APPLIED PROBABILITY[1]

Contents

[1]Grateful acknowledgements to James Norris who wrote the original version of these notes, and to Jeff Gaydon who developed and latexed them. Over the years the course has also been taught by Frank Kelly, Geoffrey Grimmett and Dan Crisan.

1 Supplementary notes

1.1 Measure spaces

The following notes describe some general notions from measure and integration theory used in the course. The definitions are easy to understand; and the course will not call on any detailed understanding of measure theory. A rigorous introduction to the concepts in this section may be found in the probability and measure course.

Definitions Let E be an arbitrary set and $\mathcal{E}$ be a set of subsets of E. We call $\mathcal{E}$ a *σ-algebra* on E if the following conditions hold:

(i) $\emptyset \in E$.

(ii) $A \in \mathcal{E}$ implies $A^c \in \mathcal{E}$.

(iii) $A_n \in \mathcal{E}$, for all $n \in \mathbb{N}$, implies $\bigcup_n A_n \in \mathcal{E}$.

Here $A^c = E \backslash A$ denotes the complement of A. Thus a σ-algebra is closed under countable set operations. The pair $(E, \mathcal{E})$ is called a *measurable space.*

If (X, τ) is a topological space then the *Borel σ-algebra* $\mathcal{B}(X)$ on X is the σ-algebra generated by τ; that is, $\mathcal{B}(X)$ is the smallest σ-algebra containing all the open sets in X. By convention, we write $\mathcal{B} = \mathcal{B}(\mathbb{R})$.

A *measure* μ is a countably additive, nonnegative set function defined on a σ-algebra $\mathcal{E}$; that is, for any countable collection of disjoint sets $(A_n : n \in \mathbb{N})$ in $\mathcal{E}$, a measure $\mu : \mathcal{E} \longmapsto [0, \infty]$ satisfies

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n).$$

We say that $A \in \mathcal{E}$ has *finite measure* if $\mu(A) < \infty$. We say also that μ is *σ-finite* if there exists a countable collection of sets of finite measure $(E_n : n \in \mathbb{N})$ such that $E = \bigcup_n E_n$. The measure μ is a *probability measure* if $\mu(E) = 1$. A *measure space* $(E, \mathcal{E}, \mu)$ is a triple which consists of a set E, a σ-algebra of subsets of E, and a measure μ defined on E. We say that $(E, \mathcal{E}, \mu)$ is a *probability space* if μ is a probability measure. The usual notation for a probability space is $(\Omega, \mathcal{F}, \mathbb{P})$.

Example Take $E = \mathbb{R}^d$ and $\mathcal{E} = \mathcal{B}\big(\mathbb{R}^d\big)$. The *Lebesgue measure* μ is the unique measure defined on $\mathcal{E}$ with the property that

$$\mu(A) = \prod_{i=1}^{d} (b_i - a_i).$$

Here A is the rectangle given by

$$A = \prod_{i=1}^{d} (a_i, b_i].$$

The fact that such a measure exists and is unique is an important theorem of measure theory.

1.2 Measurability and integration

Definitions Let $(E, \mathcal{E}, \mu)$ be a measure space. A function $f : E \longmapsto \mathbb{R}$ is *measurable* (with respect to $\mathcal{E}$) if $f^{-1}(B) \in \mathcal{E}$, for all Borel sets B (or, equivalently, just for all the open sets B in $\mathbb{R}$). The indicator function 1_A of any set $A \in \mathcal{E}$ is measurable. The *integral* of the indicator function with respect to μ is defined by $\int_E 1_A \, d\mu := \mu(A)$. More generally, a *simple* function is one which may be expressed in the form

$$f = \sum_{k=1}^{n} a_k 1_{A_k}.$$

Here $(a_k : k \leq n)$ are nonnegative constants, and $A_k \in \mathcal{E}$, for each k. For such a function, we define the integral by setting

$$\mu(f) := \sum_{k=1}^{n} a_k \mu(A_k).$$

The integral has a unique extension to the set of all *nonnegative* measurable functions f, written

$$\mu(f) = \int_E f \, d\mu = \int_E f \, \mu(dx).$$

For any measurable function f, we define $f^{\pm} := (\pm f) \vee 0$. Then f^+ and f^- are measurable, $f = f^+ - f^-$, and $|f| = f^+ + f^-$. In the case $\mu(|f|) < \infty$, we will say that f is *integrable* and set

$$\mu(f) = \mu(f^+) - \mu(f^-).$$

When μ is Lebesgue measure, one usually writes

$$\mu(f) = \int_E f\,dx.$$

For a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the integral is usually called the *expectation* written $\mathbb{E}X$.

Given a right-continuous and increasing map $g : \mathbb{R} \longmapsto \mathbb{R}$ (in particular, a distribution function), there exists a unique measure dg on $\mathbb{R}$, which is finite on compact sets, such that, for all $a, b \in \mathbb{R}$, with $a < b$, we have

$$dg((a, b]) = g(b) - g(a).$$

The measure dg is called the *Lebesgue-Stieltjes measure* associated with g; and the corresponding integral is called the *Lebesgue-Stieltjes integral.*

When X is a random variable with distribution function F, measure theory shows that for any integrable function h we have

$$\mathbb{E}h(X) = \int_E h\,dF.$$

When X has a density f, so that $\mathbb{P}(X \in A) = \int_A f\,dx$, for $A \in \mathcal{B}$, this last expression may be written $\int_A hf\,dx$.

We end this section by stating some standard theorems in integration theory of which the following is the most important.

Theorem 1.1 (Monotone convergence) *Let $(E, \mathcal{E}, \mu)$ be a measure space and let $(f_n : n \in \mathbb{N})$ be a sequence of nonnegative measurable functions such that $f_n \uparrow f$. Then $\mu(f_n) \uparrow \mu(f)$.*

The linearity of the integral is a simple consequence of monotone convergence: let $\alpha, \beta \in \mathbb{R}$ and suppose f and g are integrable. Then $\alpha f + \beta g$ is integrable and

$$\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g).$$

Here is another standard result from integration theory.

Theorem 1.2 (Bounded convergence) *Let $(X_n : n \in \mathbb{N})$ and X be random variables with $X_n \to X$, in probability[2], and $|X_n| \le C$, for all $n \in \mathbb{N}$, for some constant $C < \infty$. Then $\mathbb{E}X_n \to \mathbb{E}X$.*

[2]We say that a sequence of random variables $(X_n : n \in \mathbb{N})$ converges to X, *in probability*, if $\mathbb{P}(\{|X_n - X| > \varepsilon\}) \to 0$, for all $\varepsilon > 0$. This is implied by convergence *almost surely* (or *a.s.*) in which $\mathbb{P}(\{\omega : X_n(\omega) \not\to X(\omega)\}) = 0$.

Finally, we note that an interchange in the order of integration

$$\int_{E_1} \left(\int_{E_2} f(x_1, x_2)\, d\mu_2 \right) d\mu_1 = \int_{E_2} \left(\int_{E_1} f(x_1, x_2)\, d\mu_1 \right) d\mu_2$$

is justified whenever f is nonnegative (both sides possibly being infinite); or provided one (and hence both) of the integrals of the absolute values is finite; that is, if

$$\int_{E_1} \left(\int_{E_2} |f(x_1, x_2)|\, d\mu_2 \right) d\mu_1 = \int_{E_2} \left(\int_{E_1} |f(x_1, x_2)|\, d\mu_1 \right) d\mu_2 < \infty.$$

A precise formulation of this result, which requires a knowledge of product measure, may be found in the probability and measure course.

2 Poisson random measures

2.1 Introduction

This section provides an introduction to the theory of Poisson random measures and Poisson processes, and presents some of their applications. Poisson random measures and Poisson processes are mathematical models for sets of points which are haphazardly distributed with no obvious regularity or trend in density such as stars in a patch of sky; trees in an area of woodland; or impurities in crystal.

Unless otherwise stated we will assume that all the random variables considered will be defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and for the introduction of the notion of a Poisson random measure we will use σ-finite measure spaces.

2.2 The Poisson distribution

Definition A random variable N has a Poisson distribution with parameter $\lambda \in (0, \infty)$ if

$$\mathbb{P}(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}, \quad \text{for } n \in \mathbb{N}.$$

We write $N \sim P(\lambda)$. We also write $N \sim P(0)$ to mean $\mathbb{P}(N = 0) = 1$ and $N \sim P(\infty)$ to mean $\mathbb{P}(N = \infty) = 1$.

The Poisson distribution is 'natural' in the sense that the distribution of large number of independent binary phenomena with fixed total mean is approximately Poisson. To see this, let $N(n)$ be the number of heads obtained when tossing n coins with probability $p_n = \lambda/n$ of tossing a head (we assume that the throws are independent). The mean number of heads will be

$$\mathbb{E}N(n) = np_n = \lambda,$$

and

$$\begin{aligned}\mathbb{P}\big(N(n) = k\big) &= \binom{n}{k} p_n^k (1-p_n)^{n-k} \\ &= \left(\frac{n(n-1)\cdots(n-k+1)}{n^k\left(1-\frac{\lambda}{n}\right)^k}\right)\left(1-\frac{\lambda}{n}\right)^n \frac{\lambda^k}{k!}.\end{aligned}$$

It follows that

$$\mathbb{P}(N(n) = k) \to \frac{\lambda^k}{k!} e^{-\lambda}.$$

Theorem 2.1 (Addition property) *Suppose $(N_j : j \in \mathbb{N})$ is a sequence of independent random variables with $N_j \sim P(\lambda_j)$, for each j. Then*

$$\sum_{j=1}^{\infty} N_j \sim P\left(\sum_{j=1}^{\infty} \lambda_j\right).$$

Proof The theorem is true for two independent Poisson random variables since

$$\begin{aligned}\mathbb{P}(N_1 + N_2 = n) &= \sum_{m=0}^{n} \mathbb{P}(N_1 = m, N_2 = n-m) \\ &= \sum_{m=0}^{n} e^{-\lambda_1}\frac{\lambda_1^m}{m!} e^{-\lambda_2}\frac{\lambda_2^{n-m}}{(n-m)!} \\ &= e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}.\end{aligned}$$

The result then extends by induction to any finite number of independent Poisson random variables, and to infinite sums on noting that

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^{\infty} N_j \leq n\right) &= \lim_{k\to\infty} \mathbb{P}\left(\sum_{j=1}^{k} N_j \leq n\right) \\ &= \lim_{k\to\infty} \sum_{m=0}^{n} e^{-\sum_{j=1}^{k}\lambda_j} \frac{\left(\sum_{j=1}^{k}\lambda_j\right)^m}{m!} \\ &= \sum_{m=0}^{n} e^{-\sum_{j=1}^{\infty}\lambda_j} \frac{\left(\sum_{j=1}^{\infty}\lambda_j\right)^m}{m!}.\end{aligned}$$

Theorem 2.2 (Splitting property) *Let N and $(Y_n : n \in \mathbb{N})$ be independent random variables with $N \sim P(\lambda)$, where $\lambda < \infty$, and $\mathbb{P}(Y_n = j) = p_j$, for $j \le k$ and all $n \in \mathbb{N}$. Define, for each $j \le k$,*

$$N_j := \sum_{n=1}^{N} 1_{\{Y_n = j\}}.$$

Then $(N_j : j \le k)$ are independent random variables with $N_j \sim P(\lambda p_j)$, for all j.

Proof It is enough to show that

$$\mathbb{P}(N_1 = n_1, \dots, N_k = n_k) = \prod_{j=1}^{k} e^{-\lambda p_j} \frac{(\lambda p_j)^{n_j}}{n_j!}.$$

Observing that $N = \sum_{j=1}^{k} N_j$ and N is Poisson, an elementary calculation yields

$$\mathbb{P}(N_1 = n_1, \dots, N_k = n_k) = \mathbb{P}(N_1 = n_1, \dots, N_k = n_k | N = n) e^{-\lambda} \frac{\lambda^n}{n!}.$$

Here $n := \sum_{j=1}^{k} n_j$. But then *given that* $\sum_{j=1}^{k} N_j = n$, the vector $(N_1, \dots, N_k)$ is Multinomial$(n; p_1, \dots, p_k)$, so that

$$\mathbb{P}(N_1 = n_1, \dots, N_k = n_k | N = n) = \frac{n!}{\prod_{j=1}^{k} n_j!} p_1^{n_1} \cdots p_k^{n_k}.$$

Consequently,

$$\begin{aligned} \mathbb{P}(N_1 = n_1, \dots, N_k = n_k) &= e^{-\lambda} \frac{\lambda^n}{\prod_{j=1}^{k} n_j!} p_1^{n_1} \cdots p_k^{n_k} \\ &= \prod_{j=1}^{k} e^{-\lambda p_j} \frac{(\lambda p_j)^{n_j}}{n_j!}. \end{aligned}$$

2.3 Poisson random measures

Definition A *Poisson random measure with intensity* μ is a map

$$M : \Omega \times \mathcal{E} \longmapsto \mathbb{Z}^+.$$

satisfying, for all sequences of disjoint sets $(A_k : k \in \mathbb{N})$ in $\mathcal{E}$:

(i) $M\big(\bigcup_{k=1}^{\infty} A_k\big) = \sum_{k=1}^{\infty} M(A_k)$.

(ii) $(M(A_k) : k \in \mathbb{N})$ are independent random variables.

(iii) $M(A_k) \sim P\big(\mu(A_k)\big)$, for each $k \in \mathbb{N}$.

Here we have written $M(A)$ for the map: $\Omega \longmapsto \mathbb{Z}^+$

$$\omega \longmapsto M(\omega, A).$$

Condition (ii) includes the assumption that this map is measurable. We shall write $M \sim PRM(\mu)$ to denote a Poisson random measure with intensity μ. Poisson random measures exist as the following theorem shows.

Theorem 2.3 (The construction theorem) *Suppose $(E, \mathcal{E}, \mu)$ is a finite measure space with $\lambda := \mu(E) < \infty$; and suppose N and $(X_n : n \in \mathbb{N})$ are independent random variables such that $N \sim P(\lambda)$, and each $X_n : \Omega \longmapsto E$ has distribution μ/λ, for each $n > 0$. Define M by setting*

$$M(A) = \sum_{n=1}^{N} 1_{\{X_n \in A\}}.$$

Then $M \sim PRM(\mu)$.

Proof Let $(A_k : k \in \mathbb{N})$ be a sequence of disjoint sets in $\mathcal{E}$. We check each of the properties of a Poisson random measure. It is easy to check that

$$M\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} M(A_k).$$

To show that the last two properties hold it suffices to consider the case in which $E = \bigcup_{j=1}^{k} A_j$, and to check that $(M(A_j) : j \leq k)$ are independent random variables with $M(A_j) \sim P\big(\mu(A_j)\big)$. Define, for each $n \in \mathbb{N}$, the random variable

$$Y_n = \sum_{j=1}^{k} j 1_{\{X_n \in A_j\}}.$$

Then $(Y_n : n \in \mathbb{N})$ is a sequence of independent random variables satisfying

$$M(A_j) = \sum_{n=1}^{N} 1_{\{X_n \in A_j\}} = \sum_{n=1}^{N} 1_{\{Y_n = j\}}.$$

To complete the proof observe that, by the splitting theorem, $(M(A_j) : j \leq k)$ is a sequence of independent random variables with $M(A_j) \sim P(\lambda p_j)$, for $j \leq k$, where

$$p_j = \mathbb{P}(Y_n = j) = \mathbb{P}(X_n \in A_j) = \frac{\mu(A_j)}{\lambda}.$$

Theorem 2.4 (Extension theorem) *Let* $E := \bigcup_n E_n$ *and suppose that* $M_n \sim PRM(\mu_n)$*, for each* $n \in \mathbb{N}$*, where* $\mu_n := \mu|_{E_n}$.

(i) *If the sets* $(E_n : n \in \mathbb{N})$ *are disjoint and the measures* $(M_n : n \in \mathbb{N})$ *are independent then there exists a unique Poisson random measure* M *with intensity* μ *such that* $M|_{E_n} = M_n$*, for all* $n \in \mathbb{N}$.

(ii) *The same conclusion holds if* $E_n \uparrow E$ *and* $M_n = M_{n+1}|_{E_n}$.

Proof (i) Since $M|_{E_n}$ has to be M_n, we are forced to define

$$M(A) := \sum_{n=1}^{\infty} M_n(A \cap E_n), \quad \text{for } A \in \mathcal{E}.$$

It is straightforward, using the addition property, to check that $M \sim PRM(\mu)$.

(ii) Let $\widetilde{E}_1 := E_1$, $\widetilde{M_1} := M_1$ and, for $n \geq 2$, set $\widetilde{E}_n := E_n \backslash E_{n-1}$ and $\widetilde{M_n} := M_n|_{\widetilde{E}_n}$. Then $(\widetilde{E}_n : n \in \mathbb{N})$ is a sequence of disjoint sets and the result follows from part (i).

Together, the construction and extension theorems show that a Poisson random measure with intensity μ exists on any σ-finite measure space. Moreover, for any $A \in \mathcal{E}$ with $\lambda := \mu(A) < \infty$, we may always assume that $M|_A$ has the form

$$M(B) = \sum_{n=1}^{N} 1_{\{X_n \in B\}}, \quad \text{for } B \subset A.$$

Here $N \sim P(\lambda)$ and $(X_n : n \in \mathbb{N})$ is a sequence of independent random variables with distribution $\mu|_A/\lambda$.

2.4 The Poisson process

Definitions Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and I be a subset of $\mathbb{R}$. A *process* $X = (X_t : t \in I)$ taking values in $\mathbb{R}$ is a family of random variables $X_t : \Omega \longmapsto I$. We assume $I = [0, T]$, for some $T > 0$, or $I = [0, \infty)$.

We say that a process X in $\mathbb{R}$ is *continuous* (respectively *right-continuous*) if the map $t \longmapsto X_t(\omega) : I \longmapsto E$ is continuous (respectively right-continuous), for all $\omega \in \Omega$. We will always assume that a processes is at least right-continuous. This requirement, which ensures that the whole process may be determined by its restriction to a countable dense set of times, ensures that problems of measurability do not occur.

A process $X = (X_t : t \in I)$ has *independent increments* if for all $0 = t_0 < t_1 < \cdots < t_n$ in I the increments $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent random variables.

The right-continuous process $X = (X_t : t \geq 0)$ is a *Poisson process of rate* λ if $X_0 = 0$ and the following conditions hold:

(i) $X_t - X_s \sim P\big(\lambda(t-s)\big)$, for $s < t$.

(ii) $(X_t : t \geq 0)$ has independent increments.

Here λ denotes a positive real number. We write $X \sim PP(\lambda)$ if X is a Poisson process of rate λ. Note that X is a *pure jump process*; that is, it stays constant for a while before jumping to the next value. We denote by J_n the time of the nth jump and by $S_n := J_n - J_{n-1}$ the nth holding time. Then $J_n = \sum_{j=1}^{n} S_j$, and $X_t = n$ if $J_n \leq t < J_{n+1}$.

The following result provides alternative characterisations of the Poisson process.

Theorem 2.5 (Equivalence theorem) *Let $X = (X_t : t \geq 0)$ be a right-continuous process starting from zero. Then the following are equivalent:*

(i) X is a Poisson process of rate λ.

(ii) X has independent increments and, as $h \downarrow 0$, uniformly in t,

$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h) \quad \textit{and} \quad \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h).$$

(iii) The holding times $(S_n : n \in \mathbb{N})$ of X are independent $\mathcal{E}(\lambda)$ random variables and the jump chain is given by $J_n = n$, for all $n \in \mathbb{N}$.

The Poisson process is a special case of a Poisson random measure on $(0, \infty)$. Consider the measure space $\big((0,\infty), \mathcal{B}_{(0,\infty)}, \mu\big)$, where $\mathcal{B}_{(0,\infty)}$ denotes the Borel σ-algebra generated by the open intervals (a, b), with $a, b \in (0, \infty)$, and μ satisfies $\mu\big((0,t]\big) < \infty$, for all $t \in (0, \infty)$. Let $M \sim PRM(\mu)$ and define X by setting $X_t = M\big((0,t]\big)$. Then X_t satisfies:

(i) $X_0 = 0$.

(ii) $X_t - X_s \sim P\big(\mu(s,t]\big)$, for $s < t$.

(iii) X has independent increments.

In the case of a general measure μ, we call $(X_t : t \geq 0)$ an *inhomogeneous* Poisson process. In particular, consider the following special cases:

(i) $\mu(0,t] = \lambda t \rightsquigarrow (X_t : t \geq 0) \sim PP(\lambda)$.

(ii) $\mu(0,t] = \int_0^t \lambda(s)\, ds$. In this case we call λ the *rate function*.

Proposition 2.6 *The random variables $(S_n : n \in \mathbb{N})$ are independent and exponentially distributed with parameter λ.*

Proof A stronger result is proved in the course on Markov chains.

Proposition 2.7 *Let $(X_t : t \geq 0)$ be a Poisson process of rate λ and, for each $k \leq n$, let $Y_k = J_{\sigma_k}$, where σ is a random permutation of $\{1, \dots, n\}$. Then conditional on $X_t = n$, the sequence of random variables $(Y_k : k \leq n)$ is distributed as a random sample from $U[0,t]$.*

Proof Exercise.

Theorem 2.8 (The mapping theorem) *Suppose $(E, \mathcal{E}, \mu)$ is a measure space and $(G, \mathcal{G})$ is a measurable space. If $f : E \longmapsto G$ is $\mathcal{E}$-measurable, the image measure $\mu \circ f^{-1}$ is σ-finite, and if $M \sim PRM(\mu)$, then $M \circ f^{-1} \sim PRM(\mu \circ f^{-1})$.*

Proof The properties of a Poisson random measure are easily checked.

Theorem 2.9 (The product theorem) *Let $(J_n : n \in \mathbb{N})$ denote the jump times of a Poisson process of rate λ, and $(Y_n : n \in \mathbb{N})$ be independent E-valued random variables with law μ. Then:*

(i) Define the following measures on $(0, \infty) \times E$:

$$\begin{aligned} \nu((0,t] \times B) &:= \lambda t \mu(B), \\ M(A) &:= \sum_{n=1}^{\infty} 1_{\{(J_n, Y_n) \in A\}}. \end{aligned}$$

Then $M \sim PRM(\nu)$.

(ii) Suppose instead we define, for each $n \in \mathbb{N}$,

$$Y_n \sim \mu(t, dy), \quad \text{when } J_n = t.$$

Then the same conclusion holds with ν defined by

$$\nu\big((0,t] \times B\big) = \lambda \int_0^t \mu(s, B)\, ds.$$

Proof It suffices, by the extension theorem, to prove the result over intervals of the form $(0,t]$, for $t > 0$. By proposition (*2.7*), we may assume the Poisson process is given by

$$X_s := \sum_{k=1}^{N} 1_{\{U_k \leq s\}}, \quad \text{for } s \leq t.$$

Here $N \sim P(\lambda t)$ and $(U_n : n \geq 0)$ is a sequence of independent random variables distributed as $U[0,t]$. Set

$$M'(A) = \sum_{n=1}^{N} 1_{\{(U_n, Y_n) \in A\}}, \quad \text{for } A \subset (0,t] \times E.$$

Then $M' \sim M$ and, by the construction theorem, $M' \sim PRM(\nu)$.

(ii) True by essentially the same argument.

Example Assume that at time 0 the positions $(P_n : n \in \mathbb{N})$ of cars on a motorway are distributed as the jump times of Poisson process of rate λ, and let $(V_n : n \in \mathbb{N})$ denote the corresponding speeds which are independent and identically distributed with law μ. By the product theorem, $M \sim PRM(\nu)$, where

$$\begin{aligned} M(A) &:= \sum_{n=1}^{\infty} 1_{\{(P_n, V_n) \in A\}}, \\ \nu\big((0,t] \times B\big) &:= \lambda t \mu(B). \end{aligned}$$

The number of cars in $(x_1, y_1]$ after time t is given by

$$\begin{aligned} N\big((x_1, y_1]\big) &\sim P\big(\nu(\{(x,s) : x_1 < x + st \leq y_1\})\big), \\ \nu\big(\{(x,s) : x_1 < x + st \leq y_1\}\big) &= \int_0^\infty \int_{x_1 - st}^{y_1 - st} \lambda \, dx \, \mu(ds) = \lambda(y_1 - x_1). \end{aligned}$$

The numbers in disjoint intervals are independent, so that the distribution of cars on the motorway remains constant.

Example Customers arrive at a queue with infinitely many servers (M/G/∞ queue) in the manner of a Poisson process with rate λ and have independent service times given by a distribution function F. By the product theorem, the random measure M on $(0,\infty) \times \mathbb{R}^+$ with atoms (J_n, Y_n), for $n \geq 1$, where $(J_n : n \in \mathbb{N})$ represent the arrival times and $(Y_n : n \in \mathbb{N})$ are the service times, is Poisson with intensity

$$\nu\big([0,t] \times [0,y]\big) = \lambda t F(y).$$

Assuming the queue is empty at time 0, the number of customers in the system at time t is $P\big(\nu(A_t)\big)$, where

$$\begin{aligned} A_t &= \{(r,s) : s \in [0,t], r \in [t-s, \infty)\}, \\ \nu(A_t) &= \int_0^t \int_{t-s}^\infty \lambda dF(r) \, ds = \lambda \int_0^t (1 - F(s)) \, ds. \end{aligned}$$

A similar calculation shows that the output from the queue is an inhomogeneous Poisson process of rate $\lambda F(t)$.

Theorem 2.10 (Bartlett's theorem) *Customers arrive in the set E in the manner of a Poisson process with rate λ; and move around independently. For $A \subset E$, the probability that a given customer is in the set A a time s after its arrival is given by $\pi(s, A)$. Assuming E is empty at time 0 define*

$$\begin{aligned} \nu(A) &= \lambda \int_0^t \pi(s, A)\, ds, \\ N(A) &= \text{number of customers in } A \text{ at time } t. \end{aligned}$$

Then $N \sim PRM(\nu)$.

Proof Apply part (ii) of the product theorem, with $\mu(s, \cdot) := \pi(t - s, \cdot)$; and then the mapping theorem, with $f(s, y) := y$.

Theorem 2.11 (Campbell's theorem) *Let $(E, \mathcal{E}, \mu)$ be a finite measure space with $\lambda := \mu(E) < \infty$, and let $a : E \longmapsto \mathbb{R}$ be bounded and $\mathcal{E}$-measurable. Define the random variable $X := \int_E a(y) M(dy)$, where $M \sim PRM(\mu)$. Then*

$$\mathbb{E}\big(e^{\theta X}\big) = \exp\left(\int_E \big(e^{\theta a(y)} - 1\big)\, \mu(dy) \right).$$

Proof Conditional on $M(E) = n$, the atoms $(Y_k : k \leq n)$ of M form a random sample with law μ/λ. We have

$$\mathbb{E}\big(e^{\theta X} \big| M(E) = n\big) = \mathbb{E}\big(e^{\theta a(Y_1)}\big)^n = \left(\int_E e^{\theta a(y)} \frac{\mu(dy)}{\lambda} \right)^n.$$

Hence

$$\begin{aligned} \mathbb{E}\big(e^{\theta X}\big) &= \sum_{n=0}^{\infty} \mathbb{P}\big(M(E) = n\big) \mathbb{E}\big(e^{\theta X} \big| M(E) = n\big) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \left(\int_E e^{\theta a(y)} \frac{\mu(dy)}{\lambda} \right)^n \quad = \quad \exp\left(\int_E \big(e^{\theta a(y)} - 1\big)\, \mu(dy) \right). \end{aligned}$$

Formulae for the mean and variance of X follow by differentiation. We have

$$\mathbb{E}X = \int_E a(y) \mu(dy) \quad \text{and} \quad \operatorname{var} X = \int_E a(y)^2 \mu(dy).$$

Example Shots are heard at the jump times $(J_n : n \in \mathbb{N})$ of a Poisson process of rate λ. The initial amplitudes $(A_n : n \in \mathbb{N})$ are independent random variables with common distribution μ, and decay exponentially at rate α. We compute the distribution of the total amplitude X_t at time t, given by

$$X_t = \sum_{n=1}^{\infty} A_n e^{-\alpha(t-J_n)} 1_{\{J_n \leq t\}}.$$

Fix $t < \infty$ and let $E = [0,t] \times \mathbb{R}^+$, ν, and M satisfy

$$\begin{aligned} \nu(ds \times dx) &:= \lambda ds \mu(dx), \\ M(B) &:= \sum_{n=1}^{\infty} 1_{\{(J_n, A_n) \in B)\}}. \end{aligned}$$

By the product theorem, $M \sim PRM(\nu)$. Defining $a(s,x) := xe^{-\alpha(t-s)}$, we have

$$X_t = \int_E a(s,x)\, M(ds, dx).$$

Then, by Campbell's theorem,

$$\begin{aligned} \mathbb{E}\big(e^{\theta X_t}\big) &= \exp\left(\int_E \big(e^{\theta a(s,x)} - 1\big)\, \nu(ds, dx) \right) \\ &= \exp\left(-\lambda \int_0^t \big(1 - \phi(\theta e^{-\alpha s})\big)\, ds \right), \end{aligned}$$

where

$$\phi(\alpha) = \mathbb{E}\big(e^{\alpha A_1}\big) = \int_{\mathbb{R}^+} e^{\alpha x}\, \mu(dx).$$

Example Let S be the set of infinite undirected straight lines in $\mathbb{R}^2$. A random array of lines is then just a random subset of S and will be called a *line process*. If the random subset is a realization of a Poisson random measure, it will be called a *Poisson line process*.

A Poisson line process has a mean measure μ, which is a measure on S, and calculations involving this measure require a coordinate system on S. There are several ways in which one can set up such a system. Here is one of them.

If l is any line in $\mathbb{R}^2$, there is a unique line $l^\perp$ through the origin O perpendicular to l. Let $L = l \cap l^\perp$ and denote by p the distance OL, taken as positive if L lies in the upper half-plane $\{(x,y) : y > 0\}$ or on the positive x-axis $\{(x,0) : x > 0\}$; and negative otherwise. Let θ be the angle that $l^\perp$ makes with the axis defined so that $0 \le \theta \le \pi$. The function $l \longmapsto (p, \theta)$ maps S in a one-to-one way onto the strip

$$\widetilde{S} = \{(p, \theta) : -\infty < p < \infty, 0 \le \theta < \pi\}.$$

Thus any measure on S induces a measure on $\widetilde{S}$ and viceversa. By a convenient abuse of notation we will not distinguish between these two measures.

If μ is a non-atomic measure on $\widetilde{S}$ which is finite on bounded sets, then the construction theorem shows that there is a Poisson line process with mean measure μ, and that, almost surely, only finitely many random lines meet any bounded subset of $\mathbb{R}^2$. For example, one takes the Lebesgue measure on $\widetilde{S}$:

$$\mu\big([0, p] \times [0, \theta]\big) := p\theta.$$

More generally, one may use the measure given by $\mu_\lambda := \lambda\mu$. A Poisson line process with mean measure μ_λ is called a *uniform* Poisson line process with rate λ. The word uniform is justified by an important invariance property of μ_λ which implies a stochastic invariance of the line process itself. The Euclidean transformations of the plane (that is, the mappings which preserve distance) can be built up from translations, rotations around the origin, and reflections. Each of these takes lines to lines, inducing mappings $S \longmapsto S$ and $\widetilde{S} \longmapsto \widetilde{S}$. The measure μ_λ is invariant with respect the all these mappings. Examples of invariant transformations are:

(i) Translation along through q along the x-axis:

$$(p, \theta) \longmapsto (p + q\cos\theta, \theta).$$

(ii) Reflection with respect to the x-axis:

$$(p, \theta) \longmapsto \begin{cases} (-p, \theta) & 0 < \theta < \pi \\ (p, \theta) & \theta = 0. \end{cases}$$

3 Renewal theory

3.1 Renewal and renewal-reward processes

This section introduces the notions of renewal and renewal-reward processes. Such processes provide natural models of successively occurring events such as the failure of lightbulbs.

Definitions Suppose $(S_n : n \in \mathbb{N})$ is a sequence of independent, identically distributed random variables having common distribution function F; and such that

$\mathbb{P}(S_i > 0) > 0$ and $\mathbb{P}(S_i < \infty) = 1$. Set $J_0 = 0$, $J_n = \sum_{i=1}^n S_i$, for $n \geq 1$, and define the process $X = (X_t : t \geq 0)$ by setting $X_t = \max\{n : J_n \leq t\}$; that is,

$$X_t = n \quad \text{if } J_n \leq t < J_{n+1}.$$

We call X a *renewal process* with *holding times* $(S_n : n \in \mathbb{N})$ and *jump times* $(J_n : n \in \mathbb{N})$. We call $\lambda = (\mathbb{E}S_1)^{-1}$ the *rate* of X. On replacing S_1 by a new random variable $\widetilde{S}_1$ having a distribution which differs from that of S_1, we obtain the notion of a *delayed* renewal process. Observe that, since $J_n < \infty$, almost surely, for all $n > 0$, we have $\lim_{t\to\infty} X_t = \infty$, almost surely.

Suppose now that $(S_n, W_n : n \in \mathbb{N})$ is a sequence of independent, identically distributed random variables; and define $Y = (Y_t : t \geq 0)$ by setting $Y_t = \sum_{n=1}^{X_t} W_n$. Then Y is called a *renewal-reward process.*

Remarks (i) Letting $F(t) = 1 - e^{-\lambda t}$ we obtain a Poisson process of rate λ. Moreover, the random variables $(S_n, W_n : n \in \mathbb{N})$ form the atoms of a Poisson random measure so that the distribution of $Y = (Y_t : t \geq 0)$ is given by Campbell's theorem.

(ii) In general, renewal process model the replacement of components in which the process is delayed if the first component is not new. If $F(t) = 1 - e^{-\lambda t}$ the age of the first component does not matter:

$$\mathbb{P}(S_1 > s + t | S_1 > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(S_1 > t).$$

This is the *memoryless property* of the exponential distribution. If, on the other hand, $F(t) = 1 - (1+t)^{-1}$, then old components can last longer:

$$\mathbb{P}(S_1 > s + t | S_1 > s) = \frac{1+s}{1+s+t} > \frac{1}{1+t} = \mathbb{P}(S_1 > t).$$

(iii) Let $R = (R_t : t \geq 0)$ be a *regenerative process*; that is, suppose R is a process for which there exists a sequence of random times $(\tau_n : n \in \mathbb{N})$, with $\tau_n \uparrow \infty$, such that $(R_{\tau_n + t} : t \geq 0)$ has the same distribution as $(R_t : t \geq 0)$ and is independent of $(R_t : 0 \leq t \leq \tau_n)$, for all $n \in \mathbb{N}$. For example, take R to be a recurrent Markov chain and τ_n to be the time of the nth return to state i. Renewal processes form a useful tool for determining the properties of such a process.

Example Consider the sequence of random variables $S = (S_n : n \geq 0)$ given by

$$S_n = \begin{cases} \alpha & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

Let N_k denote the number of units with which the process increases at each jump time $k\alpha$. It is easily seen that the process $N = (N_k : k \geq 0)$ satisfies the following equations:

$$\begin{aligned}
\mathbb{P}(N_0 = n) &= p(1-p)^n, \quad \text{for } n \geq 0, \\
\mathbb{P}(N_k = n) &= p(1-p)^{n-1}, \quad \text{for } k, n \geq 1, \\
\mathbb{E}N_k &= \sum_{n=1}^{\infty} np(1-p)^{n-1} = \frac{1}{p}, \quad \text{for } k \in \mathbb{N}, \\
\mathbb{E}\big(N_k^2\big) &= \sum_{n=1}^{\infty} n^2 p(1-p)^{n-1} < \infty.
\end{aligned}$$

Similarly,

$$\mathbb{E}N_0 = \frac{1-p}{p} \quad \text{and} \quad \mathbb{E}\big(N_0^2\big) < \infty.$$

Assume $N = (N_k : k \geq 0)$ is a sequence of independent random variables and define the process $X = (X_t : t \geq 0)$ by setting

$$X_t = \sum_{j=0}^{k} N_j, \quad \text{for } k\alpha \leq t < (k+1)\alpha.$$

Then

$$\mathbb{E}\big(X_t^2\big) = \mathbb{E}\big(N_0^2\big) + k\mathbb{E}\big(N_1^2\big) + 2k\mathbb{E}N_0\mathbb{E}N_1 + k(k+1)(\mathbb{E}N_1)^2.$$

Therefore

$$\lim_{t\to\infty} \frac{\mathbb{E}\big(X_t^2\big)}{t^2} = \frac{(\mathbb{E}N_1)^2}{\alpha^2} = \frac{1}{\alpha^2 p^2}.$$

In particular, for all $\varepsilon > 0$,

$$\sup_{t\geq\varepsilon} \mathbb{E}\left[\left|\frac{X_t}{t}\right|^2\right] < \infty.$$

Proposition 3.1 *Let $X = (X_t : t \geq 0)$ be any renewal process. Then, for all $\varepsilon > 0$,*

$$\sup_{t\geq\varepsilon} \mathbb{E}\left[\left|\frac{X_t}{t}\right|^2\right] < \infty.$$

Proof There exists an $\alpha > 0$ for which $\mathbb{P}(S_1 \geq \alpha) =: p > 0$. Define a new sequence of random variables $\widetilde{S} = (\widetilde{S}_n : n \geq 0)$ by setting

$$\widetilde{S}_n = \begin{cases} \alpha & S_n \geq \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widetilde{S}_n \leq S_n$, for all n, so that the corresponding renewal process $\widetilde{X}$ satisfes $\widetilde{X}_t \geq X_t$. Now refer to the previous example.

3.2 Limit theorems

In this section we establish some important asymptotic properties of renewal and reward-renewal processes, including appropriate versions of the strong law and the central-limit theorem.

Theorem 3.2 *Let $X = (X_t : t \geq 0)$ be a renewal process of rate λ. Then*

$$\lim_{t\to\infty} \frac{X_t}{t} = \lambda, \quad a.s..$$

By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{J_n}{n} = \mathbb{E}S_1 = \frac{1}{\lambda}, \quad a.s..$$

We have, consecutively,

$$\begin{aligned} J_{X_t} &\leq t \leq J_{X_t+1}, \\ \frac{1}{J_{X_t+1}} &\leq \frac{1}{t} \leq \frac{1}{J_{X_t}}, \\ \frac{X_t}{X_t+1} \cdot \frac{X_t+1}{J_{X_t+1}} &\leq \frac{X_t}{t} \leq \frac{X_t}{J_{X_t}}. \end{aligned}$$

By taking the limit as $t \to \infty$ in the last equation, we obtain the result.

Theorem 3.3 *Let $(X_t : t \geq 0)$ be a renewal-reward process over a renewal process of rate λ, for which the sequence of rewards $(W_n : n \geq 1)$ satisfy $\mathbb{E}|W_n| < \infty$, for each $n \geq 1$. Then*

$$\lim_{t\to\infty} \frac{Y_t}{t} = \lambda \mathbb{E}W_1, \quad a.s..$$

Proof By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} W_i = \mathbb{E}W_1, \quad a.s..$$

Now observe that

$$\frac{Y_t}{t} = \frac{1}{X_t}\sum_{i=1}^{X_t} W_i \cdot \frac{X_t}{t}.$$

We obtain the result on letting $t \to \infty$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined random variables X and $(X_n : n \in \mathbb{N})$. We say that $(X_n : n \in \mathbb{N})$ converges *weakly* to X, written $X_n \Longrightarrow X$, if for all bounded continuous functions $f : \Omega \longmapsto \mathbb{R}$,

$$\mathbb{E}f(X_n) \to \int_\Omega f(x)\, \mu_X(dx) \text{ as } n \to \infty.$$

Here $\mu_X = \mathbb{P} \circ X^{-1}$ denotes the law X. Equivalently, if $(F_n : n \in \mathbb{N})$ and F denote the corresponding distribution functions then $X_n \Longrightarrow X$ if $F_n(x) \to F(x)$, whenever $F(x-) = F(x)$.

Theorem 3.4 *Let $(X_t : t \geq 0)$ be a renewal process of rate λ having holding times $(S_n : n \in \mathbb{N})$ of finite variance $\sigma^2 < \infty$. Then*

$$\frac{X_t - t\lambda}{\sigma\lambda\sqrt{\lambda t}} \Longrightarrow N(0,1) \text{ as } t \to \infty.$$

Proof By the central limit theorem,

$$\frac{J_n - n\lambda^{-1}}{\sigma\sqrt{n}} \Longrightarrow N(0,1) \text{ as } n \to \infty.$$

Defining $J_r = J_n$, for $n - 1 < r \leq n$, this limit remains valid if we replace n by the continuous parameter r. Moreover, we have

$$\{J_r > t\} \subset \{X_t \leq r\} \subset \{J_r \geq t\}.$$

Fix $x \in \mathbb{R}$ and $t > 0$. Then

$$\begin{aligned}
\mathbb{P}\left(\frac{X_t - t\lambda}{\sigma\lambda\sqrt{\lambda t}} \leq x\right) &= \mathbb{P}\left(X_t \leq t\lambda + x\sigma\lambda\sqrt{\lambda t}\right), \\
\mathbb{P}(J_r > t) \leq \mathbb{P}&\left(\frac{X_t - t\lambda}{\sigma\lambda\sqrt{\lambda t}} \leq x\right) \leq \mathbb{P}(J_r \geq t), \\
\mathbb{P}\left(\frac{J_r - r\lambda^{-1}}{\sigma\sqrt{r}} > -y(t)\right) \leq \mathbb{P}&\left(\frac{X_t - t\lambda}{\sigma\lambda\sqrt{\lambda t}} \leq x\right) \leq \mathbb{P}\left(\frac{J_r - r\lambda^{-1}}{\sigma\sqrt{r}} \geq -y(t)\right).
\end{aligned}$$

Here $r =: r(t) := \lambda(t + \sigma x\sqrt{\lambda t})$, $t = r(t)\lambda^{-1} - y(t)\sigma\sqrt{r(t)}$, and

$$y =: y(t) = \frac{x\sqrt{\lambda t}}{\sqrt{\lambda(t+\sigma x\sqrt{\lambda t})}} = x\sqrt{\frac{1}{1+\sigma x\sqrt{\lambda t^{-1}}}}.$$

This gives $\lim_{t\to\infty} r(t) = \infty$ and $\lim_{t\to\infty} y(t) = x$. It follows that

$$\mathbb{P}\left(\frac{X_t - t\lambda}{\sigma\lambda\sqrt{\lambda t}} \le x\right) = \mathbb{P}\left(\frac{J_{r(t)} - r(t)\lambda^{-1}}{\sigma\sqrt{r(t)}} > -y(t)\right) \to \Phi(x).$$

Theorem 3.5 *Let $(S_n, W_n : n \ge 0)$ be a sequence of independent, identically distributed holding times and rewards. Let $\lambda = (\mathbb{E}S_1)^{-1}$, $\mu = \mathbb{E}W_1$, and*

$$\tau^2 = \operatorname{var}(W_1 - \lambda\mu S_1) < \infty.$$

Then, as $n \to \infty$,

$$\frac{\sqrt{n}}{\lambda\tau}\left(\frac{\sum_{i=1}^n W_i}{\sum_{i=1}^n S_i} - \lambda\mu\right) \Longrightarrow N(0,1).$$

Proof Observe first that

$$\frac{\sqrt{n}}{\lambda\tau}\left(\frac{\sum_{i=1}^n W_i}{\sum_{i=1}^n S_i} - \lambda\mu\right) = \frac{n}{\lambda}\frac{1}{\sum_{i=1}^n S_i}\cdot\frac{1}{\tau\sqrt{n}}\sum_{i=1}^n (W_i - \lambda\mu S_i).$$

By the strong law of large numbers,

$$\frac{n}{\lambda}\frac{1}{\sum_{i=1}^n S_i} \to 1, \quad a.s..$$

Next the central limit theorem implies

$$\frac{1}{\tau\sqrt{n}}\sum_{i=1}^n (W_i - \lambda\mu S_i) \Longrightarrow N(0,1).$$

Let us now define, for each $n \ge 1$,

$$\alpha_n := \frac{1}{\tau\sqrt{n}}\sum_{i=1}^n (W_i - \lambda\mu S_i) \quad \text{and} \quad \beta_n := \frac{n}{\lambda}\frac{1}{\sum_{i=1}^n S_i}.$$

The result now follows from the following lemma.

Lemma 3.6 (Slutsky's lemma) *Let $(\alpha_n : n \in \mathbb{N})$ and $(\beta_n : n \in \mathbb{N})$ be sequences of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\alpha_n \Longrightarrow \alpha$ and $\beta_n \to c$, in probability, as $n \to \infty$, where c is a constant. Then $\alpha_n\beta_n \Longrightarrow c\alpha$.*

Proof Suppose $c > 0$ and pick δ such that $0 < \delta < c$. There exists an $N(\delta)$ such that $\mathbb{P}(|\beta_n - c| > \delta) < \delta$, for $n \geq N(\delta)$. Then, for $x \geq 0$,

$$\begin{aligned} \mathbb{P}(\alpha_n\beta_n \geq x) &\leq \mathbb{P}(\alpha_n\beta_n \leq x, |\beta_n - c| \leq \delta) + \mathbb{P}(|\beta_n - c| > \delta) \\ &\leq \mathbb{P}\left(\alpha_n \leq \frac{x}{c-\delta}\right) + \delta. \end{aligned}$$

Similarly,

$$\mathbb{P}(\alpha_n\beta_n > x) \leq \mathbb{P}\left(\alpha_n > \frac{x}{c+\delta}\right) + \delta.$$

Letting $n \to \infty$ and $\delta \downarrow 0$, we find that $\mathbb{P}(\alpha_n\beta_n \leq x) \to \mathbb{P}(c\alpha \leq x)$ if x/c is a point of continuity of the distribution of α. The remaining cases are similar.

3.3 The elementary renewal theorem

Definition The *renewal function* of a renewal process $(X_t : t \geq 0)$ is defined by $m(t) = \mathbb{E}X_t$, for $t \geq 0$.

We now turn our attention to the asymptotic properties of the renewal function which, as the next result shows, is asymptotically linear.

Theorem 3.7 (The elementary renewal theorem) *Let $(X_t : t \geq 0)$ be a renewal process of rate λ. Then*

$$\lim_{t\to\infty} \frac{m(t)}{t} = \lambda, \quad a.s..$$

Proof By the strong law of large numbers,

$$\lim_{t\to\infty} \frac{X_t}{t} = \lambda, \quad a.s..$$

We also have

$$\sup_{t\geq\varepsilon} \mathbb{E}\left[\left|\frac{X_t}{t}\right|^2\right] < \infty, \quad \text{for all } \varepsilon > 0.$$

The result follows from the following lemma.

Lemma 3.8 *Suppose $\alpha_t : \Omega \longmapsto [0,\infty)$ is a process satisfying $\lim_{t\to\infty} \alpha_t = 0$, almost surely, and there exists $\delta > 0$ such that $\sup_{t\geq\delta} \mathbb{E}(\alpha_t^2) < \infty$. Then $\lim_{t\to\infty} \mathbb{E}\alpha_t = 0$.*

Proof Observe that

$$\alpha_t = \alpha_t 1_{\{\alpha_t \leq M\}} + \alpha_t 1_{\{\alpha_t > M\}} \quad \text{and} \quad \lim_{t\to\infty} \alpha_t 1_{\{\alpha_t \leq M\}} = 0.$$

By bounded convergence,

$$\lim_{t\to\infty} \mathbb{E}\big(\alpha_t 1_{\{\alpha_t \leq M\}}\big) = \mathbb{E}\Big(\lim_{t\to\infty} \alpha_t 1_{\{\alpha_t \leq M\}}\Big) = 0.$$

But

$$\mathbb{E}\big(\alpha_t 1_{\{\alpha_t > M\}}\big) \leq \frac{1}{M}\mathbb{E}\big(\alpha_t^2\big) \leq \frac{1}{M} \sup_{t\geq\delta} \mathbb{E}\big(\alpha_t^2\big).$$

Hence

$$\limsup_{t\to\infty} \mathbb{E}\alpha_t \leq \frac{1}{M} \sup_{t\geq\delta} \mathbb{E}\big(\alpha_t^2\big).$$

The result follows on taking

$$\alpha_t := \left| \frac{X_t}{t} - \lambda \right|.$$

Example Let $(L_n : n \in \mathbb{N})$ be a sequence of independent, indentically distributed random variables representing component lifetimes with continuous distribution function F. Suppose $c, C, T \in (0,\infty)$ are fixed constants with $c < C$. The replacement plan is: if the component fails before time T, replace it at cost C; otherwise, replace it anyway at time T at cost c.

Let X_t be the number of components replaced by time t. Then $(X_t : t \geq 0)$ is a renewal-reward process with holding times $(S_n : n \geq 1)$, where $S_n = L_n \wedge T$ and

$$\mathbb{E}S_n = \int_0^\infty \mathbb{P}(S > t)\, dt = \int_0^T \mathbb{P}(L > t)\, dt = \int_0^T \big(1 - F(t)\big)\, dt.$$

Hence, by the strong law for renewal processes, the long-run average number of components per unit time is

$$1 \Big/ \int_0^T \big(1 - F(t)\big)\, dt.$$

Let Y_t be the cost of the replaced components by time t. Then Y_t is a renewal-reward process with holding times $(S_n : n \geq 1)$ and rewards $(W_n : n \geq 1)$, where

$$W = \begin{cases} C & S < T \\ c & S = T. \end{cases}$$

Hence

$$\mathbb{E}W = C\mathbb{P}(L < T) + c\mathbb{P}(L \geq T) = CF(T) + c\big(1 - F(T)\big).$$

By the strong law for renewal-reward processes, the long-run average cost per unit time is

$$\frac{CF(T) + c\big(1 - F(T)\big)}{\int_0^T \big(1 - F(t)\big)\, dt}.$$

We can then find T which minimises the long-run average cost.

3.4 Properties of the renewal function

Let F be the distribution function of S_1, F_n be the distribution function of J_n, and denote by $\widetilde{F}$ and $\widetilde{F}_n$ the respective moment generating functions of S_1 and J_n. Then

$$\begin{aligned} \widetilde{F}(\theta) &= \mathbb{E}\big(e^{\theta S_1}\big) = \int_0^\infty e^{\theta t} dF(t), \quad \text{for } \theta < 0, \\ \widetilde{F}_n(\theta) &= \mathbb{E}\big(e^{\theta J_n}\big) = \widetilde{F}(\theta)^n. \end{aligned}$$

Lemma 3.9 *If $\widetilde{F} = \widetilde{G}$ and $\widetilde{F}(\theta) < \infty$, for some $\theta \neq 0$, then $F = G$ (the condition holds if $F(0) = 0$).*

Proposition 3.10 *The renewal function m satisfies the* renewal equation*:*

$$m(t) = F(t) + \int_0^t m(t - s)\, dF(s).$$

We have $m(t) = \mathbb{E}X_t = \mathbb{E}\big(\mathbb{E}(X_t | S_1)\big)$ and

$$\mathbb{E}(X_t | S_1) = (1 + \mathbb{E}X_{t - S_1}) 1_{\{S_1 \leq t\}}.$$

Hence

$$m(t) = \int_0^t (1 + \mathbb{E}X_{t-s})\, dF(s) = F(t) + \int_0^t m(t - s) dF(s).$$

Definition A random variable X and its distribution function F are called *arithmetic with span* $\lambda > 0$ if X takes values in the set $\{m\lambda : m \in \mathbb{Z}\}$, almost surely, and λ is maximal with this property.

Theorem 3.11 *If S_1 is not arithmetic then, for all $h > 0$,*

$$\lim_{t\to\infty} \big(m(t+h) - m(t)\big) = \lambda h.$$

If S_1 is arithmetic with span $\lambda > 0$ then this holds whenever h is a multiple of λ.

Proposition 3.12 *Let $\widetilde{m}(\theta) = \int_0^\infty e^{\theta t} dm(t)$. Then*

$$\widetilde{F}(\theta) = \frac{\widetilde{m}(\theta)}{1 + \widetilde{m}(\theta)}.$$

In particular, m determines F.

Proof We give two proofs. For the first proof observe that, by Fubini's theorem,

$$m(t) = \mathbb{E}X_t = \mathbb{E}\left(\sum_{n=1}^\infty 1_{\{J_n \le t\}}\right) = \sum_{n=1}^\infty \mathbb{P}(J_n \le t) = \sum_{n=1}^\infty F_n(t).$$

Hence

$$\widetilde{m}(\theta) = \sum_{n=1}^\infty \int_0^\infty e^{\theta t} dF_n(t) = \sum_{n=1}^\infty \widetilde{F}_n(\theta) = \sum_{n=1}^\infty \widetilde{F}(\theta)^n = \frac{\widetilde{F}(\theta)}{1 - \widetilde{F}(\theta)}.$$

Alternatively, we use the renewal equation:

$$\begin{aligned}
\widetilde{m}(\theta) &= \widetilde{F}(\theta) + \int_0^\infty e^{\theta t} dG(t) \\
&= \widetilde{F}(\theta) + \int_0^\infty \int_0^t e^{\theta t} dm(t-s) dF(s) \\
&= \widetilde{F}(\theta) + \int_0^\infty \int_s^\infty e^{\theta t} dm(t-s) dF(s) \\
&= \widetilde{F}(\theta) + \int_0^\infty e^{\theta s} \int_s^\infty e^{\theta(t-s)} dm(t-s) dF(s) \\
&= \widetilde{F}(\theta) + \widetilde{m}(\theta)\widetilde{F}(\theta).
\end{aligned}$$

Here $G(t) := \int_0^t m(t-s)dF(s)$, and it follows that

$$dG(t) = \int_0^t dm(t-s)dF(s).$$

3.5 Discrete renewal theory

In the following section we will assume that $(S_n : n \in \mathbb{N})$ is a sequence of positive, integer-valued random variables. In particular $S_n \geq 1$, for each $n \in \mathbb{N}$. Let X_n denote the time until the next renewal. Then $X = (X_n : n \in \mathbb{N})$ is a discrete-time Markov chain with transition probabilities

$$\begin{aligned} p_{0i} &= \mathbb{P}(X_n = i | X_{n-1} = 0) &= \mathbb{P}(S_1 = i+1), \\ p_{ii-1} &= \mathbb{P}(X_n = i-1 | X_{n-1} = i) &= 1, \quad \text{for } i \geq 1. \end{aligned}$$

Observe that X is irreducible, recurrent, with expected time to return to 0 given by $m_0 = \mathbb{E}S_1$. If $\mathbb{E}S_1 < \infty$ then X is positive recurrent. It is easy to check that the invariant distribution π of X satisfies

$$\pi_i = \frac{\mathbb{P}(S > i)}{\mathbb{E}S}.$$

Theorem 3.13 (Discrete renewal theorem) *Assume that $p_i = \mathbb{P}(S_1 = i)$ satisfy $\sum_{i=1}^{\infty} p_i = 1$ and $\gcd\{i : p_i > 0\} = 1$ (that is, S is arithmetic with span 1). Let U_n denote the event of a renewal at time n. Then $\mathbb{P}(U_n) \to \lambda$ as $n \to \infty$.*

Proof The conclusion is immediate from the renewal theorem with $t = n$ and $h = 1$. It is instructive, however, to give a direct proof. The assumptions ensure that X is irreducible and aperiodic. Thus, by convergence to equilibrium of discrete-time Markov chains,

$$\lim_{n \to \infty} \mathbb{P}(U_n) = \lim_{n \to \infty} \mathbb{P}(X_n = 0) = \pi_0 = \frac{1}{\mathbb{E}S_1} = \lambda.$$

Suppose that a sequence $L = (L_n : n \in \mathbb{N})$ of random variables is observed and we keep track of the number of times that a given subsequence of outcomes or patterns occurs. This is a renewal process. Let $\mathbb{P}(L_1 = A) = p$ and $\mathbb{P}(L_1 = B) = q$. If the pattern is $BABA$ the corresponding process X is a delayed renewal process from which we can construct genuine, non-delayed renewal process in the following way. Consider a possible outcome:

$$BACCCBBABABACB \cdots.$$

We attach the pattern at the beginning of this string and start counting from the bar:

$$BABA | BACCCBBABABACB \cdots.$$

The probability of getting $BABA$ at time n is

$$\mathbb{P}(BABA \text{ occurs at time } n) = \begin{cases} 0 & n = 1 \\ pq & n = 2 \\ 0 & n = 3 \\ p^2 q^2 & n \geq 4. \end{cases}$$

Hence if X is the process defined in the previous section then the discrete renewal theorem implies

$$p^2q^2 = \lim_{n\to\infty} \mathbb{P}(X_n = 0) = \pi_0 = \frac{1}{\mathbb{E}S_1} = \lambda.$$

Hence the expected value of the holding times $1/p^2q^2$. Let T be the *first time* the pattern $BABA$ occurs. In order to find T we express it in terms of the holding times $(S_n : n \in \mathbb{N})$. We then have

$$T = S_1 + S_2 1_{\{S_1=2\}} \rightsquigarrow \mathbb{E}T = \frac{1}{p^2q^2}(1+pq).$$

If the pattern is $BABAB$ then

$$T = S_1 + S_2 1_{\{S_1=2\}} + S_3 1_{\{S_1=2, S_2=2\}} \rightsquigarrow \mathbb{E}T = \frac{1}{p^2q^3}\left(1+pq+p^2q^2\right).$$

3.6 Equilibrium renewal processes

Let $(X_t : t \geq 0)$ be a renewal process with holding times $(S_n : n \in \mathbb{N})$ and let F be the distribution of S_1. Suppose we replace the first holding time by S_1^D with distribution function $F^D(t)$ and define, as before,

$$X^D(t) = \max\{n : S_1^D + S_2 + \cdots + S_n \leq t\}.$$

We thereby obtain a delayed a renewal process. Denote by $m^D(t) := \mathbb{E}X^D(t)$ be the renewal function associated to X^D.

Proposition 3.14 *The renewal function m^D satisfies the* renewal equation*:*

$$m^D(t) = F^D(t) + \int_0^t m^D(t-s)dF(s).$$

If S_1 and S_1^D have the same distribution we recover a genuine renewal process. A delayed renewal process is therefore a generalisation of a renewal process.

Definitions A process $X = (X_t : t \geq 0)$ has *stationary increments* if, for all $s \geq 0$,

$$(X_{t+s} - X_s : t \geq 0) \sim (X_t : t \geq 0).$$

A delayed renewal process with stationary increments is called an *equilibrium* renewal process.

Theorem 3.15 *A delayed renewal process is an equilibrium or stationary renewal process if and only if*

$$F^D(t) = \lambda \int_0^t \big(1 - F(s)\big) ds.$$

Proof Suppose X^D has stationary increments. Then $m^D(t) = ct$, for some $c > 0$. By substituting this into the renewal equation for delayed processes, we obtain

$$F^D(t) = c \int_0^t \big(1 - F(t)\big) ds.$$

We obtain $c = \lambda$ on letting $t \to \infty$ and using the fact that

$$\frac{1}{\lambda} = \mathbb{E} S_1 = \int_0^\infty \big(1 - F(s)\big) ds.$$

Conversely, we substitute the formula for F^D into the renewal equation for delayed processes and use the integration-by-parts formula

$$tF(t) = \int_0^t F(s)\, ds + \int_0^t s\, dF(s).$$

We deduce

$$n(t) = \int_0^t n(t-s)\, dF(s), \quad \text{where } n(t) := m^D(t) - \lambda t.$$

Consequently, the generating function associated to n satisfies

$$\widetilde{n}(\theta) = \widetilde{n}(\theta)\widetilde{F}(\theta), \quad \text{for all } \theta < 0.$$

Hence $\widetilde{n} = 0$ and $m^D(t) = \lambda t$. Denote by $\widetilde{X}$ the process X conditioned on the event $\{X_s = n\}$. Then $\widetilde{X}_t$ is a delayed renewal process with holding times $S_{n+1} - s$ and $(S_{n+k} : k > 1)$. Hence we need to prove that $G^D(t) = F^D(t)$, where G^D is the distribution function of the excess lifetime of the current holding time:

$$G^D(t) := \mathbb{P}(J_{X_s+1} - s \leq t).$$

This is an elementary calculation:

$$\begin{aligned}
1 - G^D(t) &= \mathbb{P}(J_{X_s+1} - s > t) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(J_{k+1} - s > t, X_s = k) \\
&= \mathbb{P}(S_1^D - s > t, X_s = 0) \\
&\quad + \sum_{k=1}^{\infty} \int_0^s \mathbb{P}(J_{k+1} - s > t, X_s = k | J_k = x) dF_k(x) \\
&= 1 - F^D(t+s) + \int_0^s \big(1 - F(t+s-x)\big) dm^D(x) \\
&= 1 - F^D(t+s) + \lambda \int_0^s \big(1 - F(t+s-x)\big) dx \\
&= 1 - \lambda \int_0^{t+s} \big(1 - F(x)\big) dx + \lambda \int_t^{t+s} \big(1 - F(x)\big) dx \\
&= 1 - \lambda \int_0^t \big(1 - F(x)\big) dx \;=\; 1 - F^D(t).
\end{aligned}$$

Here we have used the fact that, by Fubini's theorem,

$$m^D(t) = \mathbb{E}X_t^D = \mathbb{E}\left(\sum_{n=1}^{\infty} 1_{\{J_n^D \le t\}}\right) = \sum_{n=1}^{\infty} \mathbb{P}(J_n \le t) = \sum_{n=1}^{\infty} F_n^D(t).$$

3.7 Regenerative processes

Let $(\alpha_t : t \ge 0)$ be an integer-valued process having the property that there exist time points at which the process restarts itself (probabilistically). In other words, suppose that, almost surely, there exists a (random) time τ_1 such that the continuation of the process beyond τ_1 is a probabilistic replica of the whole process starting at 0; that is, $(\alpha_{t+\tau_1} : t \ge 0) \sim (\alpha_t : t \ge 0)$ independently of $(\alpha_t : 0 \le t \le \tau_1)$. Observe that this property implies the existence of further times τ_n, for $n > 1$, with the same properties. We assume also that

$$\mathbb{P}(\tau_1 > 0) > 0 \quad \text{and} \quad \mathbb{E}\tau_1 < \infty.$$

Theorem 3.16 (Regenerative average theorem) *Let $(\alpha_t : t \ge 0)$ be an integer-valued regenerative process with restarting stopping times $(\tau_n : n \in \mathbb{N})$ (with the convention $\tau_0 = 0$) such that, almost surely,*

$$\lim_{n\to\infty} \frac{\tau_{n+1}}{\tau_n} = 1 \quad \textit{and} \quad \lim_{t\to\infty} \mathbb{P}(\alpha_t = i) = \pi_i.$$

Then

$$\lim_{t\to\infty} \frac{\textit{time spent in } i \textit{ during } [0,t]}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t 1_{\{\alpha_s=i\}}\,ds = \pi_i.$$

Proof Denote by $X = (X_t : t \geq 0)$ the renewal process having holding times $S_n = \tau_n - \tau_{n-1}$, for each $n \in \mathbb{N}$, so that

$$X_t = \max\{n : \tau_n \leq t\}.$$

Let Y_t be a renewal-reward process over X_t with rewards $W_i = \int_{\tau_{i-1}}^{\tau_i} 1_{\{\alpha_s=n\}}\,ds$. We have

$$\frac{1}{t}\int_0^t 1_{\{\alpha_s=i\}}\,ds = \frac{\tau_n}{t}\cdot\frac{1}{\tau_n}\int_0^{\tau_n} 1_{\{\alpha_s=i\}}\,ds + \frac{1}{t}\int_{\tau_n}^t 1_{\{\alpha_s=i\}}\,ds. \tag{1}$$

But

$$0 \leq \frac{1}{t}\int_{\tau_n}^t 1_{\{\alpha_s=i\}}\,ds \leq \frac{t-\tau_n}{t} \leq \frac{\tau_{n+1}-\tau_n}{\tau_n}.$$

The second term in equation (*1*) converges to 0, almost surely. Also,

$$1 \leq \frac{\tau_n}{t} \leq \frac{\tau_n}{\tau_{n+1}} \to 1.$$

Next observe that, by the strong law of large numbers and the strong law for renewal and renewal-reward processes,

$$\frac{1}{\tau_n}\int_0^{\tau_n} 1_{\{\alpha_s=i\}}\,ds = \frac{Y_t}{J_{X_t}} = \frac{Y_t}{t}\cdot\frac{t}{X_t}\cdot\frac{X_t}{J_{X_t}} \to \lambda\mathbb{E}W_1.$$

Thus we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t 1_{\{\alpha_s=i\}}\,ds = \lambda\mathbb{E}W_1.$$

Then, by bounded convergence and Fubini's theorem,

$$\lambda\mathbb{E}W_1 = \mathbb{E}\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t 1_{\{\alpha_s=i\}}\,ds\right) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}(\alpha_s=i)\,ds = \pi_i.$$

Remarks If the integer-valued regenerative process $(\alpha_t : t \geq 0)$ is in equilibrium then $\mathbb{P}(\alpha_t = i) = \pi_i$ and the theorem holds. Moreover, the invariant distribution satisfies

$$\pi_i = \frac{\mathbb{E}(\textit{reward during a cycle})}{\mathbb{E}(\textit{time of a cycle})}.$$

4 Queueing theory

4.1 Queues

The theory of queues grew out of the observation of telephone traffic engineers (notably A. K. Erlang in Copenhagen) that the times at which calls arrive at a telephone switchboard are well described by a Poisson process of constant (or slowly varying) rate. The general model for queues is the following.

Customers arrive at a service point or counter at which a number of servers are stationed. An arriving customer may have to wait until one of these servers becomes available. He then moves to the head of the queue and is served; he leaves the system on completion of his service. We must specify a number of details about the queueing system before we are able to model it adequately. For example:

(*i*) In what manner do customers enter the system?

(*ii*) In what order are they served?

(*iii*) How long are their service times?

We will assume the following:

(*1*) The times between arrivals are independent, identically distributed random variables.

(*2*) Arriving customers join the end of a single line of people who receive attention on a FCFS basis.

(*3*) The service times of customers are nonnegative, independent, and identically distributed random variables which do not depend on the arriving stream of customers.

The state of the queue at time t is described by the number X_t of waiting customers, where X_t includes customers who are in the process of being served. We call $X = (X_t : t \geq 0)$ *stable* if the distribution of X_t settles down as $t \to \infty$ in some 'well-behaved' manner; otherwise, we call X *unstable*. We will distinguish only between those queues which grow beyond bound or enjoy large wild fluctuations in length, and those in which the queue lengths converge in distribution to some 'equilibrium' distribution.

We define the *traffic density* to be the ratio

$$\rho = \frac{\mathbb{E}(\textit{service time})}{\mathbb{E}(\textit{inter-arrival time})}$$

For all the queues studied below, we will encounter the following theorem.

Theorem 4.1 *Let X be a queue with a single server and traffic density ρ.*

(i) If $\rho < 1$ then X is stable.

(ii) If $\rho > 1$ then X is unstable.

(iii) If $\rho = 1$ and either the service times or the inter-arrival times have strictly positive variance then X is unstable.

We end this section by recalling a standard result from the theory of Markov chains.

Theorem 4.2 (Ergodic theorem for Markov chains) *Suppose $(X_n : n \geq 0)$ is an irreducible Markov chain. Then, almost surely,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{\{X_k = i\}} = \frac{1}{m_i},$$

where m_i is the expected time to return to 0.

4.2 The M/M/1 queue

This is the simplest queue of all – the code means

***M**emoryless inter-arrival times/**M**emoryless service times/**1** server.*

It is easy to see that $X = (X_t : t \geq 0)$ is a Markov chain. The M/M/1 queue evolves like a random walk which does not take jumps below 0. If $\rho > 1$ then X is transient; that is, $X_t \to \infty$ as $t \to \infty$; and the queue *grows without limit* in the long term. When $\rho < 1$, the queue is positive recurrent having invariant distribution π given by

$$\pi_i = (1-\rho)\rho^i.$$

Hence when $\rho < 1$ the average number of customers in the queue in equilibrium is given by

$$\mathbb{E}_\pi X_t = \sum_{i=1}^{\infty} \mathbb{P}_\pi(X_t \geq i) = \sum_{i=1}^{\infty} \rho^i = \frac{\rho}{1-\rho}.$$

The long-run proportion of idle time is $\pi_0 = 1 - \rho$.

4.3 The M/M/s queue

This is a variation of the M/M/1 queue in which there are s servers instead of a single one. Suppose the arrival rate is λ and that the service rate is μ. When i

servers are occupied, the first service is completed at the minimum of i independent exponential times of parameter μ, so that the first service time is exponential of parameter $i\mu$. The total service rate increases to a maximum $s\mu$ when all servers are working. Consequently, we obtain a birth-death chain which is transient in the case $\lambda > s\mu$ and is otherwise recurrent. An invariant distribution may be found by solving the detailed balance equations:

$$\pi_i q_{i,i+1} = \pi_{i+1} q_{i+1,i}.$$

It follows that

$$\frac{\pi_i}{\pi_0} = \begin{cases} \rho^i/i! & \text{for } i = 0, 1, \dots, s \\ \rho^i/(s^{i-s}s!) & \text{for } i > s. \end{cases}$$

The queue is positive recurrent when $\lambda < s\mu$. When $s = 1$ we recover the M/M/1 queue in which the invariant distribution is geometric with parameter ρ; and in the case $s = \infty$ we normalise π by taking $\pi_0 = e^{-\rho}$ to give

$$\pi_i = \frac{\rho^i e^{-\rho}}{i!}.$$

The invariant distribution is therefore Poisson of parameter ρ.

4.4 The telephone exchange

In a variation of the M/M/s queue, customers that cannot be served immediately are turned away. This provides a simple model of a telephone exchange in which the maximum number of calls that can be connected at any particular time is s.

We compute the invariant distribution of this finite state-space Markov chain by solving the detailed balance equations. This time we obtain a *truncated Poisson distribution*:

$$\pi_i = \frac{\rho^i}{i!} \Big/ \sum_{j=0}^{s} \frac{\rho^j}{j!}.$$

By the ergodic theorem for Markov chains, the long-run proportion of time in which the exchange is full, and hence the long-run proportion of calls that are lost, is given by

$$\pi_s = \frac{\rho^s}{s!} \Big/ \sum_{j=0}^{s} \frac{\rho^j}{j!}.$$

This is known as *Erlang's formula.*

4.5 The M/G/1 queue

For this queue, the inter-arrival times are exponential with parameter λ (thus customers arrive in a Poisson process with intensity λ) and the service time is given by a general distribution F. If A_n is the inter-arrival time of the nth customer and S_n is the service time of the nth customer then

$$A_n \sim \mathcal{E}(\lambda) \quad \text{and} \quad \mathbb{P}(S_n \in dt) = F(dt), \quad \text{where } \lambda, \mathbb{E}S_n \in (0,\infty).$$

The name comes from

***M**emoryless inter-arrival times/**G**eneral service times/**1** server.*

The M/M/1 queue is a special case of an M/G/1 queue and is the only such queue which is a Markov chain; the analysis of the general M/G/1 queue requires more ingenuity. Let X_n be the queue size after the nth departure. Then

$$X_{n+1} = X_n + Y_{n+1} - 1_{\{X_n > 0\}}.$$

Here Y_{n+1} denotes the number of customers which arrive during the service time of the $(n+1)$th customer. The case $X_n = 0$ is different because then we get an extra arrival before the $(n+1)$th service time begins. By the properties of the Poisson process, $Y = (Y_n : n \in \mathbb{N})$ is a sequence of independent, identically distributed random variables – Y_n depends only on the service time S_{n+1} and is independent of X_n, whence X is a Markov chain. Conditional on $S_n = t$, we have $Y_n \sim P(\lambda t)$, so that

$$\begin{aligned}
\mathbb{E}Y &= \int_0^\infty \mathbb{E}(Y|S=t)\,dF(t) = \int_0^\infty \lambda t\,dF(t) = \lambda \mathbb{E}S, \\
G_Y(z) &= \mathbb{E}\big(z^Y\big) = \int_0^\infty \mathbb{E}\big(z^Y\big|S=t\big)dF(t) = \int_0^\infty e^{\lambda t(z-1)}dF(t) \\
&= G_S\left(e^{\lambda(z-1)}\right) = M_S\big(\lambda(z-1)\big).
\end{aligned}$$

Here G_Y is the generating function of Y, G_S is the generating function of S, and M_S is the moment generating function of S.

Theorem 4.3 *If the traffic density $\rho := \mathbb{E}Y = \lambda\mathbb{E}S < 1$, then $X = (X_n : n \geq 0)$ is positively recurrent.*

Proof We have

$$X_n = X_0 + (Y_1 + \cdots + Y_n) - n + Z_n.$$

Here Z_n denotes the number of visits to 0 before time n. By taking $X_0 = 0$, we obtain

$$\mathbb{E}\big(n^{-1}Z_n\big) = \mathbb{E}\big(n^{-1}X_n\big) + 1 - \rho \quad \text{and} \quad \mathbb{E}\big(n^{-1}Z_n\big) \geq 1 - \rho > 0.$$

By the ergodic theorem for Markov chains,

$$\lim_{n\to\infty} \mathbb{E}\big(n^{-1}Z_n\big) = \frac{1}{m_0},$$

where m_0 is the expected time to return to 0. Hence

$$m_0 \leq \frac{1}{1-\rho} < \infty.$$

Thus the process X has a unique invariant distribution which we denote by π.

Proposition 4.4 *Let X be a random variable with invariant distribution π. Then $\pi_0 = 1 - \rho$,*

$$G_X(z) = \frac{(1-\rho)(1-z)G_Y(z)}{G_Y(z) - z}, \quad \textit{and} \quad \mathbb{E}X = \rho + \frac{\lambda^2 \mathbb{E}\big(S^2\big)}{2(1-\rho)}.$$

Proof Observe that

$$\begin{aligned} zG_X(z) &= \mathbb{E}\big(z^{1+X_{n+1}}\big) = \mathbb{E}\big(z^{Y_{n+1}}\big)\mathbb{E}\big(z^{X_n + 1_{\{X_n=0\}}}\big) \\ &= G_Y(z)\left(\pi_0 z + \sum_{i=1}^{\infty} \pi_i z^i\right) = G_Y(z)\big(\pi_0 z + G_X(z) - \pi_0\big). \end{aligned}$$

Hence

$$\big(G_Y(z) - z\big)G_X(z) = G_Y(z)\pi_0(1-z) \Longrightarrow G_X(z) = \frac{G_Y(z)\pi_0(1-z)}{G_Y(z) - z}.$$

This implies

$$\lim_{z\to 1} \frac{G_Y(z) - z}{\pi_0(1-z)} = 1.$$

By l'Hôpital's rule,

$$\lim_{z\to 1} \frac{G_Y(z) - z}{\pi_0(1-z)} = \frac{1 - G_Y'(1)}{\pi_0} = \frac{1-\rho}{\pi_0} = 1.$$

This yields the first two equations. For the last equation, we have

$$\lim_{z\to 1} G_X'(z) = \lim_{z\to 1} \frac{vu' - uv'}{v^2},$$

where $u = (1-\rho)(1-z)G_Y(z)$ and $v = G_Y(z) - z$. By l'Hôpital's rule again and the fact that $u(1) = v(1) = 0$, we obtain

$$\begin{aligned}
\mathbb{E}X &= \lim_{z\to 1} \frac{vu'' - uv'' + v'u' - u'v'}{2vv'} \\
&= \lim_{z\to 1} \frac{vu'' - uv''}{2vv'} \\
&= \lim_{z\to 1} \frac{vu''' - uv''' + v'u'' - u'v''}{2(v')^2 + 2vv''} \quad = \quad \left.\frac{v'u'' - u'v''}{2(v')^2}\right|_{z=1}.
\end{aligned}$$

It is easy to check that the following equations hold:

$$\begin{aligned}
v'(1) &= G_Y'(1) - 1 = \rho - 1, \\
u'(1) &= \rho - 1, \\
v''(1) &= G_Y''(1) \quad = \quad \lambda^2 M_S''(1) \quad = \quad \lambda^2 \mathbb{E}(S^2), \\
u''(1) &= 2\rho(\rho - 1).
\end{aligned}$$

It follows that

$$\mathbb{E}X = \frac{(\rho-1)\left(2\rho(\rho-1) - \lambda^2\mathbb{E}(S^2)\right)}{2(\rho-1)^2} = \rho + \frac{\lambda^2\mathbb{E}(S^2)}{2(1-\rho)}.$$

Proposition 4.5 *Consider the chain $X = (X_n : n \in \mathbb{N})$ in equilibrium. Suppose that the customer who leaves at time 0 has spent time V queueing to be served, and time S being served. Then*

$$M_V(w) = \mathbb{E}\left(e^{wV}\right) = \frac{(1-\rho)w}{w + \lambda\left(1 - M_S(w)\right)}.$$

Proof Conditional on $V + S = t$, X_0 is Poisson of parameter λt, since the customers in the queue at time 0 are precisely those who arrived during the queueing and service times of the departing customer. Hence

$$G_X(z) = M_V\left(\lambda(z-1)\right)M_S\left(\lambda(z-1)\right).$$

On substituting for $G_X(z)$ we obtain the required formula.

Differentiation and l'Hôpital's rule lead to a formula for the mean queueing time:

$$\mathbb{E}V = \frac{\lambda\mathbb{E}(S^2)}{2(1-\rho)}.$$

We now turn to the busy period T. Let $B(w) = \mathbb{E}\left(e^{wT}\right)$ and S denote the service time of the first customer in the busy period. Then, conditional on $S = t$, we have $T = t + \sum_{i=1}^{N} T_i$, where $N \sim P(\lambda t)$ is the number of customers arriving while the first customer is served, and where $T_i \sim T$ are independent. Hence

$$\begin{aligned} B(w) &= \int_0^\infty \mathbb{E}\left(e^{wT} \middle| S = t\right) dF(t) \\ &= \int_0^\infty e^{wt} e^{-\lambda t(1-B(w))} dF(t) \;=\; M_S\big(w - \lambda(1 - B(w))\big). \end{aligned}$$

Differentiating for $w < 0$ and taking the limit as $w \to 0$, we obtain

$$\begin{aligned} \mathbb{E}T &= \lim_{w\to 0} B'(w) \\ &= \lim_{w\to 0} M_S'\big(w - \lambda(1 - B(w))\big)\big(1 + \lambda B'(w)\big) \\ &= (1 + \lambda\mathbb{E}T)\mathbb{E}S. \end{aligned}$$

Hence

$$\mathbb{E}T = \frac{\mathbb{E}S}{1 - \rho}.$$

4.6 Heavy traffic

Let V_λ be the distribution of the time spent by a customer in the queue with arrival rate λ and service time S (we keep the distribution of the service time fixed and vary λ). Assume that $\mathbb{E}\big(S^2\big) < \infty$. We consider the limiting distribution of V_λ as $\rho := \lambda\mathbb{E}S \to 1$. More precisely, we study the convergence of

$$\widetilde{V}_\lambda := (1 - \lambda\mathbb{E}S)V_\lambda \text{ as } \rho \to 1.$$

Proposition 4.6 *$\widetilde{V}_\lambda$ converges in distribution to an exponential random variable with rate $\alpha = 2(\mathbb{E}S)\mathbb{E}\big(S^2\big)^{-1}$.*

Proof We have

$$\begin{aligned} M_{\widetilde{V}_\lambda}(w) &= M_{V_\lambda}\big((1-\rho)w\big) \\ &= \frac{(1-\rho)^2 w}{(1-\rho)w + \lambda\big(1 - M_S((1-\rho)w)\big)} \\ &= \frac{(1-\rho)^2 w}{(1-\rho)w - \lambda\Big((1-\rho)wM_S'(0) + \frac{1}{2}\big((1-\rho)w\big)^2 M_S''(0)\big(1 + o(1)\big)\Big)} \\ &= \frac{(1-\rho)^2 w}{(1-\rho)^2 w - \frac{1}{2}\lambda\big((1-\rho)w\big)^2 M_S''(0)\big(1 + o(1)\big)} \;\to\; \frac{\alpha}{\alpha - w}. \end{aligned}$$

The result follows, since convergence of moment generating functions implies convergence in distribution.

Remark In a regime of heavy traffic we can approximate the distribution of V_λ by an exponential random variable with rate

$$\frac{2(1-\rho)\mu_S}{\mu_S^2+\sigma_S^2}.$$

4.7 Little's formula

Recall that a process $\alpha = (\alpha_t : t \geq 0)$ is a regenerative if there exists a (random) time τ_1 such that $(\alpha_{t+\tau_1} : t \geq 0) \sim (\alpha_t : t \geq 0)$ independently of $(\alpha_t : 0 \leq t \leq \tau_1)$. Consider a queue $X = (X_t : t \geq 0)$, starting from 0, which is regenerative. For example, X could be an M/G/1 queue in which τ_1 is the end of the first busy period. Let R_t denote the number of regenerations by time t (this is a renewal process). Define N_t to be the number of arrivals by time t and W_n to be the waiting time of the nth customer. Then N_{τ_1} is the number of arrivals by time τ_1. For $\tau_n \leq t < \tau_{n+1}$, set

$$Y_t = \int_0^{\tau_n} X_s\, ds \quad \text{and} \quad \widetilde{Y}_t = N_{\tau_n}.$$

Then $Y = (Y_t : t \geq 0)$ and $\widetilde{Y} = (\widetilde{Y}_t : t \geq 0)$ are renewal-reward processes over $R = (R_t : t \geq 0)$. Since $X_0 = X_{\tau_n} = 0$, the service of all customers arriving before time τ_n must be completed by τ_n. Hence

$$Y_{\tau_n} = \sum_{i=1}^{N_{\tau_n}} W_i.$$

Theorem 4.7 (Little's formula) *If $\mathbb{E}\tau_1 < \infty$ and $\mathbb{E}N_{\tau_1} < \infty$ the following limits exist almost surely and are non-random:*

(i) Long-run average queue size:

$$L := \lim_{t\to\infty} \frac{1}{t}\int_0^t X_s\, ds.$$

(ii) Long-run average waiting time:

$$W := \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} W_i.$$

(iii) Long-run average arrival rate:

$$\lambda = \lim_{t\to\infty} \frac{N_t}{t}.$$

Moreover, $L = \lambda W$*.*

Proof For $\tau_n \le t < \tau_{n+1}$, we have

$$\frac{Y_{\tau_n}}{\tau_{n+1}} \le \frac{1}{t}\int_0^t X_s\, ds \le \frac{Y_{\tau_{n+1}}}{\tau_n}.$$

By the strong law of large numbers, we have, almost surely,

$$\begin{aligned}
\lim_{n\to\infty} \frac{Y_{\tau_n}}{\tau_{n+1}} &= \lim_{n\to\infty} \frac{Y_{\tau_n}}{n}\cdot\frac{n}{n+1}\cdot\frac{n+1}{\tau_{n+1}} = \frac{\mathbb{E}Y_{\tau_1}}{\mathbb{E}\tau_1},\\
\lim_{n\to\infty} \frac{Y_{\tau_{n+1}}}{\tau_n} &= \lim_{n\to\infty} \frac{Y_{\tau_{n+1}}}{n+1}\cdot\frac{n+1}{n}\cdot\frac{n}{\tau_n} = \frac{\mathbb{E}Y_{\tau_1}}{\mathbb{E}\tau_1}.
\end{aligned}$$

This implies

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_s\, ds =: L = \frac{\mathbb{E}Y_{\tau_1}}{\mathbb{E}\tau_1}.$$

Similarly,

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda := \frac{\mathbb{E}\widetilde{Y}_{\tau_1}}{\mathbb{E}\tau_1}.$$

Finally, for $N_{\tau_n} \le k < N_{\tau_{n+1}}$,

$$\frac{Y_{\tau_n}}{N_{\tau_{n+1}}} \le \frac{1}{k}\sum_{i=1}^k W_i \le \frac{Y_{\tau_{n+1}}}{N_{\tau_n}}.$$

As before, we have, almost surely,

$$\begin{aligned}
\lim_{n\to\infty} \frac{Y_{\tau_n}}{N_{\tau_{n+1}}} &= \lim_{n\to\infty} \frac{Y_{\tau_n}}{n}\cdot\frac{n}{n+1}\cdot\frac{n+1}{\tau_{n+1}}\cdot\frac{\tau_{n+1}}{N_{\tau_{n+1}}} = \frac{L}{\lambda},\\
\lim_{n\to\infty} \frac{Y_{\tau_{n+1}}}{N_{\tau_n}} &= \lim_{n\to\infty} \frac{Y_{\tau_{n+1}}}{n+1}\cdot\frac{n+1}{n}\cdot\frac{n}{\tau_n}\cdot\frac{\tau_n}{N_{\tau_n}} = \frac{L}{\lambda}.
\end{aligned}$$

4.8 The G/M/1 queue

For this queue, the inter-arrival times have a general distribution ν_A and the service times are exponential with parameter μ; as usual, all inter-arrival times and service times are independent. If A_n denotes the inter-arrival time of the nth customer and S_n is the service time of the nth customer then

$$\mathbb{P}(A_n \in dt) = \nu_A(dt) \quad \text{and} \quad S_n \sim \mathcal{E}(\mu), \quad \text{where } \mu, \mathbb{E}S \in (0, \infty).$$

The name comes from

***G**eneral inter-arrival times/**M**emoryless service times/**1** server.*

We look for a Markov chain imbedded in the queue. Let $(X_n : n \geq 0)$ be the queue size just before the nth arrival. We have

$$X_{n+1} = (X_n - Y_{n+1} + 1) \vee 0.$$

Here Y_n is the number of *potential* departures in the nth inter-arrival time. Conditional on $A_n = t$, we have $Y_n \sim P(\mu t)$. By the memoryless property of the exponential distribution, $Y = (Y_n : n \in \mathbb{N})$ are independent, identically distributed random variables. Hence $X = (X_n : n \in \mathbb{N})$ is a Markov chain,

$$\mathbb{E}Y = \mu \mathbb{E}A, \quad \text{and} \quad G_Y(z) = \mathbb{E}\big(z^Y\big) = M_A(\mu(z-1)).$$

Proposition 4.8 *If the traffic intensity $\rho := (\mu \mathbb{E}A)^{-1} < 1$ then $X = (X_n : n \geq 0)$ has a unique invariant distribution π given by*

$$\pi_k = (1-\eta)\eta^k, \quad \textit{for } k \geq 0,$$

where η is the unique root of the equation $\eta = M_A\big(\mu(\eta - 1)\big) = G_Y(\eta)$.

Proof We have $M_A\big(\mu(z-1)\big) = G_Y(z)$, $G_Y(0) = \mathbb{P}(Y = 0) > 0$, and $G_Y'(1) = \mathbb{E}Y = \rho^{-1} > 1$. Hence there exists a unique solution of the equation $\eta = G_Y(\eta)$ in $(0, 1)$. For $k \geq 1$, we must have

$$\pi_k = \mathbb{P}(X_{n+1} = k) = \mathbb{P}(X_n - Y_{n+1} = k - 1) = \sum_{i=0}^{\infty} \pi_{k-1+i} p_i,$$

where $p_i = \mathbb{P}(Y_n = i)$. The equation for π_0 is redundant. We try the suggested solution $\pi_k = (1-\eta)\eta^k$. We require

$$(1-\eta)\eta^k = \sum_{i=0}^{\infty} (1-\eta)\eta^{k-1+i} p_i, \quad \text{for } k \geq 1.$$

This is equivalent to

$$\eta = \sum_{i=0}^{\infty} \eta^i p_i = G_Y(\eta).$$

Hence π is the invariant distribution for X, which is necessarily unique, by the irreducibility of X.

4.9 Reservoir model

Consider a reservoir or storage facility of finite capacity c and let X_n be the reservoir stock at time n. We assume that 1 unit of stock is withdrawn in each period (if possible) and a random number of units is added in each period. The random additions are modelled by a sequence of independent, identically distributed, integer-valued random variables $Y = (Y_n : n \in \mathbb{N})$, which satisfy

$$\mathbb{P}(Y_n = 0) > 0 \quad \text{and} \quad \mathbb{P}(Y_n > 1) > 0.$$

(These conditions ensure irreducibility of X.) We further assume that Y_{n+1} – the stock added at time n – cannot be used at time n and that any overflow is lost. Then

$$X_{n+1} = \left(X_n + Y_{n+1} - 1_{\{X_n > 0\}}\right) \wedge c.$$

Since X is irreducible on $\{0, 1, \dots, c\}$, it has a unique invariant distribution which we denote by π. When $c = \infty$ we obtain the M/G/1 queue. Define the sequence $\nu = (\nu_n : n \geq 0)$ by setting

$$\sum_{i=0}^{\infty} \nu_i z^i = \frac{(1-z)G_Y(z)}{G_Y(z) - z} \quad (\Longrightarrow \nu_0 = 1).$$

Proposition 4.9 *The long-run proportion of time in which the reservoir is empty is given by*

$$\frac{\nu_0}{\nu_0 + \cdots + \nu_c}.$$

Proof We have

$$\sum_{i=0}^{\infty} \nu_i z^i \left(\sum_{i=0}^{\infty} a_i z^i - z \right) = (1-z) \left(\sum_{i=0}^{\infty} a_i z^i \right) \nu_0, \quad \text{where } a_i = \mathbb{P}(Y_n = i).$$

Equating coefficients of z^k we obtain

$$\sum_{i=0}^{k} \nu_i a_{k-i} - \nu_{k-1} = \nu_0(a_k - a_{k-1}),$$

$$\nu_k = \nu_0 a_k + \sum_{j=1}^{k+1} \nu_j a_{k-j+1}, \quad \text{for } k = 0, 1, \ldots, c-1.$$

These are the equilibrium equations for X_n. All but one – the equation for state c – is redundant. Hence

$$\pi_i = \frac{\nu_i}{\nu_0 + \cdots + \nu_c}.$$

The conclusion now follows from the ergodic theorem.

Observe that, for the case in which $\mathbb{E}Y > 1$ and $c = \infty$, the chain X is transient so that $\sum_i \nu_i = \infty$. Thus there is always a sufficiently large capacity c which makes the probability of the reservoir being empty acceptably low.

4.10 Epidemics model

We now consider a model for the evolution of an infectious disease in a fixed-size population. Let N be the size of the population, I be the number of infected individuals, S be the number of susceptible individuals, and R be the number of recovered individuals. The model for the evolution of the disease is the continuous-time Markov chain $(S_t, I_t : t \geq 0)$ with non-zero transition rates

$$q_{(s,i)(s-1,i+1)} = \lambda_{s,i} \quad \text{and} \quad q_{(s,i)(s,i-1)} = \mu_i, \quad \text{for } i \geq 1.$$

Lemma 4.10 *If $\widetilde{\lambda}_{s,i} \geq \lambda_{s,i}$, for all (s, i), then $\mathbb{P}(\widetilde{R}_\infty > r) \geq \mathbb{P}(R_\infty > r)$, for all r.*

Proof We use a coupling argument. Define the jump chains $((S_n, I_n) : n \geq 0)$ and $((\widetilde{S}_n, \widetilde{I}_n : n \geq 0)$, associated to the infections. We can construct realisations of these chains so that whenever

$$\big((S_n, I_n) : n \geq 0\big) = \big((\widetilde{S}_n, \widetilde{I}_n) : n \geq 0\big),$$

we have $S_{n+1} \geq \widetilde{S}_{n+1}$ and $I_{n+1} \leq \widetilde{I}_{n+1}$. This forces $\widetilde{S}_\infty \leq S_\infty$. Since $\widetilde{I}_\infty = I_\infty = 0$, we obtain the corresponding inequality for $\widetilde{R}_\infty$ and R_∞.

Consider the standard model in which

$$\lambda_{s,i} = \lambda \left(\frac{s}{N}\right) i \quad \text{and} \quad \mu_i = \mu i.$$

Here each infected individual encounters other individuals at rate λ and recover at rate μ. We state, without proof, the following result.

Theorem 4.11 (Threshold theorem) *(i) If $\lambda \leq \mu$ then, for all $\varepsilon > 0$, we have $\lim_{N\to\infty} \mathbb{P}(R_\infty > \varepsilon N) = 0$.*

(ii) If $\lambda > \mu$ then we can choose $\varepsilon, \delta > 0$ such that $\lambda(1-\varepsilon)(1-\delta) = \mu$ and then $\mathbb{P}(R_\infty > \varepsilon N) \geq \delta$, for all N.

4.11 Closed migration process

Let I be a finite set representing the state-space of a Markov chain with irreducible Q-matrix Q. We know there exists a unique invariant distribution π satisfying $\pi Q = 0$. We may think of the holding times of this chain as service times by a single server at each node $i \in I$.

Suppose now that there are N particles in the state-space, which move according to the Q-matrix Q and queue for service at each node I. We assume that the service time at node $i \in I$ is exponentially distributed with parameter $q_i := \sum_{j\neq i} q_{ij}$; and the particles move from i to j with probability q_{ij}/q_i. We regard this as a new process $X = (X_t : t \geq 0)$ with state-space $\widetilde{I} = \mathbb{N}^I$, where $X_t = (n_i : i \in I)$, if at time t there are n_i particles at state i. This new process is also a Markov chain. To describe its Q-matrix $\widetilde{Q}$ we introduce the functions $\delta_i : \widetilde{I} \longmapsto \widetilde{I}$ given by

$$(\delta_i n)_j = n_j + \delta_{ij}.$$

Thus δ_i adds a particle at i. Then, for $i \neq j$, the non-zero transition rates are given by

$$\widetilde{q}(\delta_i n, \delta_j n) = q_{ij}, \quad \text{where } n \in \widetilde{I}, i, j \in I.$$

We can write the invariant measure equation $\pi Q = 0$ in the form

$$\pi_i \sum_{j\neq i} q_{ij} = \sum_{j\neq i} \pi_j q_{ji}.$$

For $n = (n_i : i \in I)$, we set

$$\widetilde{\pi}(n) = \prod_{i\in I} \pi_i^{n_i}.$$

Then

$$\begin{aligned}
\widetilde{\pi}(\delta_i n)\sum_{j\neq i}\widetilde{q}(\delta_i n,\delta_j n) &= \prod_{k\in I}\pi_k^{n_k}\left(\pi_i\sum_{j\neq i}q_{ij}\right)\\
&= \prod_{k\in I}\pi_k^{n_k}\left(\sum_{j\neq i}\pi_j q_{ji}\right)\\
&= \sum_{j\neq i}\widetilde{\pi}(\delta_j n)\widetilde{q}(\delta_j n,\delta_i n).
\end{aligned}$$

Given $m\in\widetilde{I}$, we can put $m=\delta_i n$ in the last identity whenever $m_i\geq 1$. On summing the resulting equations we obtain

$$\widetilde{\pi}(m)\sum_{n\neq m}\widetilde{q}(m,n)=\sum_{n\neq m}\widetilde{\pi}(n)\widetilde{q}(n,m),$$

whence $\widetilde{\pi}$ is an invariant measure for $\widetilde{Q}$. The total number of particles is conserved so that $\widetilde{Q}$ has communicating classes

$$C_N=\left\{n\in\widetilde{I}:\sum_{i\in I}n_i=N\right\},$$

and the unique invariant distribution for the N-particle system is given by normalising $\widetilde{\pi}$ restricted to C_N.

4.12 Open migration process

We consider next a modification of the last example in which new customers arrive at each node $i\in I$ at rate λ_i and leave the state space at rate μ_i when they finish their service at node i. We suppose also that customers receiving service at node i leave the network at rate μ_i. Thus customers enter the network, move from queue to queue according to a Markov chain and eventually leave. This model includes the closed system of the last example. Let $X_t=(X_t^i : i\in I)$, where X_t^i denotes the number of customers at node i at time t. Then $X=(X_t:t\geq 0)$ is a Markov chain in $\widetilde{I}=\mathbb{N}^I$ having non-zero transitition rates given by

$$\widetilde{q}(n,\delta_i n)=\lambda_i,\quad \widetilde{q}(\delta_i n,\delta_j n),\quad \text{and}\quad \widetilde{q}(\delta_j n,n)=\mu_j,$$

for $n\in\widetilde{I}$ and distinct states $i,j\in I$. We shall assume that $\lambda_i>0$, for some i, and $\mu_j>0$, for some j, so that $\widetilde{Q}$ is irreducible on $\widetilde{I}$.

Consider the system of equations for an invariant measure:

$$\pi_i \sum_{j \neq i} q_{ij} = \sum_{j \neq i} \pi_j q_{ji}$$

We replace these by the equivalent set

$$\pi_i \left(\mu_i + \sum_{j \neq i} q_{ij} \right) = \lambda_i + \sum_{j \neq i} \pi_j q_{ji}.$$

This system has a unique solution, with $\pi_i > 0$, for all i – these are, in fact, the detailed balance equations for the extended Q-matrix $\overline{Q}$ on $I \cup \{\partial\}$ with off-diagonal entries given by

$$\overline{q}_{\partial j} = \lambda_j, \quad \overline{q}_{ij} = q_{ij}, \quad \text{and} \quad \overline{q}_{i\partial} = \mu_i.$$

Summing the system over $i \in I$, we obtain

$$\sum_{i \in I} \pi_i \mu_i = \sum_{i \in I} \lambda_i.$$

As in the closed migration case, for $n = (n_i : i \in I)$, we set

$$\widetilde{\pi}(n) = \prod_{i \in I} \pi_i^{n_i}.$$

Transitions from $m \in \widetilde{I}$ may be divided into those in which a new particle is added and, for each $i \in I$ with $m_i \geq 1$, those in which a particle is moved from i to somewhere else. We have, for the first sort of transition,

$$\begin{aligned}
\widetilde{\pi}(m) \sum_{j \in I} \widetilde{q}(m, \delta_j m) &= \widetilde{\pi}(m) \sum_{j \in I} \lambda_j \\
&= \widetilde{\pi}(m) \sum_{j \in I} \pi_j \mu_j \quad = \quad \sum_{j \in I} \widetilde{\pi}(\delta_j m) \widetilde{q}(\delta_j m, m).
\end{aligned}$$

For the second sort, we find that

$$\begin{aligned}
\widetilde{\pi}(\delta_i n) \left(\widetilde{q}(\delta_i n, n) + \sum_{j \neq i} \widetilde{q}(\delta_i n, \delta_j n) \right) &= \prod_{k \in I} \pi_k^{n_k} \left(\pi_i \mu_i + \pi_i \sum_{j \neq i} q_{ij} \right) \\
&= \prod_{k \in I} \pi_k^{n_k} \left(\lambda_i + \sum_{j \neq i} \pi_j q_{ji} \right) \\
&= \widetilde{\pi}(n) \widetilde{q}(n, \delta_i n) + \sum_{j \neq i} \widetilde{\pi}(\delta_j n) \widetilde{q}(\delta_j n, \delta_i n).
\end{aligned}$$

On summing the resulting equations we obtain

$$\widetilde{\pi}(m) \sum_{n \neq m} \widetilde{q}(m,n) = \sum_{n \neq m} \widetilde{\pi}(n)\widetilde{q}(n,m).$$

It follows that $\widetilde{\pi}$ is an invariant measure for Q. If $\pi_i < 1$, for all i, then $\widetilde{\pi}$ has finite total mass $\prod_{i \in I}(1-\pi_i)$; otherwise the total mass is infinite. Consequently $\widetilde{Q}$ is positive recurrent if and only if $\pi_i < 1$, for all i, and in that case, in equilibrium, the individual queue lengths $(X_t^i : i \in I)$ are *independent* geometric random variables with

$$\mathbb{P}(X_t^i = n_i) = (1-\pi_i)\pi_i^{n_i}.$$