## Example Sheet 2

**1**. The rooted binary tree is an infinite graph $T$ with one distinguished vertex $R$ from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on $T$ jumps from a vertex along each available edge with equal probability. Show that the random walk is transient.

**2.** Show that for the Markov chain $(X_n)_{n\geq 0}$ in Example 12 from Sheet 1

$$\mathbb{P}(X_n \to \infty \text{ as } n \to \infty) = 1\,.$$

Suppose the transition probabilities satisfy instead

$$p_{ii+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{ii-1}$$

for some $\alpha \in (0,\infty)$. What then is the value of $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty)$?

**3**. Show that the simple symmetric random walk in $\mathbb{Z}^4$ is transient.

**4**. Find all invariant distributions of the transition matrix

$$P = \begin{pmatrix} \frac12 & 0 & 0 & 0 & \frac12 \\ 0 & \frac12 & 0 & \frac12 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac14 & \frac14 & \frac14 & \frac14 \\ \frac12 & 0 & 0 & 0 & \frac12 \end{pmatrix}.$$

**5**. Gas molecules move about randomly in a box which is divided into two halves symmetrically by a partition. A hole is made in the partition. Suppose there are $N$ molecules in the box. Show that the number of molecules on one side of the partition just after a molecule has passed through the hole evolves as a Markov chain. What are the transition probabilities? What is the invariant distribution of this chain?

**6**. A particle moves on the eight vertices of a cube in the following way: at each step the particle is equally likely to move to each of the three adjacent vertices, independently of its past motion. Let $i$ be the initial vertex occupied by the particle, $o$ the vertex opposite $i$. Calculate each of the following quantities:

(i) the expected number of steps until the particle returns to $i$;

(ii) the expected number of visits to $o$ until the first return to $i$;
(iii) the expected number of steps until the first visit to $o$.

**7**. Find the invariant distributions of the transition matrices in Example 7 from Sheet 1, parts (a), (b) and (c), and use them to check your answers.

**8**. A fair die is thrown repeatedly. Let $X_n$ denote the sum of the first $n$ throws. Find

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13).$$

**9**. Each morning a student takes one of the three books she owns from her shelf. The probability that she chooses book $i$ is $\alpha_i, 0 < \alpha_i < 1, i = 1, 2, 3$, and choices on successive days are independent. In the evening she replaces the book at the left-hand end of the shelf. If $p_n$ denotes the probability that on day $n$ the student finds the books in the order 1,2,3, from left to right, show that, irrespective of the initial arrangement of the books, $p_n$ converges as $n \to \infty$, and determine the limit.

**10**. In each of the following cases determine whether the stochastic matrix $P$ is reversible:

(a) $\begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}$ ; (b) $\begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$ ;

(c) $I = \{0, 1, \ldots, N\}$ and $p_{ij} = 0$ if $|j - i| \geq 2$ ;

(d) $I = \{0, 1, 2, \ldots\}$ and $p_{01} = 1$, $p_{ii+1} = p$, $p_{ii-1} = 1 - p$ for $i \geq 1$.

**11**. A professor has $N$ umbrellas, which he keeps either at home or in his office. He walks to and from his office each day, and takes an umbrella with him if and only if it is raining. Throughout each journey, it either rains, with probability $p$, or remains fine, with probability $1 - p$, independently of the past weather. What is the long run proportion of journeys on which he gets wet?

**12. (Renewal theorem)** Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables with values in $\{1, 2, \ldots\}$. Suppose that the set of integers

$$\{n : \mathbb{P}(Y_1 = n) > 0\}$$

has greatest common divisor 1. Set $\mu = \mathbb{E}(Y_1)$. Show that the following process is a Markov chain:

$$X_n = \inf\{m \geq n : m = Y_1 + \ldots + Y_k \text{ for some } k \geq 0\} - n.$$

Determine

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0)$$

and hence show that as $n \to \infty$

$$\mathbb{P}(n = Y_1 + \ldots + Y_k \text{ for some } k \geq 0) \to 1/\mu.$$

**13**. An opera singer is due to perform a long series of concerts. Having a fine artistic temperament, she is liable to pull out each night with probability 1/2. Once this has happened she will not sing again until the promoter convinces her of his high regard. This he does by sending flowers every day until she returns. Flowers costing $x$ thousand pounds, $0 \leq x \leq 1$, bring about a reconciliation with probability $\sqrt{x}$. The promoter stands to make £750 from each successful concert. How much should he spend on flowers?