PROBABILITY

Example Sheet 4

1. A shot is fired at a circular target. The vertical and horizontal coordinates of the point of impact (taking the centre of the target as origin) are independent random variables each distributed normally $N(0,1)$.

(i) Show that the distance from the centre to the point of impact has p.d.f. $re^{-r^2/2}$ for $r \geqslant 0$.

(ii) Show that the mean of this distance is $\sqrt{\pi/2}$, that the median is $\sqrt{\log 4}$, and that the mode is 1.

2. (i) The random variable X has a **log-normal distribution** if $Y = \log X$ has a normal distribution. When $Y \sim N(\mu, \sigma^2)$, calculate the mean and variance of X. [The log-normal distribution is often used to model stock prices–why?]

(ii) Random variables X_1 and X_2 have a bivariate log-normal distribution if $Y_1 = \log X_1$ and $Y_2 = \log X_2$ have a bivariate normal distribution. Show that when X_1 and X_2 have a bivariate log-normal distribution then they are independent if and only if $\mathbb{C}\mathrm{ov}\ (X_1, X_2) = 0$.

3. A radioactive source emits particles in a random direction (with all directions being equally likely). It is held at a distance d from a vertical infinite plane photographic plate.

(i) Show that, given the particle hits the plate, the horizontal coordinate of its point of impact (with the point nearest the source as origin) has p.d.f. $d/\pi(d^2 + x^2)$. [This distribution is known as the **Cauchy distribution**].

(ii) Can you compute the mean of this distribution?

4. Suppose that $X_1, \ldots, X_{2n+1}$ are i.i.d. random variables forming a random sample from the $U(0,1)$ distribution. Suppose that the values are arranged in increasing order as

$$X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(2n+1)};$$

these are known as the **order statistics** of the sample. Calculate expressions for the distribution function and for the probability density function of the random variable $X_{(n+1)}$ (the **sample median**).

5. Suppose that n items are being tested simultaneously and that the items have independent lifetimes, each having the exponential distribution with parameter $\lambda > 0$. Determine the mean and variance of the time until r items have failed.

6. Let X be a real-valued random variable. Suppose that the moment-generating function $m(\theta) = E(e^{\theta X})$ is finite for some $\theta > 0$. Show that for all $n \geqslant 0$, $\lim_{x\to\infty} x^n P(X \geqslant x) = 0$.

7. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then the geometric mean lies between the harmonic mean and the arithmetic mean:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1} \leqslant \left(\prod_{i=1}^{n} x_i\right)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^{n} x_i.$$

The second inequality is the arithmetic mean-geometric mean inequality; establish the first inequality.

8. Find approximately the probability that the number of 6's in 12000 rolls of a fair die is between 1900 and 2150.

9. Find the number c such that the probability is about $\frac{1}{2}$ that in 1000 tosses of a fair coin the number of heads lie between 490 and c.

10. Derive the distribution of the sum of n independent, identically distributed variables each having the Poisson distribution with parameter 1. Use the Central Limit Theorem to prove that

$$e^{-n}\left(1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) \to \frac{1}{2} \quad \text{as} \quad n \to \infty.$$

11. For Buffon's needle, calculate the probability that the needle intersects a line in the case $r > d$, where r is the length of the needle and d is the distance between the lines.

12. Bertrand's Paradox II Suppose that a chord is drawn at random in a given circle. Determine the probability that the length of the chord will be greater than the length of the side of the equilateral triangle inscribed in that circle.

Additional exercises:

13. Laplace proposed a refinement of Buffon's method of estimating π by supposing that the surface on which the needle of length r is thrown has two arrays of parallel lines at right angles so that they form a rectangular grid with sides of length c and d, where $r < \min(c, d)$. Let Z denote the number of lines that are intersected when the needle is thrown, so that Z takes the values 0, 1 or 2. Show that

$$\mathbb{P}(Z = 2) = \frac{r^2}{\pi c d}\,.$$

Hence derive the distribution of Z and calculate $\mathbb{E}\,Z$ and $\mathbb{V}\mathrm{ar}\,(Z)$.

Suppose that $r = c = d$. Buffon throws his needle 1000 times and estimates π. How many times must Laplace throw his needle to achieve at least the same accuracy for his estimate of π?

14. Suppose that X, Y and Z are independent random variables each uniformly distributed on $(0, 1)$. Show that $(XY)^Z$ is also uniformly distributed on $(0, 1)$.

15. Suppose that $X_1, X_2, \ldots$ form a sequence of independent random variables, each uniformly distributed on $(0, 1)$. Let

$$N = \min\left\{n : X_1 + \cdots + X_n \geqslant 1\right\}.$$

Calculate $\mathbb{P}\left(N \geqslant k\right)$ for each $k \geqslant 1$ and hence show that $\mathbb{E}\,N = e$.

Comments on this example sheet may be sent to: d.p.kennedy@statslab.cam.ac.uk

9 January 2007