Lucas Economy with Trading Constraints

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1 Introduction

We are interested in a general equilibrium economy under leverage constraints. In the classical representative-agent Lucas economy, there is a unique stock price derived from the unique state-price density. In our economy, agents have diverse beliefs about future performance so (following Brown and Rogers [2]) they may hold different portfolios due to differing leverage constraints. Since the state price density gives the optimal portfolio, the presence of portfolio constraints prohibits a universal state-price density. As a consequence, the market cannot be complete. Another result of our general-equilibrium model is that the imposition of leverage constraints increases stock values.

We are particularly interested in the no-leverage constraint case. This can be interpreted as a cash-in-advance requirement where assets must be purchased with ready cash. Indeed, it is this need to have cash on hand that gives value to otherwise worthless money – more tender allows the agent greater investment opportunities and he may be prepared to exchange real assets for cash in order to relax the cash constraint and gain utility from consumption. Cash-in advance has a long history in economics. One branch of the literature takes an inventory-theoretic approach. Baumol [1] and Tobin [10] introduce models where agents trade-off exogenously imposed interest earnings and banking costs to determine equilibrium cash holdings. Romer [7] presents a discrete-time general-equilibrium version of this. Closer in line with our model are settings where cash is modeled as a transactions medium in an equilibrium economy without imposing arbitrary exogenous costs. Svensson [9] has a discrete-time cash-in-advance story where agents must decide on cash holdings before their consumption is known. Hence, they hold extra precautionary cash despite a positive interest rate. We will show our results in an simple and elegant continuous-time, infinite-horizon framework. For mathematical simplicity, we will first derive our model in an economy where bonds are tender. This case will then be shown to have an interpretation as a cash-market with an endogenously determined (real) inflation rate.

There has been some work in the literature on the effects of market imperfections on market equilibrium. Ross [8] examines how short-sale constraints can lead to violations of CAPM. Milne and Neave [5] investigate a similar problem in an intricate discrete-time, finite-horizon equilibrium model. They show how transactions costs and trading constraints lead to market incompleteness. Similar results were obtained by Jouini and Kallal [4] using no-arbitrage arguments

The solution we obtain is mathematically similar to Cvitanic and Karatzas's solution [3] for the Merton problem under portfolio constraints. They show that the support function of the feasible portfolio set plays a critical role in affecting the state-price density. When an agent hits his constraint, this support function simultaneously distorts the change-of-measure and discount factor in his state price density, leading the agent to keep his portfolio within the feasible region when his unconstrained optimum would otherwise be outside. However, in Cvitanic and Karatzas's model, the stock dynamics are fixed. In our model, it is the dividend process that is specified while the equity price is derived. Hence, it is not clear from the outset how the stock price can remain a discounted martingale under different agent measures. Also, by not working in an equilibrium setting, Cvitanic and Karatzas are unable to draw any conclusions about market incompleteness.

The structure of the article is as follows. We work out the problem for general leverage constraint and obtain expressions for the stock and wealth processes in terms of discounted dividend and consumption processes under different measures. Next, we solve the special case of log agents, obtaining a near closed-form solution and give a monetary interpretation. Finally, we show the results of a simulation for GBM dividend dynamics and a Bayesian Kalman-filtering model and comment about conclude with some observations and further directions for exploration.

2 General Setup

Consider a Lucas Tree Model with multiple agents and multiple 'trees' but a single production good. The agents are optimizing their own utility from consumption under diverse beliefs. Agent i thinks future dynamics obey a law given by the measure $\mathbb{P}^i$ where

$$\left.\frac{d\mathbb{P}_i}{d\mathbb{P}}\right|_t = \Lambda_t^i$$

and Λ_t^i is a Radon-Nikodym derivative with respect to a reference measure $\mathbb{P}$. The diverse beliefs framework is a tractable means of obtaining trading volume in a Lucas model and are "orthogonal" to the constraint problem at hand. Hence, we will not discuss them further. For a more in-depth development of the theory, see Brown and Rogers [2]. Agent i's objective is therefore given by an additive utility function of his consumption c_t^i,

$$\mathrm{E}\left[\int_0^\infty U^i(t, c_t^i)\Lambda_t^i \, dt\right] .$$

Let S_t be the stock price, r_t be the interest rate, and δ_t the dividend. The agent's wealth equation is given by

$$w_t^i = \pi_t^i \cdot S_t + \phi_t^i \qquad dw_t^i = \pi_t^i \cdot dS_t + \pi_t^i \cdot \delta_t \, dt + \phi_t^i r_t \, dt - c_t^i \, dt \tag{1}$$

where π_t^i is a vector representing the portfolio and ϕ_t^i is a scalar representing cash holdings invested in a risk-free bond paying at interest rate r_t. We now impose a general constraint,

$$L^i w_t^i \geq \pi_t^i \cdot S_t \iff \pi_t^i \cdot S_t \geq -\frac{L^i}{L^i - 1}\phi_t^i =: -\frac{1}{K^i}\phi_t^i \qquad L^i \in [1, \infty] \qquad K^i \in [0, 1]$$

where L^i is a (possibly agent-dependent) leverage constraint. The special case $L^i = 1$, $K^i = 0$ corresponds to the no-borrowing constraint

$$w_t^i \geq \pi_t^i \cdot S_t \iff \phi_t^i \geq 0 \, .$$

We restrict the total initial supply of bonds to be $\sum_i \phi_0^i = A$. We can interpret this as the total outstanding government debt. A special case is $A = 0$ (i.e. the bond market clearing) although this is trivial for the no-borrowing case. The leverage constraints L^i, dividend dynamics δ_t, discount factor ρ, diverse beliefs Λ_t^i, initial wealths w_t^i and initial bond level A_0 are given. The problem is to solve for the other variables in an equilibrium economy.

We will employ the Lagrange-Pontryagin method to solve the model. Adding in constraint terms yields our new objective

$$\begin{aligned}
&\sup_{\pi^i,\phi^i} \mathrm{E}\bigg[\int_0^\infty \tilde{U}^i(t,\zeta_t^i/\Lambda_t^i)\Lambda_t^i\,dt + \left(\zeta_t^i dw_t^i + w_t^i d\zeta_t^i + d[w^i,\zeta^i]_t\right) + \zeta_t^i(K^i\pi_t^i\cdot S + \phi_t^i)d\eta_t^i\bigg] + w_0^i\zeta_0^i \\
= &\sup_{\pi^i,\phi^i} \mathrm{E}\bigg[\int_0^\infty \tilde{U}^i(t,\zeta_t^i/\Lambda_t^i)\Lambda_t^i\,dt + \pi_t^i\cdot(\zeta_t^i dS_t + \zeta_t^i\delta_t\,dt + S_t d\zeta_t^i + d[S,\zeta^i]_t + K^i\zeta_t^i S_t\,d\eta_t^i) \\
&\quad + \phi_t^i(\zeta_t^i r_t\,dt + d\zeta_t^i + \zeta_t^i d\eta_t^i)\bigg] + w_0^i\zeta_0^i
\end{aligned}$$

where ζ_t^i is the Lagrange multiplier for the wealth dynamics and η_t^i is a non-decreasing dual process for the trading constraint. Hence, $d\eta$ is finite-variation, which gives us that

$$e^{K^i\eta_t^i}\zeta_t^i S_t + \int_0^t e^{K^i\eta_u^i}\zeta_u^i\delta_u\,du \tag{2}$$

is a local martingale. Conflating local and UI martingales and assuming $e^{K^i\eta_t^i}\zeta_t^i S_t$ vanishes as t approaches ∞, we have the familiar expression for a stock price,

$$e^{K^i\eta_t^i}\zeta_t^i S_t = \mathrm{E}_t\left[\int_t^\infty e^{K^i\eta_u^i}\zeta_u^i\delta_u\,du\right]. \tag{3}$$

We also have that

$$e^{\int_0^t r_u\,du + \eta_t^i}\zeta_t^i \tag{4}$$

is a local martingale, which gives the relation to the bond-price. We can also work out

$$\begin{aligned}
d(\zeta_t^i w_t^i) &= w_t^i d\zeta_t^i + \zeta_t^i dw_t^i + d[\zeta^i, w^i]_t \\
&= (\pi_t^i\cdot S_t + \phi_t^i)d\zeta_t^i + \zeta_t^i(\pi_t^i\cdot(dS_t + \delta_t\,dt) + \phi_t^i r_t\,dt - c_t^i\,dt) + \pi_t^i\cdot d\left[\zeta^i, S\right]_t \\
&= -\zeta_t^i[(K^i\pi_t^i\cdot S_t + \phi_t^i)d\eta_t^i - c_t^i\,dt)]
\end{aligned}$$

and since the coefficient of $d\eta_t^i$ is zero when η_t^i is increasing, we have (conflating local and UI martingales and assuming vanishing limits at ∞ again),

$$\zeta_t^i w_t^i = \mathrm{E}_t\left[\int_t^\infty \zeta_u^i c_u^i\,du\right].$$

3 Single-Asset, Log Utility

To solve the problem, we will make a few simplifying assumptions. We assume that there is one risky asset and that the utility functions are of the form $U^i(t,c) = e^{-\rho_i t}\ln(c)$. Finally, we postulate dynamics

$$\begin{aligned}
d\delta_t &= \delta_t(\mu_t\,dt + \sigma_t\,dW_t)\\
dS_t &= S_t(m_t\,dt + v_t\,dW_t)\\
d\zeta_t^i &= -\zeta_t^i(\alpha_t^i\,dt + \beta_t^i\,dW_t)\\
d\Lambda_t^i &= \Lambda_t^i\lambda_t^i\,dW_t
\end{aligned}$$

for some Brownian Motion W_t. Then at the optimum, we have

$$c_t^i = \frac{e^{-\rho_i t}\Lambda_t^i}{\zeta_t^i}$$

and therefore

$$\zeta_t^i w_t^i = \mathrm{E}_t\left[\int_t^\infty \zeta_u^i c_u\,du\right] = \frac{e^{-\rho_i t}\Lambda_t^i}{\rho_i} = \zeta_t^i\frac{c_t^i}{\rho_i}\,.$$

The tractability of the wealth integral above justifies the otherwise arbitrary restriction to log-utility. We can now compute

$$dw_t^i = -\rho_i w_t^i\,dt + w_t^i\lambda_t^i\,dW_t + w_t^i\left(\alpha_t^i\,dt + \beta_t^i\,dW_t\right) + w_t^i\beta_t^i\lambda_t^i\,dt + w_t^i(\beta_t^i)^2\,dt\,.$$

Define $\widetilde{m}_t = m_t + \delta_t/S_t$. From (1), we then have

$$\begin{aligned}
\pi_t^i &= \frac{w_t^i}{S_t}\left(\frac{\lambda_t^i+\beta_t^i}{v_t}\right)\\
\phi_t^i &= w_t^i\left(1-\frac{\lambda_t^i+\beta_t^i}{v_t}\right) = \frac{w_t^i}{r_t}\left((\beta_t^i)^2+\alpha_t^i+\beta_t^i\lambda_t^i-\frac{\lambda_t^i+\beta_t^i}{v_t}\left(\widetilde{m}_t\right)\right)
\end{aligned}$$

and so

$$(\beta_t^i)^2+\alpha_t^i+\beta_t^i\lambda_t^i-\frac{\lambda_t^i+\beta_t^i}{v_t}\widetilde{m}_t = \left(1-\frac{\lambda_t^i+\beta_t^i}{v_t}\right)r_t\,. \tag{5}$$

If the constraint is binding then $\pi_t^i S_t = L^i w_t^i$ so $\lambda_t^i+\beta_t^i = L^i v_t$; otherwise, η_t^i is not increasing and so (4) implies $\alpha = r_t$. Either way, the above equation reduces to

$$0 = \widetilde{m}_t - \beta_t^i v_t - \alpha_t^i + K^i(\alpha_t^i - r_t) \tag{6}$$

which also follows from (3). From (4), we have $\dot{\eta}_t^i = \alpha_t^i - r_t$ so that the previous equation agrees with (2). Then when the constraint is binding, we have (because $K-1\le 0$)

$$v_t(L^i v_t - \lambda_t^i) = v_t\beta_t^i = \widetilde{m}_t - \alpha_t + K^i(\alpha_t^i - r_t) \le m_t + \frac{\delta_t}{S_t} - r_t$$

from (6). Otherwise, the same equation implies

$$\widetilde{m}_t - r_t = v_t\beta_t^i \le v_t(L^i v_t - \lambda_t^i)\,.$$

Therefore, we have

$$\beta_t^i = \min\left\{L^i v_t - \lambda_t^i\,,\,\frac{1}{v_t}\left(\widetilde{m}_t - r_t\right)\right\}$$

and from (6),

$$\alpha_t^i = \max\left\{L^i\left(\widetilde{m}_t - (L^i v_t - \lambda_t^i)v_t\right) + (1-L^i)r_t, r_t\right\}\ .$$

We see that agents follow the unconstrained optimum, remain at their constraints if it is succeeded by the unconstrained optimal. This result is analogous to that of Cvitanic and Karatzas [3] for the constrained Merton problem. Note that each agent's α_t^i, β_t^i satisfy (3) and so all unconstrained agents feel the stock is fairly valued. Only the constrained agents disagree but they are unable to affect the stock price by bidding it up. Hence, the agents maintain diverse state-price densities, despite agreeing on the value of the stock. From the equation for π_t^i, we see that it is the agents with bullish beliefs (high λ_t^i) that choose to hold more stock and possibly be constrained by L^i.

Market clearing for consumption goods implies $\delta_t = \sum_i c_t^i = \sum_i \rho_i w_t^i$ from which we derive

$$\delta_t \sigma_t = \sum_i \rho_i \pi_t^i S_t v_t = \sum_i \rho_i w_t^i (\lambda_t^i + \beta_t^i) \tag{7}$$

and

$$\begin{aligned}
\delta_t \mu_t &= \sum_i \rho_i w_t^i \left((\beta_t^i)^2 + \alpha_t^i + \beta_t^i \lambda_t^i - \rho_i\right) \\
&= \sum_i \rho_i w_t^i \left(\frac{\lambda_t^i + \beta_t^i}{v_t}\widetilde{m}_t + \left(1 - \frac{\lambda_t^i + \beta_t^i}{v_t}\right) r_t - \rho_i\right) \\
&= \frac{\delta_t \sigma_t}{v_t}\widetilde{m}_t + \delta_t r_t - \sum_i \rho_i^2 w_t^i\,.
\end{aligned}$$

Market clearing for stock implies $\sum_i \pi_t^i = 1$. Taking $\rho_i = \rho$ to be constant means that

$$\delta_t \sigma_t = \rho S_t v_t \qquad \delta_t(\mu_t - r_t) = \rho S_t (m_t - r_t) \tag{8}$$

i.e. that,

$$d\left[\frac{\delta_t}{\rho}\exp\left(-\int_0^t r_u\,du\right)\right] = d\left[S_t \exp\left(-\int_0^t r_u\,du\right)\right]$$

or that

$$\sum_i w_t^i = \frac{\delta_t}{\rho} = S_t + A_0 \exp\left(\int_0^t r_u\,du\right)\ . \tag{9}$$

Since the sum of equity and debt is the total value of the economy (which is fixed by the dividend), higher stock returns translate to lower-bond returns and vice-versa. Hence, the above equation explicitly gives a trade-off between bond and stock performance, a traditional claim of investing folklore. We can also give an interpretation to A_t as the aggregate bond supply at time t, where

$$A_t := A_0 \exp\left(\int_0^t r_u\,du\right) = \sum_i \phi_t^i\,. \tag{10}$$

Hence, from δ_t, we can derive the dynamics of S_t once we are given the interest rate r_t, which we will compute next.

Finally, from $\sum_i \pi_t^i = 1$ we have

$$\begin{aligned} S_t &= \sum_i w_t^i \min\left[L^i, \frac{1}{v_t\sigma_t}(\mu_t - r_t + \rho) + \frac{\lambda_t^i}{v_t}\right] \\ &= \sum_i w_t^i \min\left[L^i, \frac{\rho S_t}{\delta_t\sigma_t^2}(\mu_t - r_t + \rho) + \lambda_t^i \frac{\rho S_t}{\delta_t\sigma_t}\right] \end{aligned} \tag{11}$$

and can numerically solve for r_t. Note that the right-hand side is monotonic in r_t so the solution is unique.

Now we can give an interpretation of our results in terms of a cash-market. Up until now, all goods have been valued in terms of the consumption good. Alternatively, we could assume the amount of legal tender is exogenously fixed at A'_t by the government and that consumption goods are valued according to a price process p_t, (i.e. 1 consumption good costs p_t units of currency at time t). Then r_t is the real interest rate, r'_t the nominal interest rate, and r_t^p the inflation rate where

$$A'_t = A'_0 \exp\left(\int_0^t r'_u \, du\right) \qquad p_t = p_0 \exp\left(\int_0^t r_u^p \, du\right).$$

The equation $A_t p_t = A'_t$ gives us the basic relationship

$$r_t + r_t^p = r'_t.$$

By predicting the real interest rate, our model fixes the relationship between inflation and the nominal interest rate. For example, we might take the money supply to be fixed $r'_t = 0$, e.g., gold coins in a world with no mining. In this case, we can interpret $-r_t = r_t^p$ as the inflation rate. For instance, in our GBM-dividend-dynamics model (see below), we obtain $r_t > 0$ and hence deflation in the economy. This is a reasonable conclusion given an increasing goods supply which can only be purchased using a fixed money supply.

Secondly, observe that we can rewrite (11) as

$$\ell_t(r_t, L) := \sum_i w_t^i \min\left[\frac{L^i}{S_t}, \frac{\rho}{\delta_t\sigma_t^2}(\mu_t - r_t + \rho) + \lambda_t^i \frac{\rho}{\delta_t\sigma_t}\right] - 1 = 0. \tag{12}$$

First note that a solution must exist since $\sum_i w_t^i \geq S_t$ and $L^i \geq 1$. Since the second case of each minimum function has a strictly negative slope, the solution can only be non-unique if the first case held for each minimum. But this corresponds to all agents being at their constrained, which can only happen in the trivial case $L^i = 1$, $A_0 = 0$ so outside this corner case, the solution is unique. Let $r_t^{(L)}$ be the unique zero of the non-increasing function $\ell_t(r_t, L)$. Note that the unconstrained case corresponds to $L = \infty$. Since $\ell_t(\cdot, L)$ is a strictly decreasing function near its zero and $\ell_t(\cdot, L) \leq \ell_t(\cdot, \infty)$ for all L, we have $r_t^{(L)} \leq r_t^{(\infty)}$. (Observe that the second argument of each minimum in (12) is expressed completely in terms of quantities specified by the model and r_t.) Hence A_t is lower and (9) implies S_t is higher after imposing a portfolio constraint.

4 Simulation

We give a brief outline of our simulation algorithm. After initializing the time zero values, we iteratively update the values by

1. Compute $d\delta_t$ (computing μ_t as necessary)
2. Compute v_t using (8)
3. Compute r_t using the following trick. Define

$$\ell_I(r) := \sum_{i \in I} w_t^i \frac{L^i}{S_t} + \sum_{i \in I^c} w_t^i \left[\frac{\rho}{\delta_t \sigma_t^2} (\mu_t - r_t + \rho) + \lambda_t^i \frac{\rho}{\delta_t \sigma_t} \right] - 1 = 0$$

   where I ranges over all 2^N possible subsets of N agents and I^c is the complement I. We solve for the zeros r_I of ℓ_I quite easily since each ℓ_I is linear. Since each ℓ_I is non-increasing and $\ell_t(r, L) = \min_I \ell_I(r)$ from (12), $r_t = \min_I r_I$.
4. Compute m_t using (8)
5. Compute dS_t
6. Compute dw_t
7. Update δ_t, S_t, w_t for the next time step.

Notice that we do not explicitly use (9) or (10). However, these equations held to within a factor of 10^{-6} in our simulation below.

5 GBM Dividend Dynamics Model

We give simulation results for the cash constrained case ($L^i = 1$) when the dividend process follows a Geometric Brownian Motion (i.e. constant μ_t and σ_t). Observable parameters for our simulation were chosen to be roughly in accordance with real-world economic parameters:

$$\begin{aligned} \rho &= 0.05 & \delta_0 &= 1.0 & \lambda_t &= [-.08, -.06, 0.07, .09] \\ \mu_t &= 0.02 & S_0 &= 10.0 & w_0 &\propto [3.0, 3.5, 2.5, 2.0] \quad \text{such that} \\ \sigma_t &= 0.12 & & & \sum_i w_0^i &= \delta_0 / \rho \end{aligned}$$

The four agents are colored [blue, green, red, turquoise]. The simulation was performed with $\Delta t = \frac{1}{4000}$ up to time $T = 4$ and output is displayed in Figs. 1 and 2.

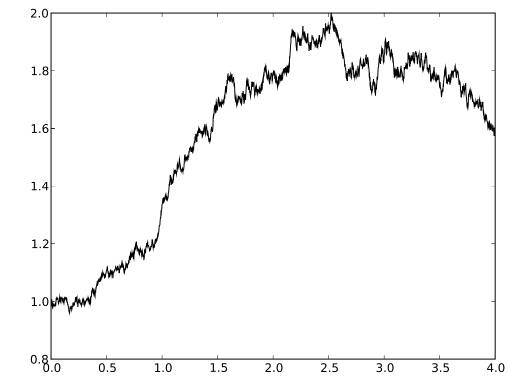
6 Bayesian Model

As another example, we show how our diverse beliefs framework can accommodate a Bayesian setup. We assume a hidden parameter model where the growth rate μ_t is an unobserved process,

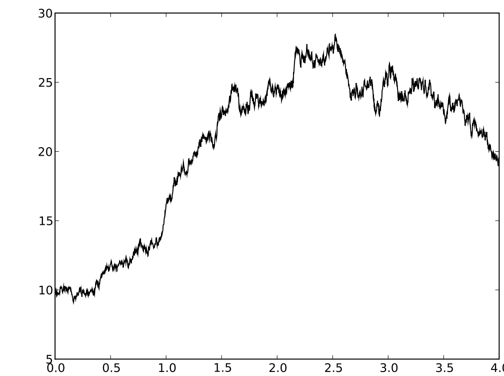
$$d\delta_t = \delta_t(\mu_t \, dt + \sigma \, dW_t) \qquad d\mu_t = -a\mu_t \, dt + \sigma' \, dB_t$$

where a, σ, σ' are positive parameters and B_t is a Brownian Motion independent of W_t. We are interested in the case when the agents have been observing dividend history for a long time. From [6], stationary equilibrium for a Kalman filter implies

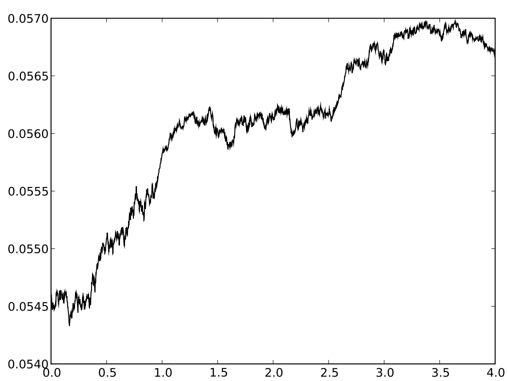
$$d\delta_t = \delta_t(\hat{\mu}_t \, dt + \sigma \, d\hat{W}_t) \qquad d\hat{\mu}_t = -a\hat{\mu}_t \, dt + \kappa \, d\hat{W}_t \tag{13}$$



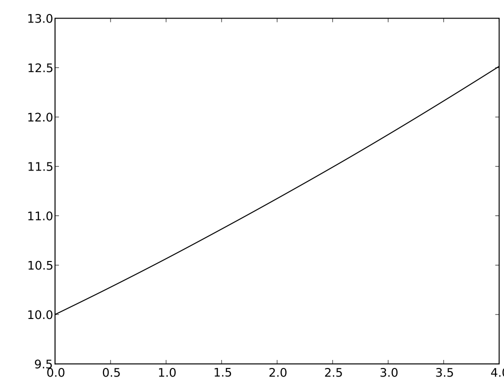
(a) δ_t: exceptionally strong dividend performance.



(b) S_t: as the dividend process increases, so does the stock value, but the latter's gains are disproportionately large.

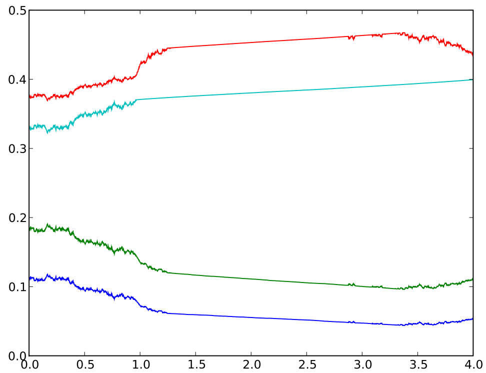


(c) r_t: reduced demand for bonds by bullish agents reduces interest rates. Notice that the interest rate (an output of the model) is reasonable.

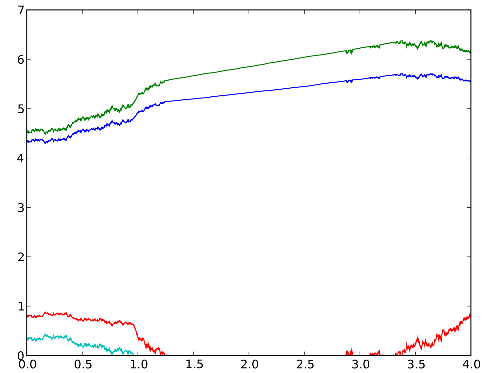


(d) $\sum_i \phi_t^i$: aggregate bonds increase smoothly.

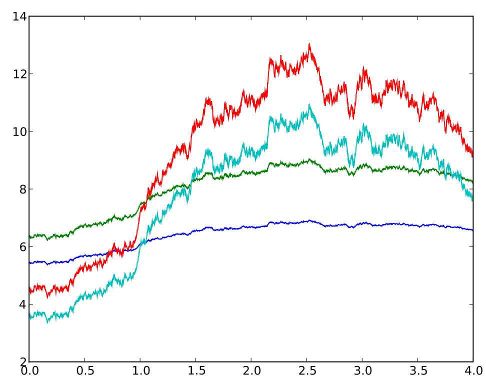
Figure 1: Market performance in a simulation with GBM Dividend Dynamics Model.



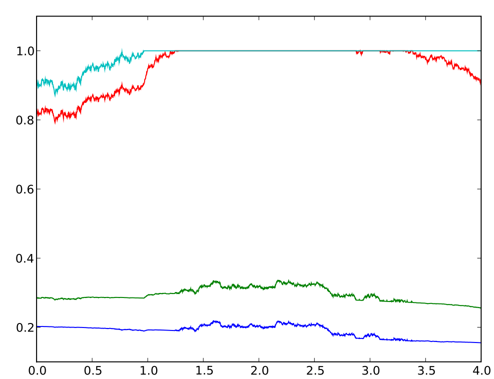
(a) π_t: from $\delta_t \sigma_t = \rho S_t v_t$, we have that higher S_t/δ_t (see Fig. 1(b)) implies a lower v_t, which makes agents want more equity.



(b) ϕ_t: reduced demand for bonds by bullish agents (red and turquoise) is picked up by bearish ones (blue and green).



(c) w_t^i: the aggressive bullish agents have done extremely well in the bull market.



(d) $\frac{\pi_t^i S_t}{w_t^i} = \frac{\lambda_t^i + \beta_t^i}{v_t} \leq L^i$. Notice that the bearish agents are able to maintain relatively flat leverage ratios by selling excess equity holdings to bullish agents. When the latter hit their trading constraints, the bearish agents must hold their own unwanted stock.

Figure 2: Agent performance in a simulation with GBM Dividend Dynamics Model.

where

$$\kappa = \frac{\sqrt{a^2\sigma^2 + \sigma'^2} - a\sigma}{\sigma'^2}\sigma > 0$$

and $\hat{W}_t$ is a $\mathbb{P}$ Brownian motion adapted to the filtration generated by δ_t. The diverse beliefs come in the form of differing beliefs about μ_t. Agent i believes that

$$d\delta_t = \delta_t(\tilde{\mu}_t^i \, dt + \sigma \, d\tilde{W}_t^i) \qquad d\tilde{\mu}_t^i = a(b^i - \tilde{\mu}_t^i) \, dt + \kappa \, d\tilde{W}_t^i \tag{14}$$

where $\tilde{W}_t^i$ is a $\mathbb{P}^i$ Brownian Motion. We see that the two dynamics coincide if

$$\tilde{\mu}_t^i = \hat{\mu}_t - \sigma\lambda_t^i \qquad d\tilde{W}_t^i = d\hat{W}_t + \lambda_t^i \, dt$$

where λ_t^i is actually constant in time and defined by

$$ab^i + \lambda_t^i \, (a\sigma + \kappa) = 0 \, .$$

Below, we have the results of sample simulations of the Bayesian Model for the cash constrained case ($L^i = 1$) with parameters

$$\begin{aligned} \rho &= 0.05 & \delta_0 &= 1.0 & \lambda_t &= [-0.20, -0.15, 0.15, 0.20] \\ \mu_0 &= 0.05 & S_0 &= 10.0 & w_0 &\propto [3.0, 3.5, 2.5, 2.0] \quad \text{such that} \\ \sigma_t &= 0.3 & \kappa &= 0.05 & \sum_i w_0^i &= \delta_0/\rho \end{aligned}$$

The four agents are colored [blue, green, red, turquoise]. The simulation was performed with $\Delta t = \frac{1}{4000}$ up to time $T = 4$ and output is displayed in Figs. 3 and 4. A similar simulation can be done with the Bayesian inference performed from the stock price, but this did not change the results significantly and is not reported here.

7 Conclusions and Future Work

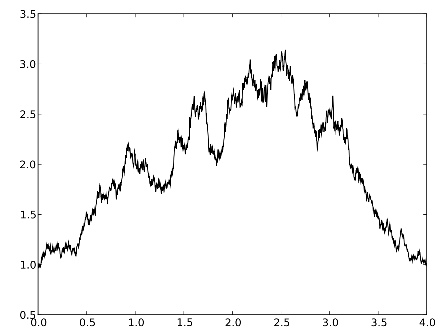
1. The imposition of trading constraints makes markets incomplete by endowing agents with individual price densities.

2. The imposition of trading constraints lowers interest rates and hence aggregate bond levels. Since the wealth level is dictated by dividends (which are determined exogenously) we have from (9) that stock values are inflated.

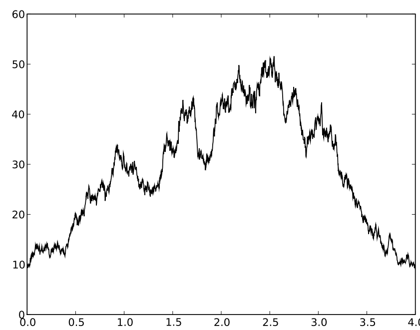
3. The no-borrowing case naturally lends itself to an interpretation as a cash-in advance model where the real-interest rate is determined endogenously.

4. We can generalize the feasible portfolio to an arbitrary convex set K, as in [3]. Then we might obtain the short-sale constraints results of Ross [8] in continuous time as a special case.

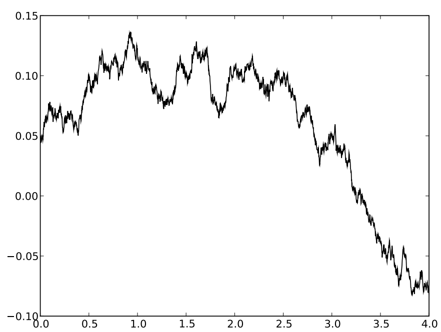
5. Verification of optimum.



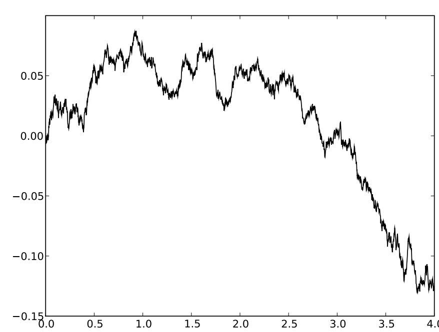
(a) δ_t: a W_t path was generated using an OU-process added to a sign curve.



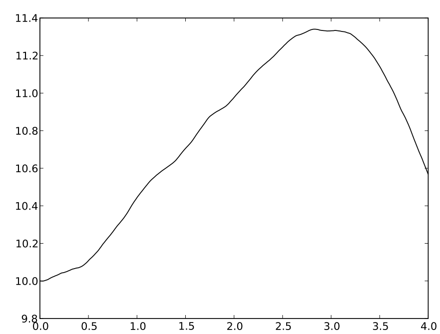
(b) S_t: the stock price responds similarly.



(c) $\hat{\mu}_t$: there is a delay in detecting the change in the growth rate. Observe that $\hat{\mu}$ crosses 0 after $t = 3.0$ but δ peaks near 2.5.

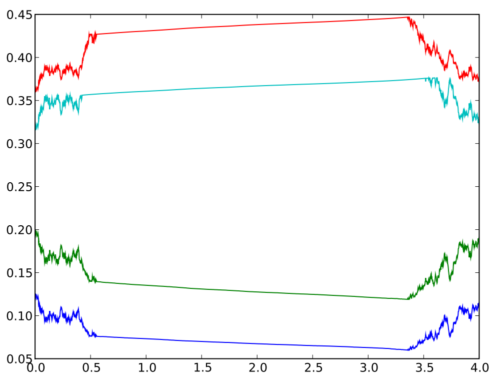


(d) r_t: similar to a scaled version of $\hat{\mu}_t$.

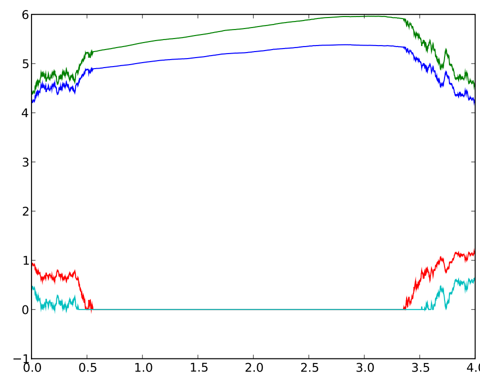


(e) $\sum_i \phi_t^i$: There is a delayed response to the quantity of bonds.

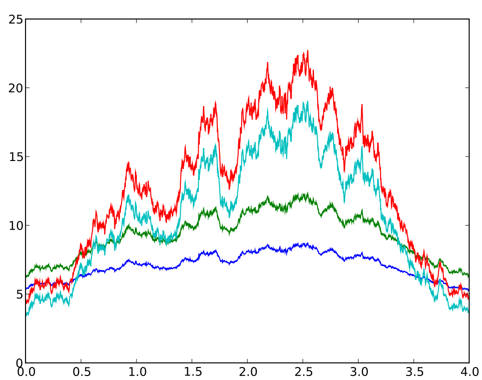
Figure 3: Market performance in a simulation of the Bayesian Model.



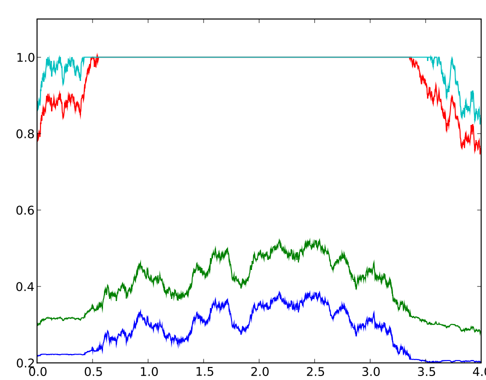
(a) π_t: portfolios become disproportionately aggressive for optimistic agents (red and teal).



(b) ϕ_t: bearish agents (green and blue) prefer to hold cash.



(c) w_t^i: the aggressive optimistic agents' fortunes wax and wane with with the market.



(d) $\frac{\pi_t^i S_t}{w_t^i} = \frac{\lambda_t^i + \beta_t^i}{v_t} \leq L^i = 1$: aggressive investors are held back by leverage constraint.

Figure 4: Agent performance in a simulation of the Bayesian Model.

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