DPK **Lent Term 2007**

PROBABILITY

Example Sheet 3

1. For a random variable X with mean μ and variance σ^2 define the function $V(x) = \mathbb{E}\,(X-x)^2$. Express the random variable $V(X)$ in terms of μ, σ^2 and X, and hence show that $\sigma^2 = \frac{1}{2}\mathbb{E}\,(V(X))$.

2. Suppose that a random variable Y has probability generating function

$$p(z) = \left[\frac{z(1-z^r)}{r(1-z)}\right]^n, \quad 0 \leqslant z < 1,$$

where $r \geqslant 1$ and $n \geqslant 1$ are fixed integers. Determine the mean and variance of Y. Show that Y has the same distribution as the sum $X_1 + \cdots + X_n$ of i.i.d.random variables $X_1, \ldots, X_n$ and use this to give an alternative calculation of the mean and variance of Y. What is the variance of the sum of the numbers on 12 throws of a standard die?

3. Let N be a non-negative integer-valued random variable with mean μ_1 and variance σ_1^2, and let $X_1, X_2, \ldots$ be random variables, each with mean μ_2 and variance σ_2^2; furthermore, assume that $N, X_1, X_2, \ldots$ are independent. Without using generating functions, calculate the mean and variance of the random variable $S_N = X_1 + \cdots + X_N$ (when $N = 0$ interpret S_N as 0).

4. At time 0, a blood culture starts with one red cell. At the end of one minute, the red cell dies and is replaced by one of the following combinations with probabilities as indicated:

$$2 \text{ red cells } \tfrac{1}{4}; \quad 1 \text{ red, } 1 \text{ white } \tfrac{2}{3}; \quad 2 \text{ white } \tfrac{1}{12}.$$

Each red cell lives for one minute and gives birth to offspring in the same way as the parent cell. Each white cell lives for one minute and dies without reproducing. Assume the individual cells behave independently.

(a) At time $n + \frac{1}{2}$ minutes after the culture began, what is the probability that no white cells have yet appeared?

(b) What is the probability that the entire culture dies out eventually?

5. (a) A mature individual produces offspring according to the probability generating function $F(z)$. Suppose we start with a population of k immature individuals, each of which grows to maturity with probability p and then reproduces, independently of the other individuals. Find the probability generating function of the number of (immature) individuals at the next generation.

(b) Find the probability generating function of the number of mature individuals at the next generation, given that there are k mature individuals in the parent generation.

(c) Show that the distributions in (a) and (b) have the same mean, but not necessarily the same variance.

6. A slot machine operates so that at the first turn the probability for the player to win is $\frac{1}{2}$. Thereafter the probability for the player to win is $\frac{1}{2}$ if he lost at the last turn, but is p $(< \frac{1}{2})$ if he won at the last turn. If u_n is the probability that the player wins at the n^{th} turn, show that, provided $n > 1$

$$u_n + \left(\tfrac{1}{2} - p\right) u_{n-1} = \tfrac{1}{2}.$$

Observe that this equation also holds for $n = 1$, if u_0 is suitably defined. Solve the equation, showing that

$$u_n = \frac{1 + (-1)^{n-1} \left(\frac{1}{2} - p\right)^n}{3 - 2p} .$$

7. A fair coin is tossed n times. Let u_n be the probability that the sequence of tosses never has 'head' followed by 'head'. Show that

$$u_n = \tfrac{1}{2}u_{n-1} + \tfrac{1}{4}u_{n-2}.$$

Find u_n, using the condition that $u_0 = u_1 = 1$. Check that your value of u_2 is correct.

8. Alice and Bob agree to meet at the Copper Kettle after their Saturday lectures. They arrive at times that are independent and uniformly distributed between 12.00 and 1.00pm. Each is prepared to wait 10 minutes before leaving. Find the probability that they meet.

9. A stick is broken in two places, independently and uniformly distributed along its length. What is the probability that the three pieces will make a triangle?

10. The radius of a circle has the exponential distribution with parameter λ. Determine the probability density function of the area of the circle.

11. Suppose that X and Y are independent random variables, with symmetric distributions, with probability density functions (p.d.f.s) f and g respectively. Show that the p.d.f. of $Z = X/Y$ is

$$h(z) = 2\int_0^\infty y f(yz) g(y) dy.$$

Now suppose that $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \tau^2)$, show that $h(z) = d/(\pi(d^2 + z^2))$ where $d = \sigma/\tau$.

12. Suppose that X and Y are independent random variables, each with the normal distribution $N(0, 1)$. Show that, for any fixed θ, the random variables

$$X \cos\theta + Y \sin\theta \qquad - X \sin\theta + Y \cos\theta$$

are independent and find their distributions.

Additional exercises:

13. Suppose that X and Y are independent random variables with $X \sim \Gamma(m, \lambda)$ and $Y \sim \Gamma(n, \lambda)$, and let $U = X + Y$ and $V = X/(X + Y)$. Find the joint probability density function of U and V and their marginal probability density functions. Are U and V independent?

14. Suppose that X and Y are independent, identically distributed random variables each with the $U(0, 1]$ distribution and let $U = X + Y$ and $V = X/Y$. Find the joint probability density function of U and V and their marginal probability density functions. Are U and V independent?

15. Let $F(z) = 1 - p(1 - z)^\beta$, where p and β are constants and $0 < p < 1$, $0 < \beta < 1$. Prove that $F(z)$, $0 \leqslant z \leqslant 1$, is a probability generating function and its iterates are

$$F_n(z) = 1 - p^{1+\beta+\cdots+\beta^{n-1}}(1 - z)^{\beta^n} \quad \text{for} \quad n = 1, 2, \ldots,$$

and find the mean m of the associated distribution and the extinction probability $q = \lim_{n\to\infty} F_n(0)$, for a branching process with offspring distribution determined by F.

16. Let $(X_n)_{n\geqslant 0}$ be a branching process such that $X_0 = 1$, $EX_1 = m$. Suppose that $Y_n = X_0 + X_1 + \cdots + X_n$, represents the total number of individuals in generations $0, 1, \ldots, n$, and for $0 \leqslant z < 1$

$$G_n(z) \equiv Ez^{Y_n}.$$

Prove that

$$G_{n+1}(z) = zF(G_n(z)),$$

where $F(z) \equiv Ez^{X_1}$. Deduce that, if $Y = \sum_{n \geqslant 0} X_n$, then $G(z) \equiv Ez^Y$ satisfies

$$G(z) = zF(G(z)), \qquad 0 \leqslant z < 1,$$

where $z^\infty \equiv 0$. If $m < 1$, prove that $EY = (1-m)^{-1}$.

Comments on this example sheet may be sent to: d.p.kennedy@statslab.cam.ac.uk

9 January 2007