Local Time, Coupling and the Passport Option [†]

Vicky Henderson and David Hobson

Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY. UK.

Emails: vh@maths.bath.ac.uk dgh@maths.bath.ac.uk

Abstract

A passport option, as introduced and marketed by Bankers Trust, is a call option on the balance of a trading account. The strategy that this account follows is chosen by the option holder, subject to position limits.

We derive a simplified form for the price of the passport option using local time. A key insight is that Tanaka's formula and the Skorokhod Lemma allow us to prove a direct relationship between the prices of passport and lookback options. Explicit calculations are provided in the case where the underlying is an exponential Brownian motion.

A further issue in the analysis of passport options is the identification of the optimal strategy. The second contribution of this article is to extend existing results on the form of the optimal strategy from the exponential Brownian motion model to a wide class of alternative price processes. We achieve this using coupling arguments.

Keywords and Phrases: Passport Option, Local Time, Coupling, Skorokhod Lemma, Option Pricing.

JEL Classification Numbers: G12, G13.

AMS (1991) Subject Classification: 60G17, 60J55, 60J60, 90A09.

[†]Manuscript received...; final version received...

1 Introduction

This article is concerned with the pricing and hedging of a new type of contingent claim, called a passport option. It is fundamentally different from most existing claims in that the underlying asset is a trading account. The buyer of the passport option pays a premium upfront and trades according to a strategy of their choice. This strategy is unrestricted except that the number of units of the risky asset held, long or short, is bounded. At the expiry date T, either the gains from this strategy are paid to the holder, or if the account lost money, the loss is borne by the seller giving the buyer a zero net position.

Such a structure could be used by active fund managers to offer products with principal protection. Whilst limiting the fund participants downside risk, the fund manager is able to engage in potentially high risk strategies with the knowledge that they will be protected in the event of loss. However, this protection comes at a cost in the form of an initial premium which the manager must be willing to bear. We calculate the cost charged by the issuer who assumes the holder chooses their strategy to maximise value.

Under the assumption that the asset price follows exponential Brownian motion, Hyer et al [10] obtain a price for the passport option by analysing the pricing partial differential equation. They construct the Hamilton-Jacobi-Bellman equation and solve the pde via Green's functions giving a complicated form for the price. They also define a non-symmetric version of the problem for which the pde is solved numerically. Andersen et al [2] price the symmetric option using a change of measure and pde methods. By deflating the trading account by the asset price they reduce the pde to a one-dimensional equation which they then solve. Shreve and Večeř [15] analyse the passport option using probabilistic techniques, again involving a change of measure. They have a direct derivation of the optimal strategy using a Mean

Comparison Theorem of Hajek [7]. The work in these papers ([2],[10],[15]) applies only in the case of an exponential Brownian motion model for the asset price.

Our aim is to use probabilistic techniques to give a concise way of pricing and hedging the passport option in the symmetric case. In doing this, we obtain a simpler form for the price than that in Hyer et al [10] by deriving an equivalent formulation of the problem which is easier to solve. This involves relating the price of a passport option to that of a lookback option. Moreover our methods extend beyond a constant volatility model. The derivation uses the concepts of local time and the Skorokhod Lemma. Local time has been used in mathematical finance by Carr and Jarrow [3] when analysing a stop-loss hedging strategy. Related ideas are used in Chesney et al [4] to price Parisian options.

As well as calculating the option price we also find the holder's optimal strategy and it is, perhaps, somewhat surprising. The holder should invest up to the allowed position limit in the underlying, *buying when the value of the trading account is negative, and selling otherwise*. For the exponential Brownian motion stock price model this can be deduced from an explicit formula for the option value. We use the tool of coupling to show that for a wide class of diffusion models, the form of the optimal strategy is unchanged. The methods of coupling of stochastic processes are dealt with in Lindvall [12] and have been used by Hobson [9] to prove results on superreplication with misspecified price processes in stochastic volatility models.

The remainder of the paper is organised as follows. Section 2 provides the model setup and defines the passport option. An equivalent expression for the price of the passport option is deduced using local time and the Skorokhod Lemma in Section 3. This allows us to write down the option price when the asset price follows exponential Brownian motion. Section 4 contains a derivation of the form of the optimal strategy and a proof of optimality for the case of exponential Brownian motion. In Section 5, two coupling results are proved and used to show the same strategy is optimal in the more general model.

2 The Passport Option and Model

We consider a continuous time model for the economy with a finite horizon T. There is a risky asset with price P_t and a bond paying a fixed constant rate of interest r. Markets are frictionless with no transactions costs or taxes and assets are infinitely divisible.

We assume that the asset price process is a diffusion and that, following Harrison and Pliska [8], the model is complete with a unique martingale measure $\mathbb{P}$. This is the pricing measure under which the discounted asset price is a martingale. As a corollary the price of any option can be written as the expectation of the discounted payoff under $\mathbb{P}$.

Our model is as follows: under $\mathbb{P}$, the price solves

$$\frac{dP_t}{P_t} = rdt + \sigma_P(P_t, t)dW_t \tag{1}$$

so that P is a diffusion process. We assume that σ_P has sufficient continuity properties to ensure that the solution to this SDE is unique in law (for example a Lipschitz condition on $x\sigma_P(x,t)$; see Rogers and Williams [14, Remark V16.4]).

Consider the discounted price process $S_t = e^{-rt}P_t$ which solves the SDE

$$dS_t = S_t\sigma(S_t, t)dW_t \tag{2}$$

where $\sigma(S_t, t) = \sigma_P(S_te^{rt}, t)$. By the martingale measure assumption, S_t is a martingale; we assume further that S_t is an element of the Hardy space H^1 of martingales (ie. $\mathbb{E}[(\int_0^T {S_t}^2\sigma(S_t,t)^2dt)^{\frac{1}{2}}] < \infty$). A simple sufficient, but far from necessary condition for this is that σ is bounded. The subsequent analysis can be extended to show that if S_t is not an element of H^1 then the price of the passport option is infinite.

Investing in the risky asset via the predictable strategy $\{q_t; 0 \leq t \leq T\}$ will generate gains or losses with value $\int_0^t q_u dP_u$ by time t. In our specification of the model the investor also receives interest rate r on any remaining funds, so overall the gains from trade process $\psi_t(q)$ is given by

$$d\psi_t(q) = r(\psi_t(q) - q_t P_t)dt + q_t dP_t$$

which simplifies to

$$d\psi_t(q) = r\psi_t(q)dt + q_t \sigma_P(P_t, t) P_t dW_t$$

on using (1).

Defining the discounted value of the trading account to be $G_t(q) = e^{-rt}\psi_t(q)$ we note that $G(q)$ is a local martingale under the measure $\mathbb{P}$ (in fact a martingale for each q, see the remarks following (7) below) and

(3) $$dG_t(q) = q_t \sigma(S_t, t) S_t dW_t = q_t dS_t.$$

A passport option with expiry T is a call option on the trading account $\psi(q)$ and is defined by the payoff

(4) $$\psi_T^+(q) \equiv \max(\psi_T(q), 0)$$

where the strategy q_t, the number of shares held, is restricted to lie in $[-K, K]$.

Assuming that the holder of the option will follow the optimal strategy, the price of the passport option at time t is given by the (discounted) expectation under the measure $\mathbb{P}$ of the payoff:

(5) $$\max_{|q_t| \leq K} e^{-r(T-t)} \mathbb{E}_t \psi_T^+(q).$$

If a suboptimal strategy is followed by the investor, the seller will make a surplus if the price given in (5) is charged.

A simple scaling argument shows that we may assume $K = 1$; define $\theta = (q/K)$, then an equivalent restriction is $\theta \leq 1$ and both the trading account

$$G_t(q) = \int_0^t q_u dS_u = K \int_0^t \theta_u dS_u = KG_t(\theta)$$

and the option payoff scale linearly in K.

3 Valuing the Passport Option using Local Time

In this section we obtain an expression for the price of the passport option under the assumptions outlined previously. The key to the result is the use of the Skorokhod lemma to give an analogous representation of the price as the expected maximum of a martingale.

Consider the case where at time 0, the initial gains from trade is assigned a non-zero value. This is, of course, equivalent to considering the intermediate problem when the purchaser of a passport option has followed some strategy over part of the lifetime of the contract. It also covers the passport option with payoff $(\psi_T(q) - k)^+$ for any $k \in \mathbb{R}$.

By Tanaka's formula for continuous semimartingales (see Revuz and Yor [13, Theorem VI.1.2])

$$G_T^+(q) = G_0^+(q) + \int_0^T I_{(G(q)>0)} dG(q) + \tfrac{1}{2} L_T^{G(q)}(0) \tag{6}$$

where $L_T^{G(q)}(0)$ is the local time of process $G(q)$ at level zero between time 0 and T.

This representation proves useful since we may write the time 0 price of the passport option, given strategy q and gains from trade to date $G_0(q)$, as

$$\mathbb{E}(e^{-rT}\psi_T^+(q)) = \mathbb{E}G_T^+(q) = \{\tfrac{1}{2}\mathbb{E}L_T^{G(q)}(0) + G_0^+(q)\}. \tag{7}$$

We use the fact that since $S_t \in H^1$ and $|I_{(G(q)>0)}q_u| \leq 1$, Exercise IV.4.22 in Revuz and Yor [13] implies that $\int_0^T I_{(G(q)>0)}q_u dS_u$ is a true martingale for all strategies q.

Equation (7) suggests that the price of the passport option is higher the greater the local time of $G(q)$ at zero - implying that intuitively the best strategy would be one that tries to force the trading account to cross zero most often, and maximises the instantaneous volatility at these moments.

In order to characterise the solution to our problem we need the following lemma (see Karatzas and Shreve [11, Lemma 3.6.14]).

Lemma 3.1 (Skorokhod's Lemma)
Let y be a real-valued continuous function on $[0,\infty), y_0 = 0$ and $z \geq 0$ a given number. Then there exists a unique pair of functions (x, l) on $[0,\infty)$ such that:

$$\begin{aligned}
(i) &\quad x = z + y + l,\\
(ii) &\quad x \geq 0,\\
(iii) &\quad l \text{ is increasing, continuous, } l_0 = 0, \text{flat off } \{x = 0\}.
\end{aligned}$$

Moreover, l is given by:

$$l_u = \max\left[\sup_{s\leq u}[-(z+y_s)], 0\right] = \left\{\sup_{s\leq u}[-(z+y_s)]\right\}^+.$$

$\square$

Let $M_u(q) = \int_0^u -q_r \mathrm{sgn}(G_r(q))dS_r$ and define $M_r^*(q) = \sup_{0\le u\le r} M_u(q)$. Skorokhod's Lemma allows us to rewrite the local time of $G(q)$ at zero in terms of the martingale M.

Lemma 3.2 $L_T^{G(q)}(0) = (M_T^*(q) - |G_0(q)|)^+$.

Proof: Tanaka's formula gives

$$|G_u(q)| = |G_0(q)| + \int_0^u \mathrm{sgn}(G_r(q))dG_r(q) + L_u^{G(q)}(0), \tag{8}$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1; & x > 0 \\ -1; & x \le 0. \end{cases}$$

From the definition of M we find

$$\int_0^u \mathrm{sgn}(G_r(q))dG_r(q) = \int_0^u q_r \mathrm{sgn}(G_r(q))dS_r(q) = -M_u(q).$$

Then from (8), $|G_u(q)| = |G_0(q)| - M_u(q) + L_u^{G(q)}(0)$ and the pair $(|G_u(q)|, L_u^{G(q)}(0))$ is a solution to the Skorokhod problem for $-M$ giving

$$L_T^{G(q)}(0) = \max[\sup_{0\le u\le T}\{M_u(q) - |G_0(q)|\}; 0] = (M_T^*(q) - |G_0(q)|)^+.$$

□

Linking Lemma 3.2 with (7) we obtain

$$\mathbb{E}G_T^+(q) = \tfrac{1}{2}\mathbb{E}(M_T^*(q) - |G_0(q)|)^+ + G_0^+(q).$$

Finally we notice that if we set $v = -q\mathrm{sgn}(G(q))$, the constraint for v is the same as that for q and we can rewrite the problem in its simplest probabilistic form as:

Proposition 3.3 (A Transformation of the pricing problem) *The passport option problem is to:*

(3.3a) find the strategy v, with $|v| \leq 1$, such that $\mathbb{E}(M_T^*(v)-g)^+$ is maximised, where $M_r^*(v) = \sup_{0\leq s\leq r} M_s(v)$ and $M_s(v) = \int_0^s v_u dS_u$,

(3.3b) find the associated value for the passport option as given by the formula $[\frac{1}{2}\mathbb{E}(M_T^*(\tilde{v}) - |G_0(q)|)^+ + G_0^+(q)]$, where $\tilde{v}$ is the optimal strategy.

Once the optimal strategy has been identified, the problem in *(3.3b)* is a standard pricing problem, see Example 1 for the case of exponential Brownian motion.

The following theorem will be proved in § 4 and § 5 under mild regularity conditions on σ (see the comments before Lemma 4.1).

Theorem 3.4 *If $x\sigma(x,t)$ is non-decreasing in x then the optimal strategy is $v = 1$ and the time 0 price of the passport option is $\frac{1}{2}\mathbb{E}(\sup_{0\leq r\leq T} S_r - S_0 \ - |G_0(q)|)^+ + G_0(q)^+$.*

This strategy corresponds to taking $q = -\text{sgn}(G(q))$ which says that the asset is held long when the portfolio value is below zero, and sold short when the value is positive. Moreover, with this strategy the associated price is directly related to the price of a lookback call option, with strike $(S_0 + |G_0(q)|)$.

Example 1 - Option price under Exponential Brownian Motion

In the specific case of exponential Brownian motion, it is straightforward to calculate the price. We state the result here and refer the reader to Goldman et al [6] for lookback option calculations. For this example, take the volatility to be constant, $\sigma(S,t) = \sigma$.

The price of the passport option at time t; $0 \le t \le T$ is given by

$$\psi_t(q)^+ + \tfrac{1}{2}\left[\, P_t\{N(d) - N(d-\sigma\sqrt{\tau}) + \sigma\sqrt{\tau}(N'(d) + dN(d))\} - |\psi_t(q)|N(d-\sigma\sqrt{\tau}) \,\right]$$

where $\tau = T - t$,

$$d = \frac{-\ln(1 + |\psi_t(q)|/P_t) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

and N is the cumulative normal distribution function.

4 The Optimal Strategy

The aim of this section is to prove Theorem 3.4 by finding an expression for the price of a passport option and showing that this price is a martingale under the optimal strategy and a supermartingale under any other strategy.

In the simple case of exponential Brownian motion, this can be done via explicit calculation, as outlined later. In the case of a general diffusion, this is more involved and we achieve the result using two lemmas proved by coupling in § 5. In the second lemma, we need the assumption that the diffusion coefficient is non-decreasing in the asset price to achieve the result.

Let $N_t = N_0 + M_t(v)$ or equivalently $N_t = N_0 + \int_0^t v_u dS_u$ be the discounted gains from trade process, with a general initial value N_0, which results from using strategy v. Given also an initial value $N_0^* \ge N_0$, let $N_t^* = \sup_{0\le s\le t} N_s \vee N_0^*$. Consider the gains process which results from using strategy v until time t, and thereafter using the conjectured optimal strategy 1. Let Z_t be the conditional expectation of the maximum of this process given the information available at time t.
Then if $S_t^* = \sup_{0\le u\le t} S_u$ we have

$$
\begin{aligned}
Z_t &= \mathbb{E}_t(N_t^* \vee \sup_{t<p\le T}(N_t + \int_t^p 1dS_u)) \\
&= \mathbb{E}_t(\max[N_t^*, N_t + \sup_{t<p\le T}(S_p - S_t)]) \\
&= \mathbb{E}_t(\max[N_t^* - N_t + S_t, \sup_{t<p\le T} S_p]) + (N_t - S_t) \\
&= f(N_t^* - N_t + S_t, S_t, T-t) + (N_t - S_t)
\end{aligned}
$$

where f is the expected value of S_T^* given the information available at time t:

$$
f(x, y, T-t) = \mathbb{E}_t(S_T^* \mid S_t^* = x, S_t = y). \tag{9}
$$

We show below that Z_t is a supermartingale, and a martingale under the optimal control $v = 1$. It follows that

$$
\mathbb{E}N_T^* = \mathbb{E}Z_T \le Z_0 = f(N_0^* - N_0 + S_0, S_0, T) + (N_0 - S_0)
$$

with equality for the optimal control, hence $v = 1$ is optimal and the value of the passport option is as given in Theorem 3.4.

Under our assumption that $S \in H^1$ we have that $\mathbb{E}(S_T^*) < \infty$ and f exists. Moreover from the representation

$$
f(S_t^*, S_t, T-t) = S_t^* + \int_{S_t^*}^{\infty} \mathbb{P}^{S_t,t}(\tau_a < T)da
$$

it follows that the differentiability of f reduces to questions about the differentiability of $\mathbb{P}^{x,t}(\tau_y < T)$. This is an initial-boundary value problem; sufficient conditions for f to be in $C^{1,2,1}(((y,x); y \ge x) \times [0,T))$ are that $x\sigma(x,t)$ is Holder continuous and bounded above and below by positive constants, see Friedman [5, Chapter 3]. Assuming f is sufficiently differentiable, we prove

Lemma 4.1 *Z_t is a $\mathbb{P}$-supermartingale and a martingale for $v = 1$.*

Proof:

Applying Ito's formula and using the simplification $dN_t = v dS_t$ we obtain

$$dZ_t = f_1 dN_t^* + \{(v-1) + (1-v)f_1 + f_2\}dS_t + (\tfrac{1}{2}f_{22}(dS_t)^2 - f_3 dt) + (\tfrac{1}{2}(1-v)^2 f_{11} + (1-v)f_{12})(dS_t)^2.$$

To prove that Z_t is a supermartingale under any strategy, and a martingale under the optimal strategy, we need to show that

(i) $f_1 dN_t^* = 0 = (\frac{1}{2}f_{22}(dS_t)^2 - f_3 dt)$ and

(ii) $\frac{1}{2}(1-v)^2 f_{11} + (1-v)f_{12} \le 0$ for $|v| \le 1$, with equality for $v = 1$.

(i) This follows immediately from the representation (9) of $f(S_t^*, S_t, T-t)$ as a martingale.

(ii) Given $v = 1$, equality is immediate. Since $0 \le \frac{1}{2}(1-v) \le 1$, necessary and sufficient conditions are:

$$f_{12} \le 0, \tag{10}$$

$$f_{11} + f_{12} \le 0. \tag{11}$$

For specific diffusion models, for example exponential Brownian motion, these conditions can be verified directly. More generally, (10) and (11) hold for a wide class of diffusion models. This result is the object of the next section.

□

Example 1 continued - Verification of the Optimal Strategy

In the specific case of exponential Brownian motion, it is straightforward to calculate the price as given in Example 1. Then $f(S_t^*, S_t, T-t) = \mathbb{E}_t(S_T^*)$ is given (in, for example Goldman et al [6]) by

$$f(S_t^*, S_t, T-t) = S_t^* N(\sigma\sqrt{\tau} - d) + S_t \sigma\sqrt{\tau} N'(d) + S_t(1 + \sigma\sqrt{\tau} d)N(d) \tag{12}$$

where

$$d = \frac{\ln(S_t/S_t^*) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

Calculus gives

$$\begin{aligned}
f_{11} &= \frac{S_t}{(S_t^*)^2}\left\{N(d) + \frac{2}{\sigma\sqrt{\tau}}N'(d)\right\} \\
f_{12} &= \frac{-1}{S_t^*}\left\{N(d) + \frac{2}{\sigma\sqrt{\tau}}N'(d)\right\}
\end{aligned}$$

and (10) and (11) are easily verified.

5 Two Useful Coupling Results

We first rewrite f as

$$\begin{aligned}
f(S_t^*, S_t, T-t) &= \mathbb{E}_t(S_T^* | S_t^*, S_t) \\
&= S_t^* + \mathbb{E}_t(\max_{t<r\leq T} S_r - S_t^*)^+.
\end{aligned}$$

Again in this section, for notational simplicity we take t=0 and set $S_0^* = x, S_0 = y$ and $S_r^* = x \vee \max_{0\leq u\leq r} S_u$. The general case is identical. Denote by $\mathbb{P}^y$ the measure corresponding to the discounted price process started at y.

Defining the function

$$f(x, y) = x + \mathbb{E}^y[(S_T^* - x)^+]$$

gives

$$\begin{aligned}
f_1 &= 1 - \mathbb{P}^y(\max_{0\leq u\leq T} S_u \geq x) \\
f_{12} &= -\frac{\partial}{\partial y}\mathbb{P}^y(\max_{0\leq u\leq T} S_u \geq x).
\end{aligned}$$

We require two results which are proved via coupling. The first will allow us to prove the first condition (10), namely that $f_{12} \leq 0$.

Lemma 5.1 *If S is a continuous strong-Markov process which is unique in law then $\mathbb{P}^y(\max_{0\leq u\leq T} S_u \geq x)$ is non-decreasing in y.*

Proof: Take $x > y_2 > y_1$, the other cases being trivial. We show

$$\mathbb{P}^{y_2}(\max_{0\leq u\leq T} S_u \geq x) \geq \mathbb{P}^{y_1}(\max_{0\leq u\leq T} S_u \geq x)$$

using coupling of diffusions. The argument is essentially that given in the introduction of Lindvall [12].

Take two realisations S^{y_1} , S^{y_2} of the (discounted) price process started at y_1 and y_2 respectively.

Define

$$\tau = \inf_u \{S_u^{y_2} \leq S_u^{y_1}\}$$

to be the first crossing time and let

$$\tilde{S}_u^{y_1} = \begin{cases} S_u^{y_1} & 0 \leq u \leq (\tau \wedge T) \\ S_u^{y_2} & (\tau \wedge T) \leq u \leq T. \end{cases}$$

Then using the Strong-Markov property $\tilde{S}_u^{y_1} \overset{\text{law}}{=} S_u^{y_1}$ and by construction $\tilde{S}_u^{y_1} \leq S_u^{y_2}$ for all u and all ω. Hence

$$\begin{aligned} \mathbb{P}^{y_1}(\max_{0\leq u\leq T} S_u \geq x) &= \mathbb{P}^{y_1}(\max_{0\leq u\leq T} \tilde{S}_u \geq x) \\ &\leq \mathbb{P}^{y_2}(\max_{0\leq u\leq T} S_u \geq x). \end{aligned}$$

□

Before moving to the second lemma, we re-express f_1 in the following way

$$\begin{aligned} f_1 &= 1 - \mathbb{P}(y + \max_{0\le u\le T}(S_u^y - y) \ge x) \\ &= 1 - \mathbb{P}(X^y \ge x - y) \end{aligned}$$

where $X^y = \max_{0\le u\le T}(S_u^y - y)$.

Then

$$\begin{aligned} f_{11} + f_{12} &= \frac{\partial}{\partial x}(1 - \mathbb{P}(X^y \ge x - y)) + \frac{\partial}{\partial y}(1 - \mathbb{P}(X^y \ge x - y)) \\ &= -\frac{\partial}{\partial y}\mathbb{P}(X^y \ge z) \mid_{z=x-y} . \end{aligned}$$

The following result will show the second condition holds, under conditions on the diffusion coefficient of our price process.

Lemma 5.2 *If S follows a diffusion process given by the solution to $dS_t = \eta(S_t,t)dB_t$ with $S_0 = y$, and if $\eta(S,t)$ is non-decreasing in S, then $\mathbb{P}(X^y \ge z)$ is non-decreasing in y.*

Proof: Take $y_2 > y_1$. We want to show

$$\mathbb{P}(X^{y_2} \ge z) \ge \mathbb{P}(X^{y_1} \ge z). \tag{13}$$

The approach we take is to write the continuous local martingale S_t as a time change of Brownian motion. Comparisons between the two processes S^{y_2} and S^{y_1} can then be drawn from comparisons of the time changes.

In particular, for each ω define $(\Gamma^i)_{i=1,2}$ to be the solution up to the first explosion time, if any, of the ordinary differential equation

$$\frac{d\Gamma_s^i}{ds} = \frac{1}{\eta^2(W_s + y_i, \Gamma_s^i)}.$$

Denote the inverse to Γ^i by A^i and define

(14) $$S_t^{y_i} - y_i = W_{A_t^i}.$$

Now

$$\frac{dA_t^i}{dt} = \eta^2(W_{A_t^i} + y_i, t) = \eta^2(S_t^{y_i}, t)$$

and hence $dS = \eta(S_t^{y_i}, t)dB$ for some Brownian motion B (see Karatzas and Shreve [11, Theorem 3.4.6]).

Then

$$\max_{0 \le t \le T} S_t^{y_i}(\omega) = y_i + \max_{0 \le s \le A_T^i} W_s(\omega),$$

and

$$\mathbb{P}(\max_{0 \le t \le T} S_t^{y_i}(\omega) \ge z + y_i) \equiv \mathbb{P}(\max_{0 \le s \le A_T^i} W_s(\omega) \ge z).$$

To prove (13), it will be sufficient to prove:

(15) $$A_T^2(\omega) \ge A_T^1(\omega);\ \forall \omega$$

and note that clearly $\mathbb{P}(\max_{0 \le u \le t} W_u \ge z)$ is increasing in t.

We will need to use the condition $\eta(S_t, t)$ is non-decreasing in S. Intuitively we are saying that when the pair of price processes S^{y_i} are constructed from the same trajectory $W(\omega)$ the one starting higher will be running at a faster rate and hence use more of the Brownian path. From this we deduce that the chance of moving up by z is increasing in the starting point.

Since $y_2 > y_1$ and $\eta(S_t, t)$ is non-decreasing in S,

(16) $$\left.\frac{dA_t^2}{dt}\right|_{t=0} = \eta^2(S_0^{y_2}, 0) \ge \eta^2(S_0^{y_1}, 0) = \left.\frac{dA_t^1}{dt}\right|_{t=0}.$$

Now define $\tau = \inf_t\{A_t^2(\omega) < A_t^1(\omega)\}$. From the time change result (14):

$$S_t^{y_2}(\omega) - y_2 = W_{A_t^2(\omega)}(\omega)$$

which at $t = \tau$ gives:

$$S_\tau^{y_2}(\omega) - y_2 = W_{A_\tau^2(\omega)}(\omega) = W_{A_\tau^1(\omega)}(\omega) = S_\tau^{y_1}(\omega) - y_1$$

As $y_2 > y_1$, we have $S_\tau^{y_2}(\omega) > S_\tau^{y_1}(\omega)$ and by continuity of S this must be true over $(\tau, \tau + \epsilon)$ also. Hence

$$A_{\tau+\epsilon}^2 - A_\tau^2 = \int_\tau^{\tau+\epsilon} \eta^2(S_u^{y_2}, u)du \geq \int_\tau^{\tau+\epsilon} \eta^2(S_u^{y_1}, u)du = A_{\tau+\epsilon}^1 - A_\tau^1.$$

It follows that $A_t^2(\omega) \geq A_t^1(\omega)$ for all t, uniformly in ω.

$\square$

Remark 5.3 If $\eta(S_t, t)$ is non-increasing in S, the proof adapts easily to show $\tilde{v} = -1$ and the price becomes

$$\tfrac{1}{2}\mathbb{E}\big(\sup_{0\leq r\leq T}(-S_r) + S_0 - |G_0(q)|\big)^+ + G_0(q)^+.$$

Clearly if $\eta(S_t, t)$ is constant, both $v = -1$ and $v = 1$ are optimal strategies. Of course assuming η is non-increasing is inconsistent with price processes which are necessarily non-negative.

6 Conclusion

Following the recent introduction of the passport option, fund managers may offer products with principal protection to their clients. This represents a major change to the industry, allowing 'active' fund managers to compete with traditional passive funds for risk averse investors.

In this paper, we use a probabilistic approach to price the symmetric version of the passport option. This enables us to extend the previous work of Hyer et al [10] and Andersen et al [2] to non-constant volatility models. Central to this article is a demonstration of a relationship between the prices of passport and lookback options. This means that the problem of pricing and hedging a passport option can be reduced to the problem of pricing and hedging an instrument which is already well understood. Our argument uses the concept of local time and the Skorokhod Lemma. To illustrate these ideas we calculate an explicit price when the underlying asset price follows an exponential Brownian motion. Furthermore, for a wide class of diffusion models, we provide a description of the holder's optimal strategy. This involves buying up to the position limit when the trading account is negative and selling otherwise. The proof involves the tool of coupling of stochastic processes.

References

[1] Ahn H., Penaud A., and P. Wilmott; Various Passport Options and their Valuation, Preprint OCIAM Oxford University, 1998.

[2] Andersen L., Andreasen J. and R. Brotherton-Ratcliffe; The Passport Option, Journal of Computational Finance, Vol.1, No.3, p15-36, 1998.

[3] Carr P. and R.A. Jarrow; The Stop-Loss Start-Gain Paradox and Option Valuation: A New Decomposition into Intrinsic and Time Value, The Review of Financial Studies, Vol.3, No.3, p469-492, 1990.

[4] Chesney M., Cornwall J., Geman H., Kentwell G., Jeanblanc M., Pitman J. and M. Yor; Parisian Options, Sturm-Liouville equation, Feynman-Kac formula, Preprint, Equipe d'Analyse et Probabilities, Universite d'Evry, Jan 1997.

[5] Friedman A.; Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[6] Goldman M.B., Sosin H.B. and M.A. Gatto; Path Dependent Options: "Buy at the Low, Sell at the High", Journal of Finance, Vol XXXIV, No.5, Dec 1979.

[7] Hajek B.; Mean Stochastic Comparison of Diffusions, Z.Wahrscheinlichkeits-theorie verw.Gebiete , 68, p315-329, 1985.

[8] Harrison J.M. and S.R. Pliska; Martingales and Stochastic Integrals in the Theory of Continuous Trading, Stoch.Proc. and Their Appl., 11, p215-260, 1981.

[9] Hobson D.G.; Volatility mis-specification, option pricing and super-replication via coupling, Annals of Applied Probability, 8, No.1, p193-205, 1998.

[10] Hyer T., Lipton-Lifschitz A. and D. Pugachevsky; Passport to Success, RISK Magazine, 10, No.9, p127-131, 1997.

[11] Karatzas I. and S.E. Shreve; Brownian Motion and Stochastic Calculus, Springer-Verlag, New York, 1988.

[12] Lindvall T.; Lectures on the Coupling Method, Wiley, New York, 1992.

[13] Revuz D. and M. Yor; Continuous Martingales and Brownian Motion, Springer, 1991.

[14] Rogers L.C.G. and D. Williams; Diffusions, Markov Processes, and Martingales, Vol. 2, Wiley, Chichester, 1987.

[15] Shreve S.E. and J. Večeř; Passport Option: Probabilistic Approach for Solving the Symmetric Case, Preprint, Department of Mathematical Sciences, Carnegie Mellon University, 1998.