigma^{-2}(C_j - \sum_i A_{ji}\rho_i)$ for each $j \in \mathcal{J}$. The Brownian model line sizes and delays are again given by equations (5.9) and (5.10) respectively, each with stationary distributions given by a linear combination of independent exponential random variables, one for each section of road.

A key feature of the linear network, and its generalization to tree networks, is that all traffic is bound for the same destination. In our application to a road network this ensures that all traffic in a line at a given entry point is on the same route. If traffic on different routes shared a single line it would not be possible to align the delay incurred by traffic so precisely with the sum of dual variables for the resources to be used.⁵

6 Route choices

Next consider the road network illustrated in Figure 6.1. Three parallel roads lead into a fourth road and hence to a common destination. Access to each of these roads is metered, so that their respective capacities C_1 , C_2 , C_3 , C_4 are not overloaded, and $C_1 + C_2 + C_3 < C_4$. There are four sources of traffic with respective loads ρ_1 , ρ_2 , ρ_3 , ρ_4 : the first source has access to road 1 alone, on its way to road 4; the second source has access to both roads 1 and 2; and the third source can access all three of the parallel roads. We assume that traffic arriving with access to more than one road distributes itself in an attempt to minimize its queueing delay, an assumption whose implications we shall explore.

We could view sources of traffic as arising in different geographical regions, with different possibilities for easy access to the motorway network and with real time information on delays. Or we could imagine a

⁵ The tree topology of Figure 5.2 ensures that the queueing delays in the proportionally fair Brownian network model are partially ordered. A technical consequence is that a wide class of fair capacity allocations, the α -fair allocations, share the same workload cone: in the notation of [21], the cone \mathcal{W}_α does not depend upon α .

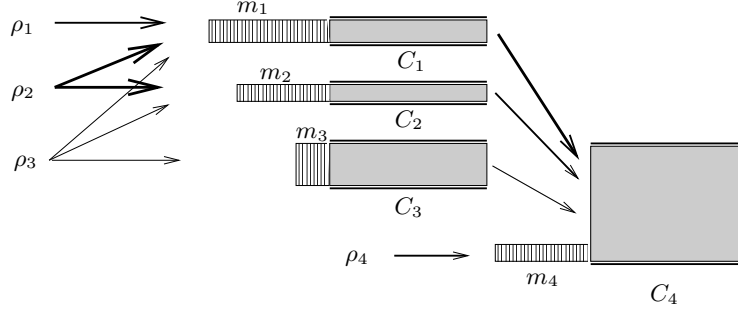


Figure 6.1 Three parallel roads lead to a fourth road and hence to a common destination. Lines of size m_1, m_2, m_3, m_4 are held on the slip roads leading to these roads. There are four sources of traffic: sources 2 and 3 may choose their first road, with choices as shown.

priority access discipline where some traffic, for example high occupancy vehicles, has a larger set of lines to choose from.

Given the line sizes $M(t) = m$, we suppose the metered rates $\Lambda(m)$ are chosen to be proportionally fair: that is, the capacity allocation policy $\Lambda(\cdot)$ solves the optimization problem (3.1)–(3.4). For this network

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}$$

and so, from relations (3.5)–(3.6),

$$\Lambda_i(m) = \frac{m_i}{q_i + q_4}, \quad i = 1, 2, 3, \quad \Lambda_4(m) = \frac{m_4}{q_4}.$$

We assume the ramp metering policy has no knowledge of the routing choices available to arriving traffic, but is simply a function of the observed line sizes m , the topology matrix A and the capacity vector C .

How might arriving traffic choose between lines? Well, traffic that arrives when the line sizes are m and the metered rates are $\Lambda(m)$ might reasonably consider the ratios $m_i/\Lambda_i(m)$ in order to choose which line to join, since these ratios give the time it would take to process the work currently in line i at the current metered rate for line i , for $i = 1, 2, 3$. But these ratios are just $q_i + q_4$ for $i = 1, 2, 3$. Given the choices available to the three sources, we would expect exercise of these choices to ensure that $q_1 \geq q_2 \geq q_3$, or equivalently that the delays through lines 1, 2, 3 are weakly decreasing.

Because traffic from sources 2 and 3 has the ability to make route choices, condition (3.8) is sufficient, but no longer necessary, for stability. The stability condition for the network of Figure 6.1 is

$$\sum_{i=1}^j \rho_i < \sum_{i=1}^j C_i, \quad j = 1, 2, 3, \quad \sum_{i=1}^4 \rho_i < C_4, \quad (6.1)$$

and is thus of the form (3.8), but with A and C replaced by \bar{A} and \bar{C} respectively, where

$$\bar{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \\ \bar{C}_4 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_1 + C_2 \\ C_1 + C_2 + C_3 \\ C_4 \end{pmatrix}.$$

The forms \bar{A} , \bar{C} capture the concept of four *virtual resources* of capacities \bar{C}_j , $j = 1, 2, 3, 4$. Given the line sizes $m = (m_i, i = 1, 2, 3, 4)$, the workloads $w = (w_i, i = 1, 2, 3, 4)$ for the four virtual resources are $w = \bar{A}m$.

For $i = 1, 2, 3, 4$, we model the cumulative inflow of work from source i over the interval $(0, t]$ as a Brownian motion $\check{E}_i = (\check{E}_i(t), t \geq 0)$ starting from the origin with drift ρ_i and variance parameter $\rho_i \sigma^2$ that can be written in the form (5.6), where $(\check{Z}_i(t), t \geq 0)$, $i = 1, 2, 3, 4$, are independent standard Brownian motions. Let $\check{E} = (\check{E}_i, i = 1, 2, 3, 4)$ and let $\check{X} = (\check{X}_j, j \in \mathcal{J})$ be defined by

$$\check{X}(t) = \bar{A}\check{E}(t) - \bar{C}t, \quad t \geq 0.$$

Then \check{X} is a four-dimensional Brownian motion starting from the origin with drift $\bar{A}\rho - \bar{C}$ and covariance matrix $\Gamma = \sigma^2 \bar{A}[\rho]\bar{A}'$. We assume the stability condition (6.1) is satisfied, so that all components of the drift are strictly negative. Let $\mathcal{W}, \mathcal{W}^j$ be defined by (5.7), (5.8) respectively, with A replaced by \bar{A} .

Our Brownian network model for $\bar{A}M$ is then the process \check{W} defined by properties (i)–(iv) of Section 4 with \check{W} in place of \tilde{W} and $\bar{C} - \bar{A}\rho$ in place of θ .

Define a process of dual variables for the virtual resources: $\check{Q} = (A[\rho]A')^{-1}\check{W}$. Since \check{W} is our model for $\bar{A}M$, our Brownian model for the line sizes is given by

$$\check{M} = \bar{A}^{-1}\check{W} = [\rho]\bar{A}'\check{Q}.$$

Our Brownian model for the delays at each line is given by

$$\check{D} = \bar{A}'\check{Q},$$

which from the form of \bar{A} becomes

$$\check{D}_i = \sum_{j=i}^4 \check{Q}_j, \quad i = 1, 2, 3, 4.$$

Thus at any time t when $\check{Q}_1(t) > 0$, $\check{Q}_2(t) > 0$ then $\check{D}_1(t) > \check{D}_2(t) > \check{D}_3(t)$, and the incentives for arriving traffic are such that traffic from source i joins line i . However if $\check{Q}_1(t) = 0$, and so $\check{D}_1(t) = \check{D}_2(t)$, then arriving traffic from stream 2 may choose to enter line 1, and thus contribute to increments of the workload for virtual resource 1, whilst still contributing to the workload for virtual resources 2, 3 and 4. Our model corresponds to the assumption that no more traffic does this than is necessary to keep \check{Q}_1 non-negative, or equivalently to keep $\check{D}_1 \geq \check{D}_2$. Similarly if $\check{Q}_2(t) = 0$ then $\check{D}_2(t) = \check{D}_3(t)$, and arriving traffic from stream 3 may choose to enter line 2, and thus contribute to increments of the workload for virtual resource 2, whilst still contributing to the workload for virtual resources 3 and 4; we suppose just sufficient traffic does this to keep \check{Q}_2 non-negative, or equivalently to keep $\check{D}_2 \geq \check{D}_3$. Finally if $\check{Q}_3(t) = 0$ or $\check{Q}_4(t) = 0$ then (real) resource 3 or 4 respectively may not be fully utilized, as in earlier examples, and our model corresponds to the assumption that there is no more loss of utilization at (real) resources 3 and 4 than is necessary to prevent \check{W} from leaving \mathcal{W} .

If \check{W}^s is a random variable with the stationary distribution of \check{W} , then the components of $\check{Q}^s = (A[\rho]A')^{-1}\check{W}^s$ are independent and for $j = 1, 2, 3, 4$, \check{Q}_j^s is exponentially distributed with parameter ζ_j where

$$\begin{aligned} \zeta_1 &= \frac{2}{\sigma^2}(C_1 - \rho_1), \\ \zeta_2 &= \frac{2}{\sigma^2}(C_1 + C_2 - \rho_1 - \rho_2), \\ \zeta_3 &= \frac{2}{\sigma^2}(C_1 + C_2 + C_3 - \rho_1 - \rho_2 - \rho_3), \\ \zeta_4 &= \frac{2}{\sigma^2}(C_4 - \rho_1 - \rho_2 - \rho_3 - \rho_4). \end{aligned}$$

Under the Brownian network model, the stationary distribution for line sizes and for delays at each line are given by the distributions of \check{M}^s and

\check{D}^s , respectively, where

$$\check{M}_i^s = \rho_i \left(\sum_{j=i}^4 \check{Q}_j^s \right), \quad \check{D}_i^s = \sum_{j=i}^4 \check{Q}_j^s \quad i = 1, 2, 3, 4.$$

The Brownian network model thus corresponds to natural assumptions about how arriving traffic from different sources would choose their routes. The results on the stationary distribution for the network are intriguing. The ramp metering policy has no knowledge of the routing choices available to arriving traffic, and hence of the enlarged stability region (6.1). Nevertheless, under the Brownian model, the interaction of the ramp metering policy with the routing choices available to arriving traffic has a performance described in terms of dual random variables, one for each of the virtual resources of the enlarged stability region; when a driver makes a route choice, the delay facing a driver on a route is a sum of dual random variables, one for each of the virtual resources used by that route; and under their stationary distribution, the dual random variables are independent and exponentially distributed.

7 Concluding remarks

The design of ramp metering strategies cannot assume that arriving traffic flows are exogenous, since in general drivers' behaviour will be responsive to the delays incurred or expected. In this paper we have presented a preliminary exploration of an approach to the design of ramp metering flow rates informed by earlier work on Internet congestion control. A feature of this approach is that it may prove possible to integrate ideas of fairness of a control policy with overall system optimization.

There remain many areas for further investigation. In particular, we have seen intriguing examples, in the context of a single queue and of Internet congestion control, of remarkably good approximations produced for the stationary distributions of queue length and workload by use of the direct Brownian modelling approach. Furthermore, in the context of a controlled motorway, where a detailed model for arriving traffic is not easily available, use of a direct Brownian model has enabled us to develop an approach to the design and performance of ramp metering and in the context of that model to obtain insights into the interaction of ramp metering with route choices. Nevertheless, we expect that the use of direct Brownian network models will not always produce good re-

sults. Indeed, it is possible that such models may be suitable only when the scaled workload process can be approximated in heavy traffic by a reflecting Brownian motion that has a product-form stationary distribution. We believe that understanding when the direct method is a good modelling approach and when it is not, and obtaining a rigorous understanding of the reasons for this, is an interesting topic worthy of further research.

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