DPK **Easter Term 2005**

OPTIMIZATION (D2)

Additional Example Sheet

These questions are for further practice and revision.

1. Use the Lagrangian Sufficiency Theorem to:

$$\begin{aligned} &\text{minimize} && \frac{v_1}{x_1}+\frac{v_2}{x_2} \\ &\text{subject to} && c_1x_1+c_2x_2=c, \\ & && x_1\geqslant a_1,\ x_2\geqslant a_2, \end{aligned}$$

where c_1, c_2, a_1, a_2, v_1, v_2 and c are positive constants, with $c \geqslant a_1c_1 + a_2c_2$.

2. A gambler at a horse race has an amount b to bet. The gambler assesses p_i, the probability that horse i will win, and knows that s_i has been bet on horse i by others, for $i = 1, \ldots, n$. The total amount bet on the race is shared out in proportion to the bets on the winning horse, and so the gambler's optimal strategy is to choose $(x_1, \ldots, x_n)$ to

$$\text{maximize} \sum_{i=1}^{n} \frac{p_i x_i}{s_i + x_i} \quad \text{subject to} \quad \sum_{i=1}^{n} x_i = b, \quad x_i \geqslant 0,\ 1 \leqslant i \leqslant n,$$

where x_i is the amount the gambler bets on horse i.

Find the form of the gambler's optimal strategy. Deduce that if b is small enough, the optimal strategy is to bet only on the horses for which the ratio p_i/s_i is maximal.

3. Consider the problem:

$$\begin{aligned} &\text{maximize} && 3x_1+6x_2-\ x_3 \\ &\text{subject to} && -x_1+4x_2-\ x_3\leqslant 2 \\ & && 2x_1+\ x_2+\ x_3\leqslant 5 \\ & && -x_1+\ x_2\ +2x_3\leqslant 1 \\ & && x_1,\ x_2,\ x_3\geqslant 0. \end{aligned}$$

Write down the dual problem and find its solution. Now replace the second constraint by $2x_1 + x_2 + x_3 \leqslant 5 + t$. Find the new maximum for small values of t. For what range of values of t is this solution valid?

4. Use the simplex algorithm to solve:

$$\begin{aligned} &\text{minimize} && 5x_1-3x_2 \qquad\ \ = f \\ &\text{subject to} && 2x_1-\ x_2+3x_3\leqslant 4 \\ & && x_1+\ x_2+2x_3\leqslant 5 \\ & && 2x_1-\ x_2\ +x_3\leqslant 1 \\ & && x_1,\ x_2,\ x_3\geqslant 0. \end{aligned}$$

Write down the dual problem, and solve it by inspection of the final tableau for the primal. If the constraints on the right-hand side of the above problem are changed to $4+\epsilon_1$, $5+\epsilon_2$, $1 + \epsilon_3$ respectively, for small ϵ_1, ϵ_2, ϵ_3, by how much does the optimal value of f change?

5. A linear programming problem in the form maximize $\boldsymbol{c}^\top\boldsymbol{x}$ subject to $A\boldsymbol{x} \leqslant \boldsymbol{b}$, $\boldsymbol{x} \geqslant 0$, is solved using the simplex algorithm, starting from the initial b.f.s. given by setting the slack variables $\boldsymbol{z} = \boldsymbol{b}$ and $\boldsymbol{x} = 0$, to give the optimal tableau

	$\ast$ x_1	$\ast$ x_2	$\ast$ x_3	z_1	z_2	z_3	
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{5}{2}$
x_3	0	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{5}{2}$
x_1	1	0	0	$-\frac{3}{2}$	$\frac{2}{3}$	$\frac{7}{6}$	$\frac{5}{2}$
Payoff	0	0	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	$-\frac{85}{2}$

Determine the matrix A and the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

6. A classical example of a degenerate linear programming problem (due to Beale) which 'cycles' in the simplex algorithm is

$$\begin{aligned} &\text{Minimize} && -\tfrac{3}{4}x_1+150x_2-\tfrac{1}{50}x_3+6x_4 \\ &\text{subject to} && \tfrac{1}{4}x_1-\ 60x_2-\tfrac{1}{25}x_3+9x_4+x_5 \qquad\qquad = 0 \\ & && \tfrac{1}{2}x_1-\ 90x_2-\tfrac{1}{50}x_3+3x_4 \qquad +x_6 \qquad = 0 \\ & && \qquad\qquad\qquad x_3 \qquad\qquad\qquad\qquad +x_7=1, \quad x_i\geqslant 0 \end{aligned}$$

Show that the algorithm can cycle through the bases:

$$\begin{array}{ccccc} (x_5,x_6,x_7) & \rightarrow & (x_1,x_6,x_7) & \rightarrow & (x_1,x_2,x_7) \\ \uparrow & & & & \downarrow \\ (x_4,x_5,x_7) & \leftarrow & (x_3,x_4,x_7) & \leftarrow & (x_2,x_3,x_7). \end{array}$$

Find the optimal solution.

7. Consider the problem

$$\text{minimize} \sum_{i=1}^{n} |x_i| \quad \text{subject to } A\boldsymbol{x} \leqslant \boldsymbol{b},$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$, A is an $m \times n$ matrix and $\boldsymbol{b} \in \mathbb{R}^m$. Show how to convert the problem so that the optimal solution may be found by solving a standard linear programming problem. What happens if you replace minimize by maximize?

8. In another version of the game Undercut, each player selects a number from 1, 2, 3, 4. The players reveal their numbers and the player with the smaller number wins a number of pounds equal to the absolute value of the difference in the numbers, unless the numbers are either adjacent, when the player with the larger number wins £4, or equal, when the game is tied with payoff zero. Find **all** the optimal strategies for the game.

9. Formulate the problem of finding a maximum flow through a network as a linear programming problem. How many variables and constraints may be needed for a problem with n nodes? Show that the dual problem has a solution in which the variables take only two values and explain the significance of this result.

10. (**König-Egerváry Theorem**) Consider an $m \times n$ matrix A in which each entry is either 0 or 1. Say that a set of lines (rows or columns of the matrix) *covers* the matrix if each 1 belongs to some line of the set. Say that a set of 1's is *independent* if no pair of 1's of the set lies in the same line. Use the max-flow min-cut theorem to show that the maximal number of independent 1's equals the minimum number of lines that cover the matrix.

11. (**Menger's Theorem**) Derive the vertex form of Menger's Theorem which states that if A and B are nodes of an undirected network then the maximum number of node-disjoint paths from A to B which can be chosen simultaneously is equal to the minimum number of nodes whose removal disconnects A and B. [Two paths from A to B are node disjoint if the only two nodes that they have in common are A and B. The removal of a set of nodes S disconnects A and B if any path from A to B passes through at least one node of S.]

12. Suppose that N is a network with vertices $0, 1, 2, \ldots, 2n, 2n+1$, where 0 is the source and $2n+1$ is the sink, such that

(a) for each $i = 1, \ldots, n$, there is an edge $(0, i)$ of capacity 1;

(b) for each $j = n+1, \ldots, 2n$, there is an edge $(j, 2n+1)$ of capacity 1;

(c) the only other edges have capacity n and are of the form (i, j) with $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, 2n\}$,

and for each subset $I \subset \{1, \ldots, n\}$ the number of distinct vertices j such that an edge (i, j) exists for some $i \in I$ is not less that $|I|$, the number of elements in I. Prove that any maximal flow in N has value n.

Hence show (the **Hall 'Marriage Theorem'**) that if we have a set of n boys and a set of n girls, such that every subset B of the boys between them know at least $|B|$ of the girls, then they can pair off, each boy with a girl he knows.

13. A manufacturer has to supply $\{5, 7, 9, 6\}$ units of a good in each of the next four months. He can produce up to 8 units each month on ordinary time at costs $\{1, 3, 4, 2\}$ per unit, and up to 3 extra each month on overtime at costs $\{2, 5, 7, 4\}$ per unit (where costs are given for each of the next four months). Storage costs are 1 per unit per month. He desires to schedule production to minimize costs over the four-month period. Formulate his problem as a transportation problem (with 8 sources and 5 destinations) and solve it.

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