CHAPTER 6

Reversible Migration Processes

In this chapter we shall consider an adaptation of the migration processes of Chapter 2 which allows the probability intensity that a customer moves from one colony to another to depend upon the number in the receiving colony. The adapted process thus permits blocking, but in order to make any analytical progress reversibility must be assumed. In Section 6.2 we shall consider an application of the resulting process to the modelling of social grouping behaviour. Finally in Section 6.3 we shall contrast various processes, including those introduced in Chapters 2 and 5, in which at most one individual is allowed in each colony.

6.1 MIGRATION PROCESSES REVISITED

In Section 2.3 we considered a closed migration process with state given by $\mathbf{n}=(n_1, n_2, \ldots, n_J)$, where n_j is the number of individuals in colony j. If the probability intensity that an individual moves from colony j to k is

$$q(\mathbf{n}, T_{jk}\mathbf{n}) = \lambda_{jk}\phi_j(n_j) \tag{6.1}$$

then we saw that the equilibrium distribution takes a simple form (Theorem 2.3). A limitation of the transition rate (6.1) is that it does not depend upon n_k, the number of individuals already present at colony k. Perhaps the simplest form incorporating such a dependence is

$$q(\mathbf{n}, T_{jk}\mathbf{n}) = \lambda_{jk}\phi_j(n_j)\psi_k(n_k) \tag{6.2}$$

where $\phi_j(0)=0$ and for simplicity $\lambda_{jj}=0$. To ensure that $\mathbf{n}$ is irreducible within the state space

$$\mathcal{S} = \left\{\mathbf{n} \,\middle|\, n_j \geq 0, j=1,2,\ldots,J; \sum_{j=1}^{J} n_j = N\right\}$$

we require that $\phi_j(n)>0$ if $n>0$, $\psi_j(n)>0$ if $n\geq 0$, and that the parameters λ_{jk} allow an individual to pass between any two colonies, either directly or indirectly via a chain of other colonies. With transition rates (6.2) the equilibrium equations become

$$\pi(\mathbf{n}) \sum_{j=1}^{J}\sum_{k=1}^{J} \lambda_{jk}\phi_j(n_j)\psi_k(n_k) = \sum_{j=1}^{J}\sum_{k=1}^{J} \pi(T_{jk}\mathbf{n})\lambda_{kj}\phi_k(n_k+1)\psi_j(n_j-1)$$

In general these equations do not have a simple solution. We might hope, however, that we could solve the equations in the special case when the process $\mathbf{n}$ is reversible. The detailed balance conditions are then

$$\pi(\mathbf{n})\lambda_{jk}\phi_j(n_j)\psi_k(n_k)=\pi(T_{jk}\mathbf{n})\lambda_{kj}\phi_k(n_k+1)\psi_j(n_j-1)$$

and it is easy to check that a solution to these is

$$\pi(\mathbf{n})=B\prod_{j=1}^{J}\left\{\alpha_j^{n_j}\prod_{r=1}^{n_j}\frac{\psi_j(r-1)}{\phi_j(r)}\right\} \tag{6.3}$$

provided

$$\alpha_j\lambda_{jk}=\alpha_k\lambda_{kj} \tag{6.4}$$

Thus although the transition rates (6.2) appear more general than (6.1) to deal with them we have to impose the restriction (6.4) on the parameters λ_{jk}. The normalizing constant B is, as usual, chosen so that the distribution $\pi(\mathbf{n})$ sums to unity over the state space $\mathcal{S}$. We can summarize the above results in the following theorem.

Theorem 6.1. *A stationary closed migration process with transition rates* (6.2) *is reversible if there exist positive constants* $\alpha_1, \alpha_2, \ldots, \alpha_J$ *satisfying condition* (6.4). *In this case the equilibrium distribution takes the form* (6.3).

For an open migration process we need to specify additional transition rates to complement (6.2). The most obvious choices are

$$q(\mathbf{n}, T_{j.}\mathbf{n})=\mu_j\phi_j(n_j) \tag{6.5}$$

for the probability intensity that an individual leaves the system from colony j and

$$q(\mathbf{n}, T_{.k}\mathbf{n})=\nu_k\psi_k(n_k) \tag{6.6}$$

for the probability intensity that an individual enters the system to join colony k. Assume that the parameters λ_{jk}, μ_j, and ν_k allow an individual to reach any colony from outside the system and to leave the system from any colony, either directly or indirectly via a chain of other colonies. This, together with the earlier assumption that $\phi_j(0)=0$, $\phi_j(n)>0$ if $n>0$, $\psi_j(n)>0$ if $n\geq 0$, ensures that the process $\mathbf{n}$ is irreducible within the state space $\mathbb{N}^J$. It is easy to check that the form (6.3) will again satisfy the detailed balance conditions provided

$$\alpha_j\mu_j=\nu_j \tag{6.7}$$

and condition (6.4) are satisfied. We thus have the following result.

Theorem 6.2. *A stationary open migration process with transition rates* (6.2), (6.5), *and* (6.6) *is reversible if there exist positive constants*

$\alpha_1, \alpha_2, \ldots, \alpha_J$ satisfying conditions (6.4) and (6.7). In this case the equilibrium distribution takes the form (6.3) and in equilibrium $n_1, n_2, \ldots, n_J$ are independent.

We shall call the processes introduced in this section reversible migration processes. This should not cause confusion with the migration processes of Chapter 2 since if a migration process as defined there is reversible then it is in fact a reversible migration process as defined here with $\psi_j(n)=1$ for $n \geq 0$, $j=1,2,\ldots,J$.

The results obtained in Chapter 3 show that various modifications can be made to a migration process without affecting the equilibrium distribution $\pi(\mathbf{n})$. In particular, suppose that when an individual arrives at colony j he is assigned a nominal lifetime which has an arbitrary distribution with unit mean, that he ages through this lifetime at rate $\phi_j(n_j)\sum_i \lambda_{ji}$ while there are n_j individuals in colony j, and that when his lifetime in colony j comes to an end he moves to colony k with probability $\lambda_{jk}/\sum_i \lambda_{ji}$. Suppose for simplicity that all nominal lifetimes are independent. The case where nominal lifetimes are exponentially distributed corresponds to a closed migration process with transition rates (6.1), but the equilibrium distribution $\pi(\mathbf{n})$ is the same whatever the nominal lifetime distributions, since the colonies are examples of server-sharing queues. Consider now a closed reversible migration process with transition rates (6.2). This can be modified to allow arbitrary nominal lifetimes by supposing that an individual in colony j ages through his lifetime at rate $\phi_j(n_j)\sum_i \lambda_{ji}\psi_i(n_i)$ and that when his lifetime in colony j comes to an end he moves to colony k with probability $\lambda_{jk}\psi_k(n_k)/\sum_i \lambda_{ji}\psi_i(n_i)$. Observe that when nominal lifetimes are exponentially distributed the process $\mathbf{n}$ is Markov with transition rates (6.2). In Chapter 9 (Exercises 9.3.2 and 9.4.2) we shall see that provided equations (6.4) have a solution the equilibrium distribution $\pi(\mathbf{n})$ is the same whatever the form of the nominal lifetime distributions.

Exercises 6.1

1. Use Kolmogorov's criteria to show that a stationary migration process with transition rates (6.2) is reversible only if condition (6.4) can be satisfied.
2. Consider a Markov process $\mathbf{n}$ for which the transition rates (6.5) and (6.6) are positive, and where these are not the only positive transition rates. Show that if $\mathbf{n}$ is reversible then $n_1, n_2, \ldots, n_J$ are independent and the equilibrium distribution is of the form (6.3), whatever the form of the transition rates other than (6.5) and (6.6).
3. If we regard the open migration process discussed in Section 2.4 as a model of the movement of particles between cells then the number of

particles in a cell will have a Poisson distribution if $\phi(n)=\phi n$, a geometric distribution if $\phi(n)=\phi$, $n>0$, and a Bernoulli distribution (i.e. a distribution on the values of zero or unity) if $\phi(1)=\phi$, $\phi(n)=\infty$, $n>1$. Physicists attach the names Maxwell–Boltzmann, Bose–Einstein, and Fermi–Dirac respectively to these distributions. Observe that the suggested functions $\phi(n)$ are the only ones which will give rise to these distributions. Discuss how the distributions could arise from the reversible migration processes of this section. Observe that there are a variety of models which could lead to each of the given distributions.

4. In an open migration process with transition rates (6.2), (6.5), and (6.6) the stream of individuals entering the system at colony j will not in general be Poisson. Suppose, however, that $\psi_j(n)\leq 1$ for $n\geq 0$, $j=1,2,\ldots,J$. In this case the given transition rates could be reconciled with Poisson streams if it is assumed that an individual arriving at colony j to find n_j individuals already there is lost with probability $1-\psi_j(n_j)$. If individuals arriving at and departing from colony j are considered as customers of class j show that, counting lost customers, the system is quasi-reversible under the weaker assumptions described in Exercise 3.2.1. The weaker assumptions are needed since the class of a customer may change as he passes through the system. Observe that in this system the probability that a customer is lost can depend upon the state of an individual colony in a way which could not be allowed in Section 3.5.
5. In an open migration process with transition rates (6.2), (6.5), and (6.6) suppose that if $\nu_k>0$ then $\psi_k(n)=1$, $n\geq 0$. Show that if individuals arriving from outside the system at colony j or leaving the system from colony j are considered as customers of class j then the system is quasi-reversible under the weaker assumptions described in Exercise 3.2.1. Of course, if all customers are considered to be of the same class the system is quasi-reversible, and thus Exercise 3.2.4 shows how to produce a quasi-reversible system in which the class of a customer does not change.

6.2 SOCIAL GROUPING BEHAVIOUR

One area where reversible migration processes can be useful is in the modelling of the behaviour of individuals gathering in groups for social reasons, e.g. monkeys forming sleeping groups or children at play. To illustrate the results of the previous section we shall consider an open and closed version of a model which might be appropriate in this context.

Suppose that group j consists of those individuals at a particular geographical location. Let n_j be the number of individuals in group j and suppose that $\mathbf{n}=(n_1,n_2,\ldots,n_J)$ is a migration process with the following transition rates:

$$\begin{aligned} q(\mathbf{n}, T_{\cdot k}\mathbf{n}) &= a_k + c_k n_k \\ q(\mathbf{n}, T_{j\cdot}\mathbf{n}) &= d_j n_j \\ q(\mathbf{n}, T_{jk}\mathbf{n}) &= \lambda_{jk} d_j n_j (a_k + c_k n_k) \end{aligned} \tag{6.8}$$

where $\lambda_{jk} = \lambda_{kj}$, $a_k > 0$, and $d_k > c_k \geq 0$. Here a_k can be thought of as the attractiveness to an outsider of belonging to group k, c_k as the attractiveness to an outsider of an individual in group k, d_j as the propensity to depart from group j of an individual in group j, and λ_{jk} as a measure of the proximity of groups j and k. The transition rates (6.8) are of the form discussed in the previous section with

$$\phi(n_j) = d_j n_j \qquad \text{and} \qquad \psi_k(n_k) = a_k + c_k n_k$$

and a solution to equations (6.4) and (6.7) is $\alpha_1 = \alpha_2 = \cdots = \alpha_J = 1$. Thus Theorem 6.2 gives the form of the equilibrium distribution. To calculate the normalizing constant is not difficult, and the conclusion is that in equilibrium $n_1, n_2, \ldots, n_J$ are independent, each with a negative binomial distribution:

$$\pi(n_j) = \binom{f_j + n_j - 1}{n_j}(1 - g_j)^{f_j} g_j^{n_j} \qquad n_j = 0, 1, \ldots$$

where $f_j = a_j/c_j$ and $g_j = c_j/d_j$. The expected number of individuals in group j is thus $a_j/(d_j - c_j)$.

Consider now the closed version of the above model with transition rates

$$q(\mathbf{n}, T_{jk}\mathbf{n}) = \lambda_{jk} d_j n_j (a_k + c_k n_k) \tag{6.9}$$

where $\lambda_{jk} = \lambda_{kj}$, a_k, c_k, $d_k > 0$, and the total number of individuals in the system is N. Theorem 6.1 allows us to deduce the form of the equilibrium distribution but the normalizing constant is in general an awkward expression. It simplifies when $c_j/d_j = g$ for $j = 1, 2, \ldots, J$. Then the equilibrium distribution can be written

$$\pi(\mathbf{n}) = \binom{-\sum f_k}{N}^{-1} \prod_{j=1}^{J} \binom{-f_j}{n_j} \tag{6.10}$$

for $\mathbf{n}$ such that $\sum n_j = N$, where $f_j = a_j/c_j$.

A drawback of the model described is that it assumes a group is based at one of J geographical locations. Often this assumption is inappropriate, and in Chapter 8 we shall consider models which do not restrict the groups in this way.

Exercises 6.2

1. Show that if $c_k = 0$ in transition rates (6.8) then in equilibrium n_k has a Poisson distribution. Deduce that if $c_k = 0$ in transition rates (6.9) then in equilibrium $\mathbf{n}$ has a multinomial distribution.

2. Deduce from the distribution (6.10) that the marginal distribution for n_j is the Polya distribution

$$\pi(n_j)=\binom{-\sum f_k}{N}^{-1}\binom{-f_j}{n_j}\binom{f_j-\sum f_k}{N-n_j}$$

Suppose now that $f_1=f_2=\cdots=f_J$. Show that if $N, J\to\infty$ with N/J held constant the marginal distribution for n_j approaches a negative binomial distribution.
3. In the open model with transition rates (6.8) determine the probability flux that an individual moves from group j to group k. Use Little's result to deduce the mean time an individual stays in group j.

6.3 CONTRASTING FLOW MODELS

Section 5.2 discussed a flow model in which each site could hold at most one individual. This restriction can also be imposed on the migration processes of Chapter 2 or of this chapter, with rather different effects. In this section we shall comment briefly on the resulting flow models and introduce a further one.

Consider an open migration process with transition rates (2.8) in which

$$\phi(n)=\begin{cases}1 & n=1\\ \infty & n>1\end{cases}$$

In this process site j can hold at most one individual. If an individual arrives at site j when it is already occupied he is immediately ejected from the site and moves on to site k with probability λ_{jk}/λ_j or leaves the system with probability μ_j/λ_j. Thus there is no blocking in this system—the reverse in fact, since the more likely site j is to be occupied the faster an individual needing to visit site j will pass through the system. If $\alpha_1, \alpha_2, \ldots, \alpha_J$ is the solution to equations (2.9) then in equilibrium site j is occupied with probability

$$\frac{\alpha_j}{1+\alpha_j} \tag{6.11}$$

independently of the state of the rest of the system.

Consider now an open migration process with transition rates given by (6.2), (6.5), and (6.6) where

$$\phi(n)=1 \qquad n>0$$

$$\psi(n)=\begin{cases}1 & n=0\\ 0 & n>0\end{cases}$$

In this process site j can again hold at most one individual. While site j is occupied transitions which would bring another individual to site j are forbidden. Provided conditions (6.4) and (6.7) are satisfied we can determine the equilibrium distribution, and in equilibrium site j is again occupied with probability (6.11), independently of the rest of the system. A drawback of the model is that since it is reversible there can be no net flow of individuals in a given direction through the system. Note that in both this model and in the flow model of Section 5.2 an individual unable to move to site j from site k because site j is occupied may well end up moving from k to a site other than j.

There is a further flow model for which analytical results are available, and a closed version of it can be described as follows. Suppose that while site j is occupied and site k is free the probability intensity that the individual at j moves to k is λ_{jk} $(j, k = 1, 2, \ldots, J)$ where the parameters λ_{jk} satisfy

$$\sum_k \lambda_{jk} = \sum_k \lambda_{kj} \qquad j = 1, 2, \ldots, J \tag{6.12}$$

The equilibrium equations for this process are

$$\pi(\mathbf{n}) \sum_{j:n_j=1} \sum_{k:n_k=0} \lambda_{jk} = \sum_{j:n_j=1} \sum_{k:n_k=0} \pi(T_{jk}\mathbf{n})\lambda_{kj} \tag{6.13}$$

A solution to these equations is (Exercise 6.3.3)

$$\pi(\mathbf{n}) = \binom{J}{N}^{-1} \tag{6.14}$$

for each $\mathbf{n}$ representing a state where there are N individuals present in the system, all at different sites. Thus all possible states are equally likely.

An open version of the above model can be obtained by appending to the system a large number of additional sites, each connected to the previously existing sites in the same way. The details are given in Exercise 6.3.5 and the resulting model can be described as follows. If site j is occupied then the individual there leaves the system with probability intensity μ_j; if site k is free an individual arrives there from outside the system with probability intensity ν_k; and if site j is occupied and site k is free then the individual at j moves to k with probability intensity λ_{jk}. In place of restriction (6.12) assume that the rates satisfy the equations

$$\sum_k \lambda_{jk} + \frac{\mu_j}{1-p} = \sum_k \lambda_{kj} + \frac{\nu_j}{p} \qquad j = 1, 2, \ldots, J \tag{6.15}$$

where

$$p = \frac{\rho}{1+\rho} \tag{6.16}$$

and

$$\rho = \frac{\sum_j \nu_j}{\sum_j \mu_j} \tag{6.17}$$

The equilibrium equations for the process are

$$\pi(\mathbf{n})\left[\sum_{j:n_j=1}\sum_{k:n_k=0}\lambda_{jk} + \sum_{j:n_j=1}\mu_j + \sum_{k:n_k=0}\nu_k\right]$$

$$= \sum_{j:n_j=1}\sum_{k:n_k=0}\pi(T_{jk}\mathbf{n})\lambda_{kj} + \sum_{j:n_j=1}\pi(T_{j\cdot}\mathbf{n})\nu_j + \sum_{k:n_k=0}\pi(T_{\cdot k}\mathbf{n})\mu_k$$

The equilibrium equations are satisfied by

$$\pi(\mathbf{n}) = B\rho^{\sum_i n_i} \tag{6.18}$$

and hence in equilibrium a site is occupied with probability p, and whether the site is occupied or not is independent of the state of the rest of the system. This is an intriguing result: we are accustomed to such independence in open networks of quasi-reversible queues and in reversible migration processes, but this flow model shows that it can occur in other systems as well.

As an example of the result consider the one-dimensional flow model illustrated in Fig. 6.1, where jumps take place between adjacent sites with the probability intensities shown. If

$$\lambda_1 = \lambda_2 + \mu + \nu$$

then restriction (6.15) is satisfied and in equilibrium a site is occupied with probability $\nu/(\mu+\nu)$, independently of the other sites. The same model was considered in Exercise 5.2.2 under the restriction $\lambda_1 = \lambda_2$.

Exercises 6.3

1. Observe that in the first flow model considered in this section the probability that site j is occupied, given by expression (6.11), remains unaltered when the time a particle remains at site j is arbitrarily distributed with mean λ_j^{-1}. In the case where the distribution is exponential observe that the model is unaltered if it is the previous occupant of site j who is expelled when a second individual arrives there.
2. The organizational hierarchy illustrated in Fig. 6.2 consists of J posts, each of which is held by at most one individual. At points in time which

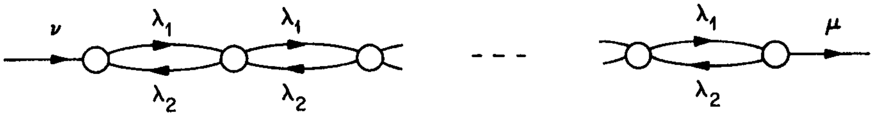

Fig. 6.1 A one-dimensional flow model

Fig. 6.2 An organizational hierarchy

form a Poisson process of rate ν the most senior individual in the organization leaves (to join the Head Office). This causes promotions within the organization in the following way. When post j becomes vacant there is a delay, exponentially distributed with mean λ_j^{-1}, before a replacement takes up the post. The replacement comes from the next most junior post in the organization which is occupied, or from outside the organization if all the more junior posts are vacant. By considering the movement of vacancies show that in equilibrium the posts are vacant or not independently and the probability post j is vacant is $\nu/(\nu+\lambda_j)$. Show that this remains true if the delay before a replacement is appointed to post j is arbitrarily distributed with mean λ_j^{-1}.

Extend the model to allow K_j individuals to hold positions at level j. Show that the probability level j has its complete complement of K_j individuals is

$$\left[\sum_{n=0}^{K_j} \frac{1}{n!}\left(\frac{\nu}{\lambda_j}\right)^n\right]^{-1}$$

3. Consider a Markov process with states $1, 2, \ldots, J$ and transition rates satisfying equation (6.12). Observe that in equilibrium each state is equally likely. Deduce from Lemma 1.4 that

$$\sum_{j\in\mathcal{A}} \sum_{k\notin\mathcal{A}} \lambda_{jk} = \sum_{j\in\mathcal{A}} \sum_{k\notin\mathcal{A}} \lambda_{kj}$$

and hence that expression (6.14) satisfies equations (6.13).

4. Consider the model of a mining operation discussed in Section 2.3. Suppose now that a machine cannot start work on a face until the next face is free. Show that machine j operates as a queue with $\phi_j(1)=0$, $\phi_j(n)=\phi_j$, $n>1$. Deduce that in equilibrium the probability machine j is working is

$$\frac{B_{N-J}}{\phi_j B_{N-J-1}}$$

where B_N is as defined for the original model of Section 2.3. Observe that if $\phi_1=\phi_2=\cdots=\phi_J=\phi$ the system can be represented by a flow model satisfying condition (6.12). Show that in this case the average time for a machine to complete one cycle of faces is

$$\frac{N(N-1)}{(N-J)\phi}$$

An alternative amendment to the model would be to suppose that after a machine has finished work on a face it cannot move on until the next face is free. None of the flow models considered can represent this behaviour, and indeed analytical results are generally unavailable for networks involving this fairly common form of blocking.

5. An open flow model satisfying (6.15) can be obtained as the limiting case of a closed flow model formed as follows. The closed flow model consists of the given J sites together with M appended sites. The flow rate from site j to each appended site is $\mu_j/M(1-p)$, and from each appended site to site j is ν_j/Mp. Show that for this closed flow model condition (6.12) becomes (6.15) provided p satisfies (6.16) and (6.17). Now let the number of appended sites M and the number of individuals N tend to infinity in such a way that N/M tends to p. Show that in the limit the open flow model leading to distribution (6.18) is obtained.
6. Using the transition rates (5.14), (5.15), and (5.16) show how the flow models leading to the equilibrium distributions (6.14) and (6.18) can be extended to allow a site to contain more than one individual. Observe that in equilibrium the number of individuals at a site in the open version will be binomially distributed.
7. Let $\mathbf{n}(t)$ be the state at time t of the flow model leading to equation (6.14). Show that the reversed process $\mathbf{n}(-t)$ corresponds to a similar flow model, but with λ_{jk} replaced by λ_{kj}. If $\mathbf{n}(t)$ is the state at time t of the open flow model leading to equation (6.18) show that $\mathbf{n}(-t)$ corresponds to a similar flow model with λ_{jk}, ν_j, and μ_j replaced by λ_{kj}, $\rho\mu_j$, and ν_j/ρ respectively. Deduce that the one-dimensional flow model considered at the end of this section is dynamically reversible.
8. A further example of an open flow model satisfying (6.15) is given by the following choice of parameters:

$$\lambda_{jk} = 0 \qquad \text{unless } k = j+1$$

$$\lambda_{j,j+1} = \lambda$$

for $j = 1, 2, \ldots, J-1$

$$\lambda_{Jk} = 0 \qquad \text{unless } k = 1$$

$$\lambda_{J1} = \lambda$$

and

$$\left.\begin{aligned} \mu_j &= \mu \\ \nu_j &= \nu \end{aligned}\right\} \qquad \text{for } j = 1, 2, \ldots, J$$

Check that in equilibrium the probability that a given site is occupied is ν/μ and hence does not depend upon λ. Show that the process is dynamically reversible. Following Exercise 6.1.4 discuss how the system might be rendered quasi-reversible.