Optimal capital structure and endogenous default

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Abstract. In a sequence of fascinating papers, Leland and Leland & Toft have investigated various properties of the debt and credit of a firm which keeps a constant profile of debt and chooses its bankruptcy level endogenously, to maximise the value of the equity. One feature of these papers is that the credit spreads tend to zero as the maturity tends to zero, and this is not a feature which is observed in practice. This defect of the modelling is related to the diffusion assumptions made in the papers referred to; in this paper, we take a model for the value of the firm's assets which allows for jumps, and find that the spreads do not go to zero as maturity goes to zero. The modelling is quite delicate, but it just works; analysis takes us a long way, and for the final steps we have to resort to numerical methods.

1 Introduction.

The burgeoning literature on credit modelling falls into two methodological classes, the *structural* approach and the *intensity-based* approach, or *reduced-form* approach. The central feature of the first is some attempt to model the evolution of the assets of the firm, whereas the second treats default as an essentially random event, whose intensity may nonetheless depend on underlying variables. This paper is an outgrowth of one of the most interesting series of papers using the first approach, by Leland (1994a,1994b) and Leland & Toft (1996). In these papers, we study a firm the value of whose assets evolves as a diffusion until falling below some critical bankruptcy level. The firm is partly financed by debt, whose maturity profile is maintained constant through time, by the simultaneous issuance of new debt and retirement of old. This debt is of equal seniority, and attracts coupons at a fixed rate. Additionally, the firm receives tax benefits on the coupon payments to bondholders, at least while the value of the assets is high enough. The shareholders set the bankruptcy level so as to maximise the value of the firm's equity, while respecting the constraints that the value of equity must remain non-negative at all times, and that the firm's assets may not be sold off to pay coupons.

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Leland & Toft's analysis gives remarkable insight into the workings of debt, and shows us (among other things) how the value of the firm depends on the leverage, what the optimal leverage should be to maximise the value of the firm, and what the credit spread will be. It is in the last of these that we find a rather unsatisfactory conclusion, that the credit spreads go to zero as maturity decreases to zero. This is certainly not an observed feature of the market, and is happening because of the diffusion assumptions in the papers. What we shall do here is to extend the analysis to allow the value of the firm's assets to make downward jumps; by so doing, we eliminate the undesirable qualitative feature of credit spreads decreasing to zero at the short end, and we also permit a stochastic loss on default. Since we are interested in default, the restriction to downward jumps is not a problem (and in any case, big downward jumps are much more likely than big upward jumps!) If we were to allow jumps of both signs, the analysis of the problem is in effect intractable except in very simple special cases.

The plan of the paper is as follows. In Section 2, we shall specify the evolution of the firm's assets, the structure of its debt, and we shall value the debt. We shall then show (following Leland & Toft (1996)) how the optimal bankrutcy level is chosen. In Section 3, we shall present the particular assumptions that we make on the evolution of the firm's assets, and show how these allow us to develop the theory further, obtaining explicit expressions for the bankruptcy level. In contrast to the works of Leland and Leland & Toft, the value of the firm and the value of its debt no longer have closed-form expressions, but we are able to find transforms of these, which can be inverted numerically without too much difficulty. This numerical inversion is the subject of Section 4, where we present some typical results for our model and discuss their interpretation. Finally, in Section 5 we summarise our conclusions. Some technical results are relegated to appendices: Appendix A develops the expression for the bankruptcy level in the case of a tax threshold, Appendix B explains the details of the bivariate Laplace transform inversion required to study the spread as a function of time, and Appendix C presents the results in the diffusion case, which is the only case where closed-form expressions can be found.

2 The structure of the firm.

Let V_t denote the value of the firm's assets [2] at time t. We shall suppose that V evolves as

$$dV_t = V_t(dZ_t + (r - \delta)dt), \tag{2.1}$$

where Z is some martingale, and r and δ are positive constants corresponding to the riskless rate [3], and the proportional rate at which profit is disbursed to investors.

[2] This is not the same as the value of the firm; the two differ by the net present value of losses on default and all future tax rebates. See for the expression for the value of the firm.

[3] So we are assuming immediately that all processes are specified in the risk-neutral measure. Since our main focus is on pricing issues, this is not a restriction. For questions of determining

Since this firm will have bondholders as well as shareholders, we cannot interpret δ as a dividend rate - from the cashflow $\delta V_t dt$ the coupons and principal repayments due to the bondholders must first be paid before the residual can be paid out to the shareholders as dividend.

The firm is partly financed by debt, which is being constantly retired and reissued in the following way. In time interval $(t, t+dt)$, the firm issues new debt with face value pdt, and maturity profile φ, where φ is non-negative and $\int_0^\infty \varphi(s)ds = 1$. Thus in time interval $(t, t+dt)$ it issues debt with face value $p\varphi(s)dtds$ maturing in time interval $(t+s, t+s+ds)$. Bearing in mind all previously issued debt, at time 0 the face value of debt maturing in $(s, s+ds)$ is therefore

$$(\int_{-\infty}^{0} p\,\varphi(s-v)dv)ds \equiv p\,\Phi(s)ds, \tag{2.2}$$

where $\Phi(s) \equiv \int_s^\infty \varphi(y)dy$ is the tail of the maturity profile. Taking $s = 0$ in (2.2), we see that the face value of debt maturing in $(0, ds)$ is pds, the same as the face value of the newly-issued debt. Thus *the face value of all debt is constant*, equal to

$$P = p\int_0^\infty \Phi(s)ds. \tag{2.3}$$

The paper of Leland & Toft (1996) takes the debt profile to be δ_T, the Dirac delta-function at T; this means that all new debt is always issued with a maturity of T. The paper of Leland (1994b) uses

$$\varphi(t) = me^{-mt} \tag{2.4}$$

for some positive m. This is the main maturity profile that we shall be discussing in this paper, though the methods extend immediately to any maturity profile of the form $\varphi(t) = \sum_{i=0}^N \theta_i \exp(-m_i t)$ for some weights θ_i and positive m_i for which φ is non-negative and integrates to 1.

All debt is of equal seniority, and attracts coupons at the fixed rate ρdt until maturity, or default if that occurs sooner. Default happens at the first time H that the value of the firm's assets falls to some level V_B or lower; the value of V_B will be determined optimally later. On default, a fraction α of the value of the firm's assets is lost in reorganisation.

A bond issued at time 0 with face value 1 and maturity t is worth

$$\begin{aligned} d_0(V, V_B, t) &= E\Big[\int_0^{t\wedge H} \rho e^{-rs}ds\Big] + E\Big[e^{-rt} : t < H\Big] \\ &\quad + \frac{1}{P}(1-\alpha)E\Big[V(H)e^{-rH} : H \le t\Big]. \end{aligned} \tag{2.5}$$

default probabilities, we would of course need to work in the real-world probability. See ...

The first term on the right of (2.5) can be interpreted as the net present value of all coupons paid up til t or the default time H, whichever is sooner. The second term is the net present value of the principal repayment, if this occurs before bankruptcy, and so the final term must be the net present value of what is recovered upon bankruptcy, if this happens before maturity. Indeed, $V(H)$ is the value of the firm's assets when bankruptcy occurs and $(1-\alpha)V(H)$ is the value that remains after bankruptcy costs are deducted. Of this, the bondholder with face value 1 gets the fraction $1/P$, since his debt represents this fraction of the total debt outstanding. Notice that if the process V were continuous, then $V(H)$ would simply be the bankruptcy level V_B, but as we shall be allowing the possibility that V has jumps, $V(H)$ may be below V_B.

The total value at time 0 of all debt outstanding is

$$\begin{aligned} D(V, V_B) &= \int_0^\infty p\,\Phi(t)\, d_0(V, V_B, t)\, dt \\ &= p\rho E\Big[\int_0^H e^{-rs}\bar{\Phi}(s)ds\Big] + pE\Big[\int_0^H e^{-rs}\Phi(s)ds\Big] \\ &\qquad + \frac{(1-\alpha)p}{P} E\Big[V(H)e^{-rH}\bar{\Phi}(H)\Big]. \end{aligned} \tag{2.6}$$

Here $\bar{\Phi}(t) = \int_t^\infty \Phi(s)ds$. Using the assumed form (2.4) of the maturity profile, we have $\Phi(t) = m\bar{\Phi}(t) = \exp(-mt)$, and so the total value of the debt simplifies to

$$D(V, V_B) = \frac{\rho P + mP}{m+r} E\Big[1 - e^{-(m+r)H}\Big] + (1-\alpha)E\Big[V(H)e^{-(m+r)H}\Big]. \tag{2.7}$$

This expression for the debt involves parameters of the problem, as well as two expectations. If the value process V were a log-Brownian motion, as in the earlier papers of Leland and Leland & Toft, then these expectations could be written in closed form: see Appendix C. For our model, there will be no closed-form solution, but we may nevertheless make progress, as we shall see in Section 3.

Write $C \equiv \rho P$ for the total coupon rate. We shall assume that there is a corporate tax rate τ, and the coupons paid can be offset against tax. The effect of this is to generate an income stream of $\tau C dt$ for the firm. It has to be realised that this is a rather idealised treatment of tax, but is better than ignoring the effect altogether. Leland & Toft (1996) also introduce a tax cutoff level V_T, whose effect is that the tax rebates are 0 while $V < V_T$, and are $\tau C dt$ when $V \geq V_T$. Again this is an idealisation, but it is intended to reflect the idea that when the coupons exceed the profits, you are unable to reclaim the tax on the coupon payments in excess of the profits[4]. Without such a tax cutoff, the numerical values of the coupons become ridiculously high, with the firm in effect promising huge returns financed by tax rebates; the tax cutoff prevents this.

[4] In Section 4, we shall use a tax cutoff $V_T = C/\delta$, which is the value of V at which the disbursed profits exactly equal the coupons.

In this set-up, assuming no cutoff for tax rebates, the value of the firm is

$$v = v(V, V_B) = V + \frac{\tau C}{r} E\Big[1 - e^{-rH}\Big] - \alpha E\Big[V(H) e^{-rH}\Big], \tag{2.8}$$

made up of the value of the firm's assets plus the net present value of all tax rebates on the coupons up to bankruptcy[5], less the net present value of the losses on default. Once again, the expectations cannot be written in closed form in the general problem, although for the log-Brownian case of Leland & Toft (1996) this is possible. In terms of (2.8) and (2.7), we can simply express the value of equity of the firm as

$$Q(V, V_B) = v(V, V_B) - D(V, V_B). \tag{2.9}$$

It is this that the shareholders will attempt to maximise by choice of V_B.

As in Leland & Toft (1996), the optimality criterion is the 'smooth pasting' condition

$$\frac{\partial Q}{\partial V}(V_B, V_B) = 0 \tag{2.10}$$

It might appear that without any explicit expression for Q it will be impossible to solve (2.10) explicitly for V_B. However, in the class of models to be considered next, it turns out (surprisingly) that we *can* do so.

3 A class of models.

We return to the dynamics (2.1) for the value of the firm's assets, and make some explicit assumptions about the abstract martingale Z appearing there. We shall assume that Z is a Lévy process with *no upward jumps*, a so-called *spectrally negative* Lévy process. This means that we can express V as

$$V_t = V_0 \exp(X_t), \tag{3.1}$$

where X is a Lévy process started at 0 with no upward jumps. For more information on Lévy processes, see for example the book of Bertoin (1996), or Chapter VI of Rogers & Williams (2000). We restrict attention to spectrally negative Lévy processes for reasons of tractability; we have already discussed why this may in any case be an acceptable modelling hypothesis, and we note that the analysis of Leland (1994b) is dealt with as a special case of what we shall develop here.

As is well known, the moment generating function of a Lévy process has the form

$$E^0 \exp(z X_t) = \exp(t\psi(z)) \tag{3.2}$$

[5]This must be modified when there is a cutoff on tax rebates: see Appendix A.

for some function ψ which is analytic in the interior of its domain of definition [6]. Here and elsewhere, the notation E^x denotes an expectation under the condition that $X_0 = x$. In order to get the correct growth rate (2.1) we see that we must have

$$E^0 \exp(X_t) = \exp(t\psi(1)) = \exp((r-\delta)t), \tag{3.3}$$

so that $\psi(1) = r - \delta$.

Re-expressing the value of debt (2.7) and the value of the firm (2.8) in terms of X, we obtain

$$D(V, V_B) = \frac{C + mP}{m + r} E^0\left[1 - e^{-(m+r)H}\right] + (1-\alpha)VE^0\left[e^{X(H)-(m+r)H}\right] \tag{3.4}$$

and

$$v(V, V_B) = V + \frac{\tau C}{r} E^0\left[1 - e^{-rH}\right] - \alpha V E^0\left[e^{X(H)-rH}\right], \tag{3.5}$$

where $H = H_x \equiv \inf\{t : X(t) < x\}$ and $x = \log(V_B/V)$. Our problem now is to find the expectation terms in the non-Brownian situation we are dealing with.

Let us begin by discussing the calculation of the terms of the form $E^0(\exp(-\lambda H_x))$; we shall return to the terms of the form $E^0(\exp(X(H_x) - \lambda H_x))$ later.

To tackle this, we introduce the familiar device of an independent random time T with an exponential(λ) distribution, so that

$$E^0\left[e^{-\lambda H_x}\right] = P^0\left[T > H_x\right] = P^0\left[\underline{X}(T_\lambda) < x\right], \tag{3.6}$$

where $\underline{X}(t) \equiv \inf\{X(u) : u \le t\}$ and $\bar{X}(t) \equiv \sup\{X(u) : u \le t\}$ Thus finding the first expectations in (3.4) and (3.5) reduces to computing the law of $\underline{X}(T_\lambda)$, which is the business of the classical *Wiener-Hopf factorisation* of a Lévy process. This states that

$$\begin{aligned} E^0 e^{zX(T_\lambda)} &= \frac{\lambda}{\lambda - \psi(z)} \\ &= \psi_\lambda^+(z).\psi_\lambda^-(z) && (3.7) \\ &= E^0\left[\exp(z\bar{X}(T_\lambda))\right].E^0\left[\exp(z\underline{X}(T_\lambda))\right]. && (3.8) \end{aligned}$$

See any of Bingham (1974), Bertoin (1996), Rogers & Williams (2000), Greenwood & Pitman (1980) for more on the Wiener-Hopf factorisation. The factorisation relates the known quantity $\lambda/(\lambda - \psi(z))$ to two unknowns, so is in general hard to make use of. However, our special situation of a spectrally negative Lévy process has the special feature that *the law of* $\bar{X}(T_\lambda)$ *is exponential*; see, for example, Bertoin (1996) for an explanation. Thus we know that for some $\beta^* \equiv \beta^*(\lambda)$ we shall have

$$\psi_\lambda^+(z) \equiv E^0\left[e^{z\bar{X}(T_\lambda)}\right] = \frac{\beta^*}{\beta^* - z}. \tag{3.9}$$

[6] In general, the domain of definition is the imaginary axis, but there are many examples (such as those we shall be studying) where the domain of definition is a strip or half-space. See the above references for more information.

Thus

$$\psi_\lambda^-(z) = \frac{\lambda}{\lambda - \psi(z)} . \frac{\beta^* - z}{\beta^*}. \tag{3.10}$$

Since this function is analytic in the right half-plane, it must be that the apparent pole which occurs when $\lambda = \psi(z)$ is in fact cancelled out by a zero in the numerator, which is to say that the unknown value of $\beta^* = \beta^*(\lambda)$ must be the solution to

$$\psi(\beta) = \lambda. \tag{3.11}$$

The conclusion then is that we may find the Wiener-Hopf factor $\psi_\lambda^-(z)$ explicitly in this example, at least up to solution of the equation (3.11). For the examples which we consider numerically, this can all be handled very easily.

Notice we want the distribution function of the random variable $\underline{X}(T_\lambda)$ evaluated at the point x, and the Wiener-Hopf factorisation only gives us the Laplace transform of the law of $\underline{X}(T_\lambda)$; this requires us then to invert the Laplace transform numerically to get the desired answer. This is a non-trivial numerical problem. The obvious approach (using Fast Fourier Transform) turned out to be insufficiently stable and accurate, so we turned to a method of Abate & Whitt (1992) and Hosono (1984), which worked very well.

This deals with the terms of the form $E(\exp(-\lambda H_x))$, but what about the terms of the form $E(\exp(X(H_x) - \lambda H_x))$? One of the many fluctuation identities for Lévy processes discovered over the years states that for any $\mu > 0$ and $\theta \geq 0$

$$\int_{-\infty}^0 \mu e^{\mu x} \, E^0 \Big[\exp(\theta X(H_x) - \lambda H_x) \Big] \, dx = \frac{\psi_\lambda^-(\theta) - \psi_\lambda^-(\theta + \mu)}{\psi_\lambda^-(\theta)} \; ; \tag{3.12}$$

see, for example, Bingham (1975). Taking $\theta = 1$ in this identity, we see that technically the terms of the form $E^0(\exp(X(H_x) - \lambda H_x))$ are actually no more difficult than the terms of the form $E^0(\exp(-\lambda H_x))$; to recover what we want, we need to do a Laplace transform inversion of a known function. Notice in passing that taking $\theta = 0$ in (3.12) gives us back the definition of ψ_λ^-. We shall often have need of the following simple variant of (3.12):

$$\int_0^\infty \mu e^{-\mu x} \, E^x \Big[\exp(\theta X(H_0) - \lambda H_0) \Big] \, dx = \frac{\mu}{\mu - \theta} \left[1 \; - \; \frac{\psi_\lambda^-(\mu)}{\psi_\lambda^-(\theta)} \right] \tag{3.13}$$

which is obtained from (3.12) by a little elementary calculus.

Using the theory of Wiener-Hopf factorisation, we can now compute the value of the firm (3.5) and the value of the debt (3.4), *provided* we know what value of V_B to use. This value is determined by the smooth pasting condition (2.10), and it is a stroke of good fortune that the value of V_B *can be computed in closed form*! Defining

$$\begin{aligned} \varphi(x, \lambda) &\equiv 1 - E^x e^{-\lambda H_0} \\ \gamma(x, \beta, \lambda) &\equiv 1 - E^x e^{\beta X(H_0) - \lambda H_0}, \end{aligned}$$

where E^x denotes expectation for the Lévy process started at x, we have alternative expressions for D and v:

$$D(V, V_B) = \frac{C + mP}{m + r}\,\varphi(x, m + r) + (1 - \alpha)V_B[1 - \gamma(x, 1, m + r)], \tag{3.14}$$

$$v(V, V_B) = V_B e^x + \frac{\tau C}{r}\,\varphi(x, r) - \alpha V_B[1 - \gamma(x, 1, r)], \tag{3.15}$$

where we write $x = \log(V/V_B)$. Now the smooth pasting condition (2.10) expresses itself as

$$\left.\frac{\partial(v - D)}{\partial x}\right|_{x=0} = 0,$$

so if we differentiate (3.14) and (3.15), and solve for V_B, we obtain

$$V_B = \frac{\frac{C+mP}{m+r}\,\varphi'(0, m + r) - \frac{\tau C}{r}\,\varphi'(0, r)}{1 + \alpha\gamma'(0, 1, r) + (1 - \alpha)\gamma'(0, 1, m + r)}. \tag{3.16}$$

Here, the dashes on φ and γ denote differentiation with respect to the first argument. The expression (3.16) can only be said to be explicit if we have explicit expressions for the derivatives of φ and γ, but this we have, as follows. Firstly, note that both φ and γ vanish at zero, so near 0 we have $\varphi(x, \lambda) \sim x\varphi'(0, \lambda)$, $\gamma(x, \theta, \lambda) \sim x\gamma'(0, \theta, \lambda)$. Thus

$$\begin{aligned}
\mu\psi_\lambda^-(\mu) &\equiv \mu E^0\left[e^{\mu \underline{X}(T_\lambda)}\right] \\
&= \int_{-\infty}^0 \mu^2 e^{\mu x} P^0\left[\underline{X}(T_\lambda) > x\right] dx \\
&= \int_{-\infty}^0 \mu^2 e^{\mu x} E^0\left[1 - e^{-\lambda H_x}\right] dx \\
&= \int_0^\infty \mu^2 e^{-\mu x} \varphi(x, \lambda) dx \\
&\to \varphi'(0, \lambda) \qquad (3.17) \\
& \qquad\qquad (3.18)
\end{aligned}$$

as $\mu \to \infty$. Thus we can identify the derivative of φ at 0 as

$$\varphi'(0, \lambda) = \lim_{\mu \to \infty} \mu\psi_\lambda^-(\mu) \tag{3.19}$$

Similarly, if we notice that for $x > 0$

$$\gamma(x, \beta, \lambda) = 1 - E^0 \exp(\beta X(H_{-x}) - \lambda H_{-x} + \beta x),$$

then we may express (taking $\mu > \beta$)

$$\int_0^\infty \mu^2 e^{-\mu x} \gamma(x, \beta, \lambda) dx = \int_0^\infty \mu^2 e^{-\mu x}\left[1 - E^x e^{\beta X(H_0) - \lambda H_0}\right] dx$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} \mu^2 e^{\mu x}\Big[1 - E^0 \exp(\beta X(H_x) - \lambda H_x - \beta x)\Big]dx \\
&= \mu - \frac{\mu^2}{\mu-\beta}\int_{-\infty}^{0} (\mu-\beta)e^{(\mu-\beta)x} E^0 \exp(\beta X(H_x) - \lambda H_x) dx \\
&= \mu - \frac{\mu^2}{\mu-\beta}.\frac{\psi_\lambda^-(\beta) - \psi_\lambda^-(\mu)}{\psi_\lambda^-(\beta)} \\
&= \frac{-\mu\beta}{\mu-\beta} + \frac{\mu^2}{\mu-\beta}.\frac{\psi_\lambda^-(\mu)}{\psi_\lambda^-(\beta)} \\
&\to -\beta + \frac{\varphi'(0,\lambda)}{\psi_\lambda^-(\beta)}
\end{aligned}
$$

as $\mu \to \infty$. Hence

$$\gamma'(0,\beta,\lambda) = \frac{\varphi'(0,\lambda)}{\psi_\lambda^-(\beta)} - \beta \tag{3.20}$$

Now the expressions (3.19) and (3.20) for the derivatives at zero of ϕ and γ can be evaluated once we have made some assumptions concerning the underlying Lévy process. We shall assume that our Lévy process is a Brownian motion with drift plus a compound Poisson process of downward jumps, so has the (Lévy-Khintchin) representation

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + bz + \int_{-\infty}^{0} (e^{zx} - 1)\nu(dx) \tag{3.21}$$

for some finite[7] measure ν on $\mathbb{R}^-$. Now by inspection of the expression (3.10) for the left Wiener-Hopf factor ψ_λ, we see from (3.21) that

$$\varphi'(0,\lambda) = \lim_{\mu\to\infty} \mu\psi_\lambda^-(\mu) = \frac{2\lambda}{\sigma^2\beta^*(\lambda)}, \tag{3.22}$$

which is an explicit expression. This allows us to compute the value (3.16) of V_B explicitly. More generally, we shall want to allow the possibility that the firm gets the tax rebates only while the value of its assets exceeds some threshold V_T (which we shall take to be C/δ, the value where all of the disbursed profits are going to pay the coupons). In this case (see Appendix A), the expression for the the value of the firm changes, and consequently the expression for the bankruptcy level changes slightly to

$$V_B = \frac{\frac{C+mP}{m+r}\,\varphi'(0,m+r) - \frac{\tau C}{r}\,e^{-\beta^*(r)b}\varphi'(0,r)}{1 + \alpha\gamma'(0,1,r) + (1-\alpha)\gamma'(0,1,m+r)}, \tag{3.23}$$

where $b = \max\{\log(V_T/V_B), 0\}$. We note that when $V_T \le V_B$ this reduces to (3.16). Note also that this is not an explicit expression for V_B, since V_B appears on the right-hand side, through the dependence on b. Nevertheless, if we *choose* a value for V_B, (3.23) exhibits a simple linear relation between C and P, and this is how we shall use it in what follows.

[7]This is almost the most general spectrally-negative Lévy process - see Bertoin (1996), for example.

A major feature of this modelling approach is that spreads do not go to zero as maturity goes to zero. Let us make precise what we mean, and what we do *not* mean by this. In the papers of Leland (1994) and Leland & Toft (1996), the credit spread is taken to be $C/P - r$. It is an aggregated credit spread for a dollar put into a sinking fund, and it depends on maturity through the parameter m; thus if we plot the spread as a function of m, we are actually comparing costs of borrowing for *completely different firms!* Such a plot is certainly of interest, but needs to be interpreted with care. We shall present some of these plots later, and also shall present plots of spreads against maturity for a single firm ABC plc with *fixed* m with varying maturity. These latter plots show the spread that should be demanded by someone lending one dollar to ABC for different fixed periods of time.

In more detail, by finding the value of $\rho = \rho^*$ for which the right-hand side of (2.5) equals 1 when $V = V_0$ and $t = T$, we find the spread $\rho^* - r$ for borrowing with fixed maturity T. Rearrangement of (2.5) yields the following expression for the spread on borrowing of maturity T:

$$\text{spread} = \frac{r}{P} \frac{E\Big[(P - (1-\alpha)V(H))e^{-rH} : H \leq T\Big]}{E\Big[1 - e^{-r(T \wedge H)}\Big]} \tag{3.24}$$

To understand the asymptotics of this as $T \downarrow 0$, we notice that for very small T we may ignore the contribution of the Brownian motion and the drift to the movement of the Lévy process X; if X has got down to the bankruptcy level $x = \log(V_B/V_0) < 0$ by time T, it is overwhelmingly likely to have got there by a jump. If we consider just the compound-Poisson part of X, then

$$P(H \leq T) = 1 - \exp(-T\nu(x)) + o(T)$$

as $T \downarrow 0$. Here, $\nu(x) \equiv \int_{-\infty}^{x} \nu(dy)$. The rationale for this is that a crossing of x before time T is overwhelmingly likely to have been by a jump, and the probability of more than one jump in $[0, T]$ is $o(T)$. Given that $H \leq T$, the law of $\log V(H)$ will be the law of a single jump conditioned to have gone below x, so

$$E[V(H)|H \leq T] \doteqdot \frac{\int_{-\infty}^{x} V_0 e^y \nu(dy)}{\nu(x)} \equiv \bar{V},$$

say. The denominator of (3.24) is easily seen to be asymptotic to rT as $T \downarrow 0$, so in conclusion

$$\text{spread} \to \frac{\nu(x)((P - (1-\alpha)\bar{V})}{P} \tag{3.25}$$

as $T \downarrow 0$.

To summarise: the limiting spread as $T \downarrow 0$ is non-zero; it has a value given by (3.25); and this can be checked against numerical results (see Section 4).

4 Numerical results.

As mentioned earlier, we shall assume that there is a cutoff level V_T such that when $V > V_T$ the firm gets tax rebates at rate τC, but when $V \leq V_T$ there is no tax rebate. Under this assumption, the value of the firm changes to

$$v(V_0, V_B) = V_0 - \alpha V_B[1 - \gamma(x, 1, r)] + \tau C g(x), \tag{4.1}$$

where the function g is given by

$$\begin{aligned} g(x) \;=\; & I_{\{b\leq 0\}}\varphi(x,r)/r + I_{\{b>0\}}\Big[\, I_{\{b\geq x\}} B e^{-\beta^* b}\,[\, e^{\beta^* x} - 1 + \gamma(x,\beta^*,r)\,] \\ & + I_{\{b<x\}}\,[\,\varphi(x-b,r)/r + B\{1 - \gamma(x-b,\beta^*,r) - e^{-\beta^* b}(1-\gamma(x,\beta^*,r))\}\,]\,\Big] \end{aligned} \tag{4.2}$$

where $b = \log(V_T/V_B)$, $x = \log(V_0/V_B)$, and $\beta^* \equiv \beta^*(r)$ for short. The constant B is given by

$$B = \frac{1}{\beta^*(r)\psi'(\beta^*(r))} = \frac{\psi_r^-(\beta^*(r))}{r}\;;$$

this and (4.2) are explained in Appendix A.

For all the computations, the values of certain parameters were held fixed: we took $\sigma = 0.2$, $r = 7.5\%$, $\delta = 7\%$, $\alpha = 50\%$, $\tau = 35\%$ and $V_0 = 100$, which are the values used by Leland (1994b) and Leland & Toft (1996), so as to aid comparison with those papers. We shall assume as in Leland & Toft (1996) (but not in Leland (1994b)) that $V_T = C/\delta$.

For our numerical examples, we assumed that the Lévy process X has Lévy exponent

$$\begin{aligned} \psi(z) &= \frac{1}{2}\sigma^2 z^2 + bz + a\int_{-\infty}^{0} ce^{cx}\,(e^{zx} - 1)\,dx \\ &= \frac{1}{2}\sigma^2 z^2 + bz - \frac{az}{c+z}, \end{aligned} \tag{4.3}$$

which means in sample-path terms that $X_t = \sigma W_t + bt + J_t$, where J is a compound Poisson process consisting of independent downward jumps with $\exp(c)$ distribution, arriving at the times of a Poisson process of rate a. The constant b is chosen to match the condition (3.3):

$$b = r - \delta - \sigma^2/2 + a/(1+c).$$

We performed the calculations for four cases. Case A was the diffusion case, with $a = 0$. In Case B, we took $a = 0.5$ and $c = 9$ (thus on average[8] once every two years the firm suffers an instantaneous loss of 10 % of its value), in Case C we took $a = 1$ and $c = 4$ (on average once a year the firm suffers an instantaneous loss of 20 %

[8] Relative to the pricing probability, of course.

of its value), and finally in Case D we took $a = 0.2$ and $c = 1$, so on average once every five years the firm instantaneously loses half its value. The values of m used varied from $m = 100$ to $m = 0.001$, so the mean of the maturity profile varied from about half a week to 1000 years.

The limiting value for the yield spread (3.25) can be given more explicitly, since $\nu(x) = ae^{cx}$ and

$$\bar{V} = \frac{cV_B}{1+c} .$$

The expression (3.25) becomes

$$\frac{ae^{-cx}(P - (1-\alpha)\bar{V})}{P} . \tag{4.4}$$

We present in Figure 1 a plot of firm value as a function of leverage for each of the cases. The plots are truncated below at firm value of 100, since we would presumably mainly be interested in leverages for which the firm value was high. The maximal value of debt varies from case to case: 111.6 for Case A, 110.4 for Case B, 106.4 for Case C, and 107.2 for Case D. The optimal leverage also varies; in each case, the optimal leverage rises as the mean maturity rises.

Figure 2 shows plots of the value of debt as a function of leverage for the four cases. The general shape of the curves is qualitatively similar to the plots in Leland (1994b) and Leland & Toft (1996), with longer mean debt maturity increasing the value of debt, and increasing leverage also increasing the value up to a point, but then decreasing the value, as the coupons have to be pushed up to induce investors to put money into the firm which is increasingly risky.

Next in Figure 3 we plot the value of equity over face value (defined as the initial value of the firm's assets $V_0 = 100$ less the initial value P of the bondholders' principal) against leverage. This is arguably what the shareholders would be most interested in when the company is set up, as it gives the factor by which their initial investment in the company is increased (as a result of the tax rebates, of course). The general shape is very similar to Figure 1. Notice how this varies substantially from case to case: 1.2471 in Case A, 1.2089 in Case B, 1.1048 in Case C and 1.1284 in Case D. The range of variability is much greater than the range in Figure 1 for the value of the firm. In all cases, the value of debt which maximises firm value is *less* than the value of debt which maximises the relatie value of equity.

Our next plots, Figure 4, show the value of debt as a function of coupon. As mean maturity increases, the value of debt generally rises, but in all cases for large enough coupon values there is a best maturity for the debt, and as the mean maturity passes above this the debt value falls back. This means that for the longer mean maturities there are often two coupon values which could fund the same level of debt.

One of the main conclusions of the paper is displayed in the next plot, Figure 5, of yield spreads as a function of maturity for a range of different values of leverage. We take leverage from 5% to 75% increasing in steps of 5%. Compare the diffusion case, Case A, with the other cases with jumps. Notice how the spreads are going quite rapidly to 0 with mean maturity in Case A, but are apparently tending to positive limits in the other cases. All of the plots exhibit the general types of behaviour found by Leland (1994b) and Leland & Toft (1996), confirming the empirical results of Sarig & Warga (1989); for firms with low levels of debt, spreads are small and increase with mean maturity, but as the level of debt rises the spread curve becomes humped. We can understand these differences as follows. For a firm with little debt, the bankruptcy level is a long way from V_0, so initally there is very little risk of bankrutcy; if bankruptcy occurs, it will only be after a considerable amount of time that the asset price process has got near to V_B. So the riskiness of such a firm will be low, but can be expected to rise with maturity. On the other hand, for a highly-levered firm, V_B is a lot closer to V_0, so debt is more risky, but why should we see the spread *falling* ultimately as maturity increases? The reason is that given that the firm has survived for the first 10 years, the value of the firm is very likely to have gone up; conditioning on survival effectively forces the firm value up, so that given that it has survived 10 years, it is much more likely then to continue to survive, since it is now at a much healthier asset value.

Our next plot, Figure 6, shows the yield spread for a dollar lent for different fixed periods, taking the debt maturity profile parameter $m = 1$. The analysis leading to these results requires some steps which may not be obvious at first sight, so we detail in Appendix B the argument on which the calculation of these figures is based. We see that the spread does not go to zero as maturity goes to zero in Cases B, C, and D, whereas the spread does go to zero in Case A, as predicted. But does the limit of the spread agree with (4.4)? In Tables I-III we present the theoretical limiting values of the spreads from (4.4) alongside the numerical values, produced by Richardson extrapolation of the two values of maturities closest to zero; the numerical agreement is good to a few percent. Bearing in mind the calculations required to arrive at these figures, this seems quite reasonable: the spread is computed (3.24) as the ratio of two numbers both of which (for small T) will be quite close to 0, and each of these in turn is the output of a numerical bivariate Laplace transform inversion. These values are then extrapolated.

The final plot, Figure 7, shows $C - \delta V_B$ against V_B. In each case, the lower the curve, the shorter the mean maturity. A situation where $C - \delta V_B > 0$ is one where the firm will continue past the point where all the cashflow from the firm is going to pay bondholders' coupons; this is rational for the shareholders, because they can reasonably expect that the value of the firm will recover in due course and they will then receive more dividends (and tax rebates). However, this is only rational if there is enough time for such a recovery to take place, in other words, if the mean maturity m^{-1} is large enough. This is exactly what the plots show. Note that this in contrast to the analysis of Kim, Ramaswamy & Sundaresan (1993), who study the situation where the shareholders will declare bankruptcy immediately the cashflow from the firm is all used to pay coupons.

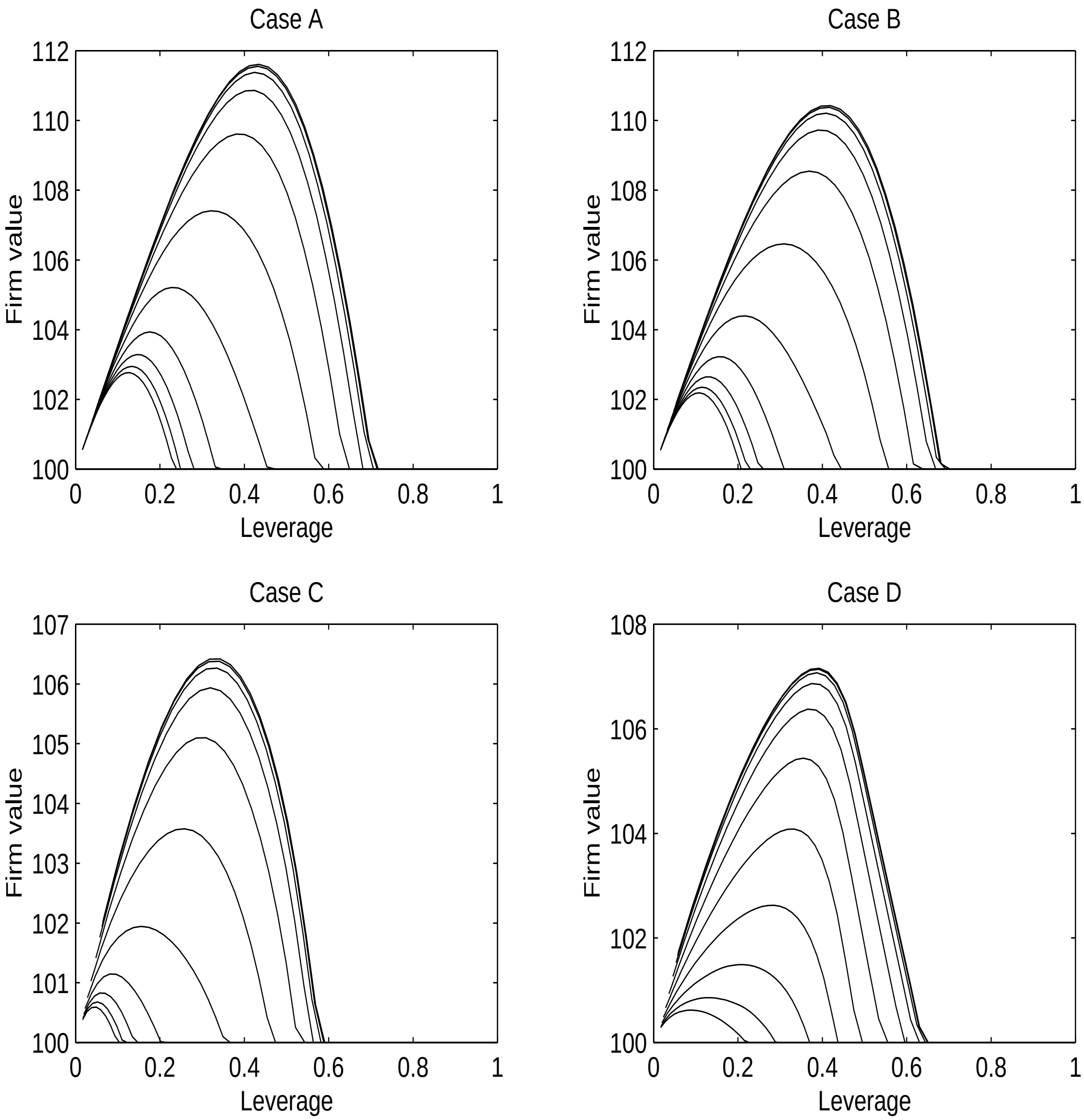

Figure 1: Firm value as a function of leverage. Higher curves correspond to higher values of mean time to maturity m^{-1}. Values for m^{-1} were 1000, 316, 100, 31.6, 10, 3.16, 1, 0.316, 0.1, 0.0316, 0.01.

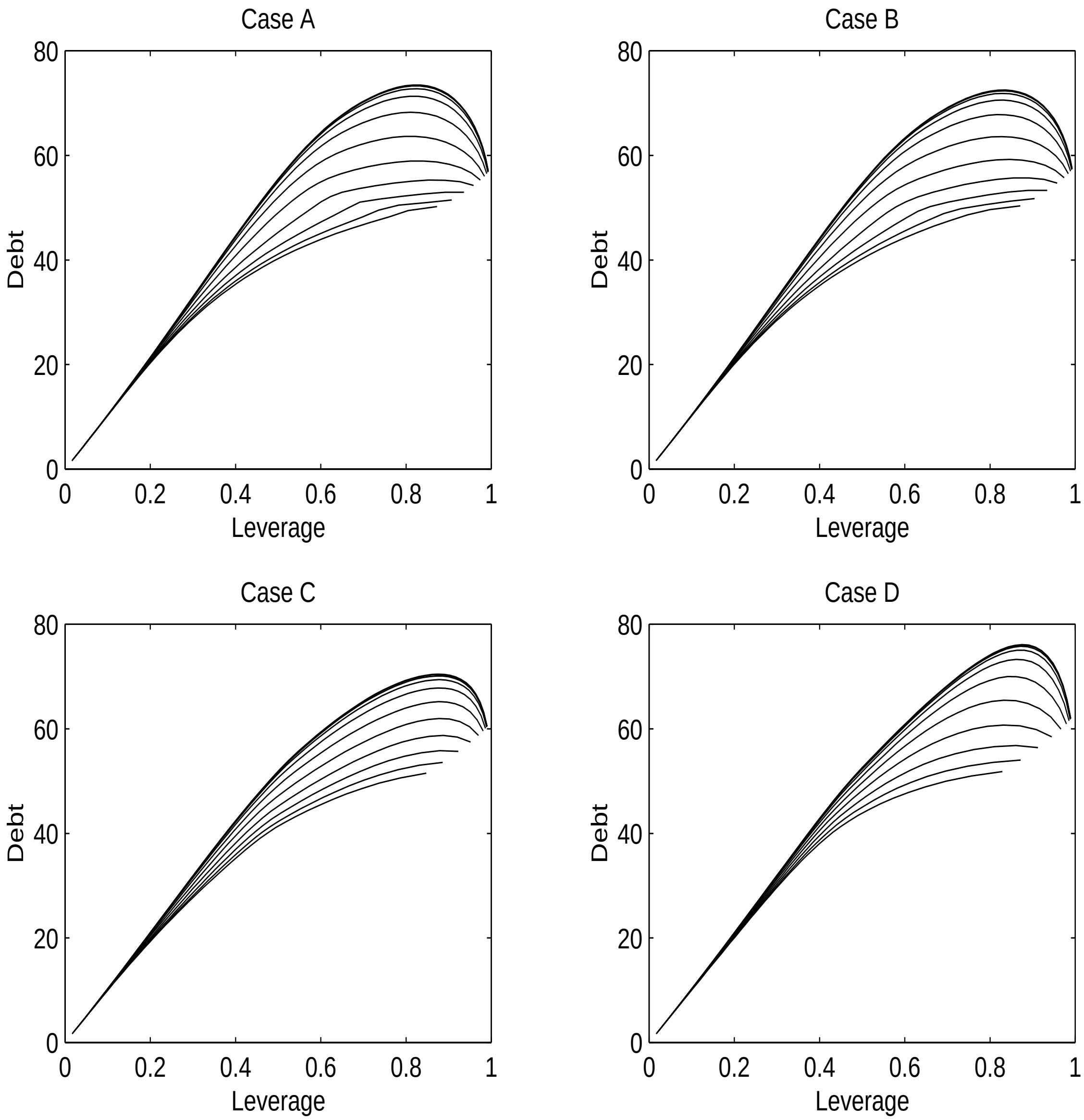

Figure 2: Debt as a function of leverage. Higher curves correspond to higher values of mean time to maturity m^{-1}.

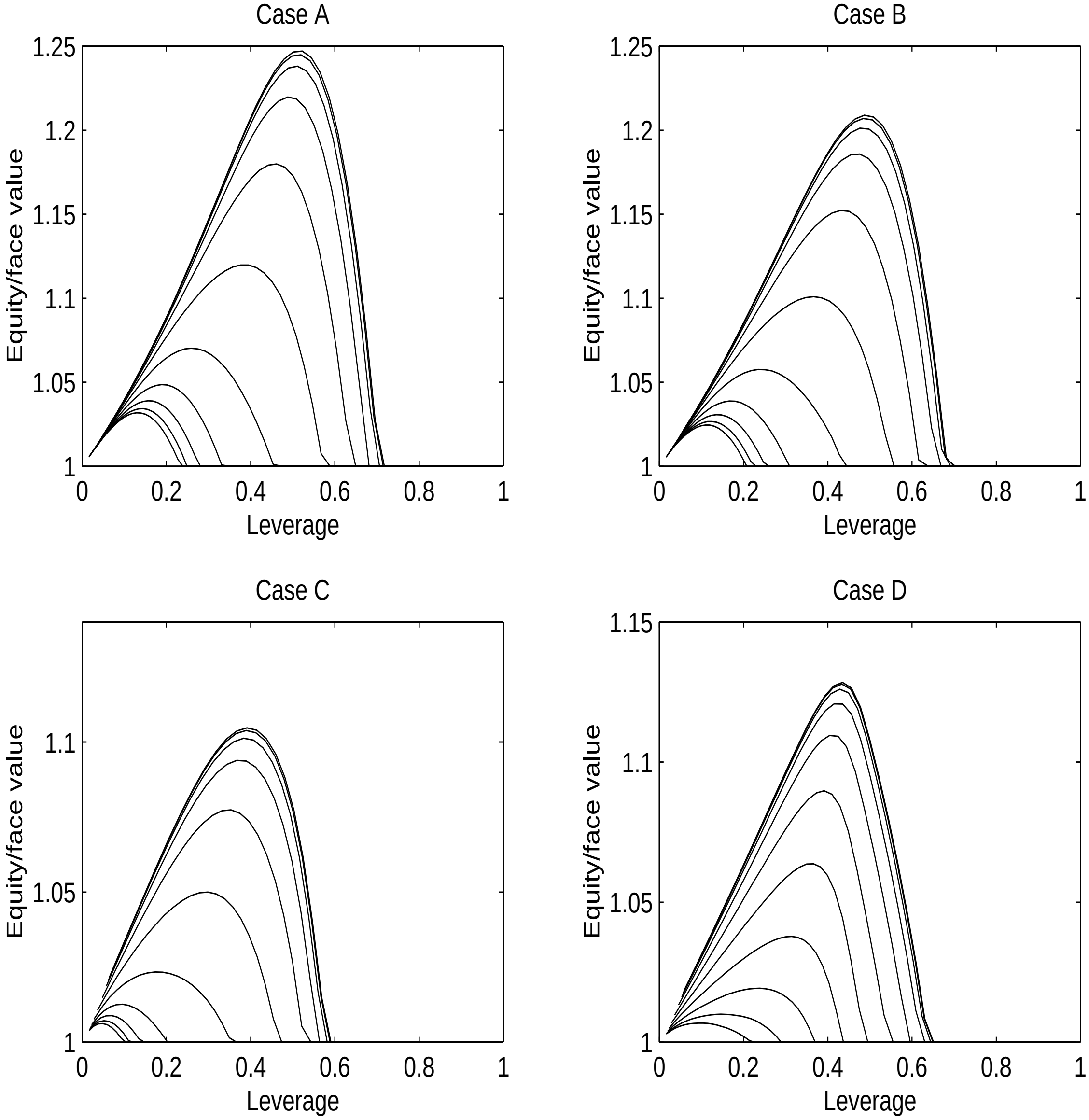

Figure 3: Equity/(face value) as a function of leverage. Higher curves correspond to higher values of mean time to maturity m^{-1}.

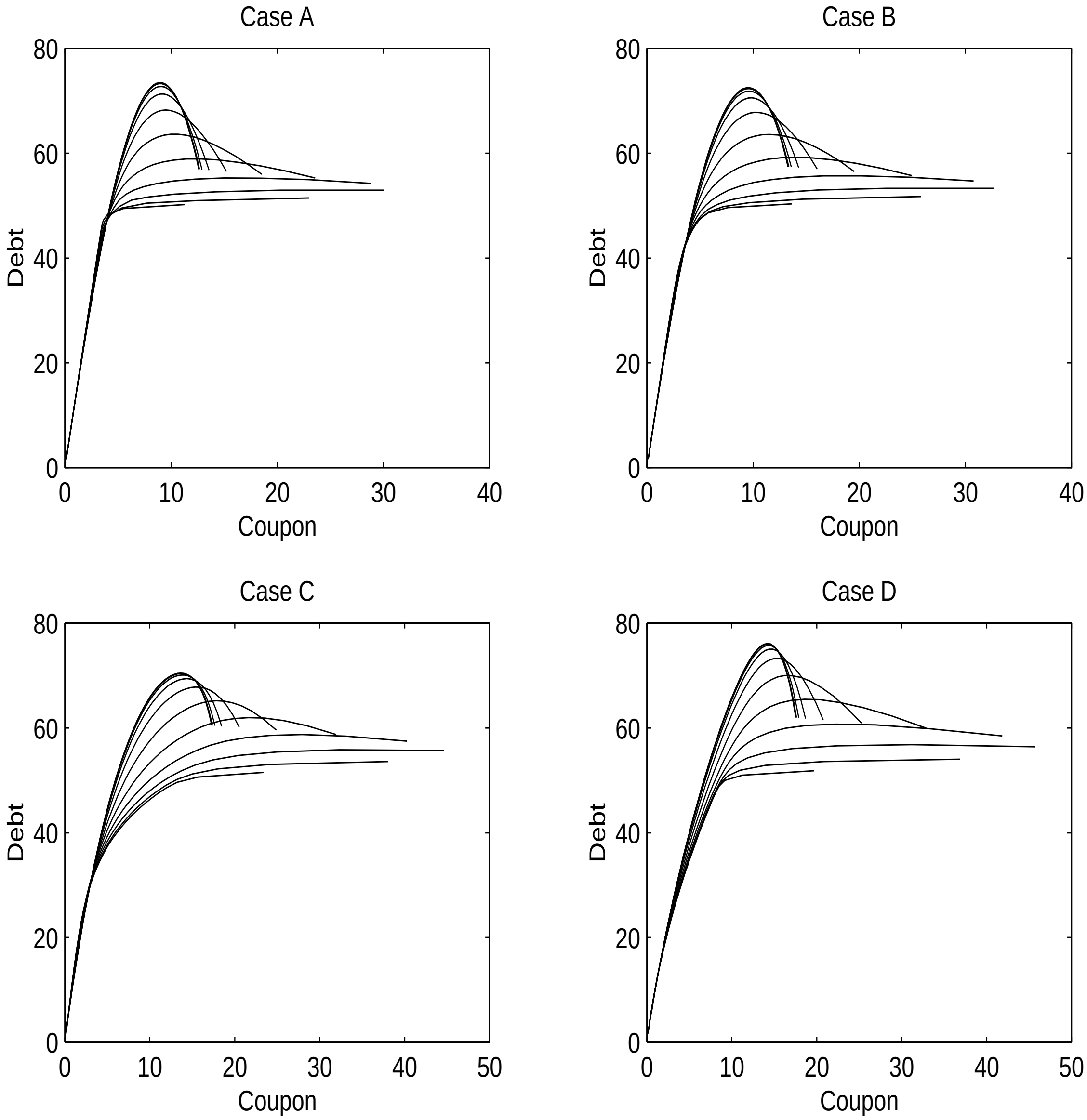

Figure 4: Debt as a function of coupon. Higher values of mean time to maturity m^{-1} give larger values of debt.

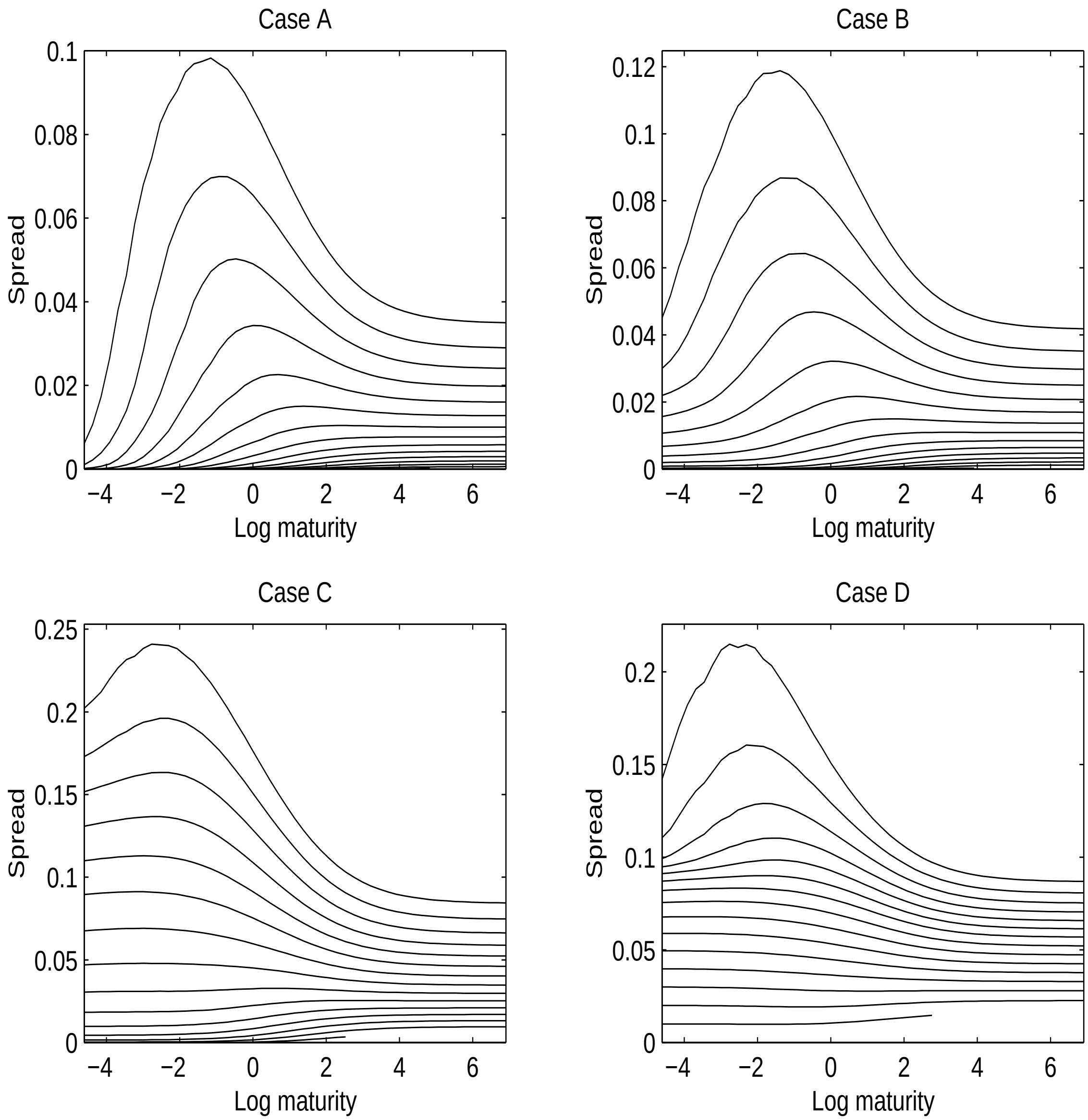

Figure 5: Spread as a function of log-maturity, for different values of leverage, running from 5% to 75% in steps of 5%. The higher the leverage, the higher the spread.

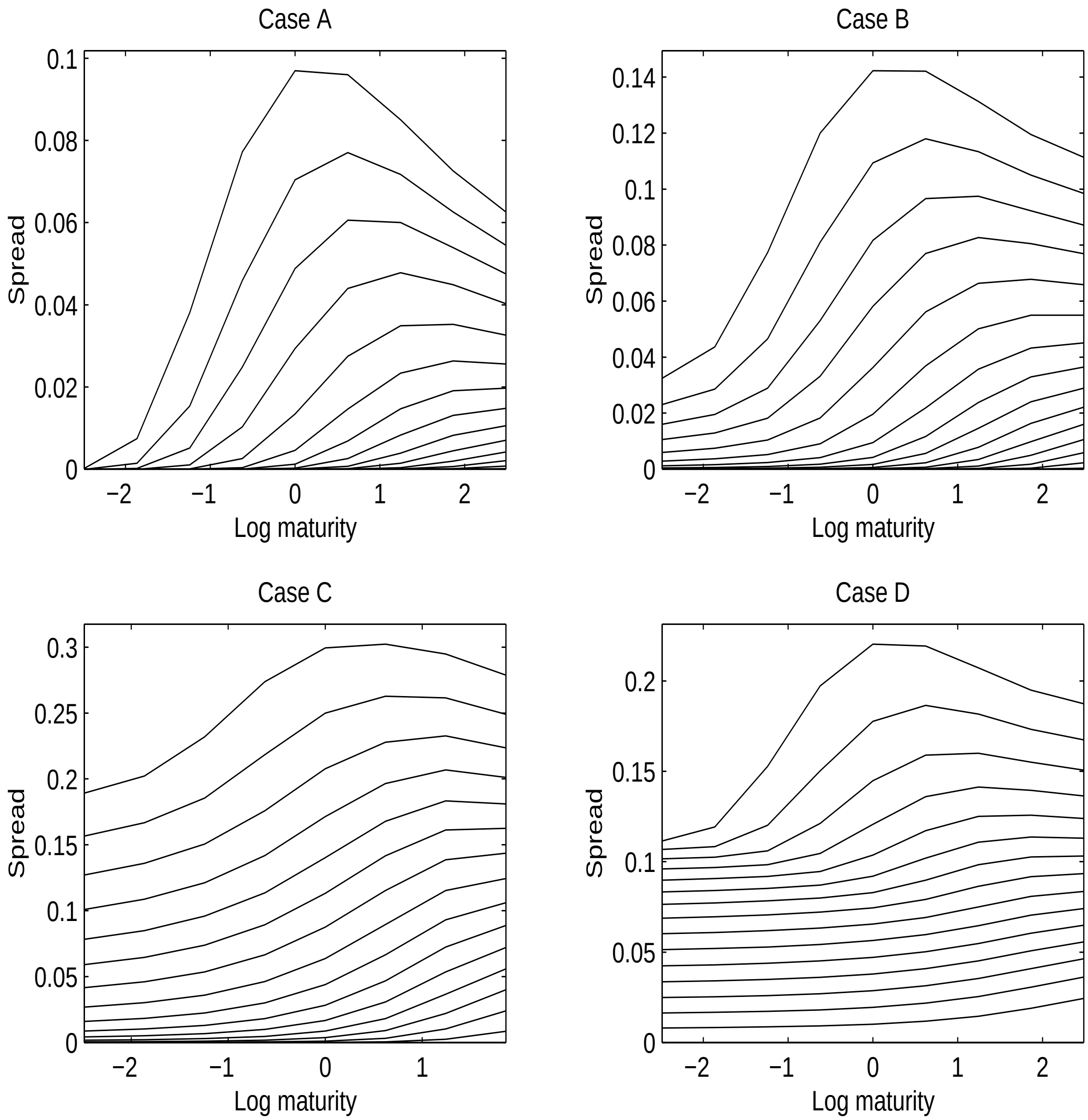

Figure 6: Spread as a function of log-maturity, for different values of leverage, running from 5% to 75% in steps of 5%. The higher the leverage, the higher the spread. In contrast to Figure 5, we are taking a firm with debt maturity profile given by $m = 1$ and considering the spread which would be needed to induce an agent to lend to that firm for a fixed time period.

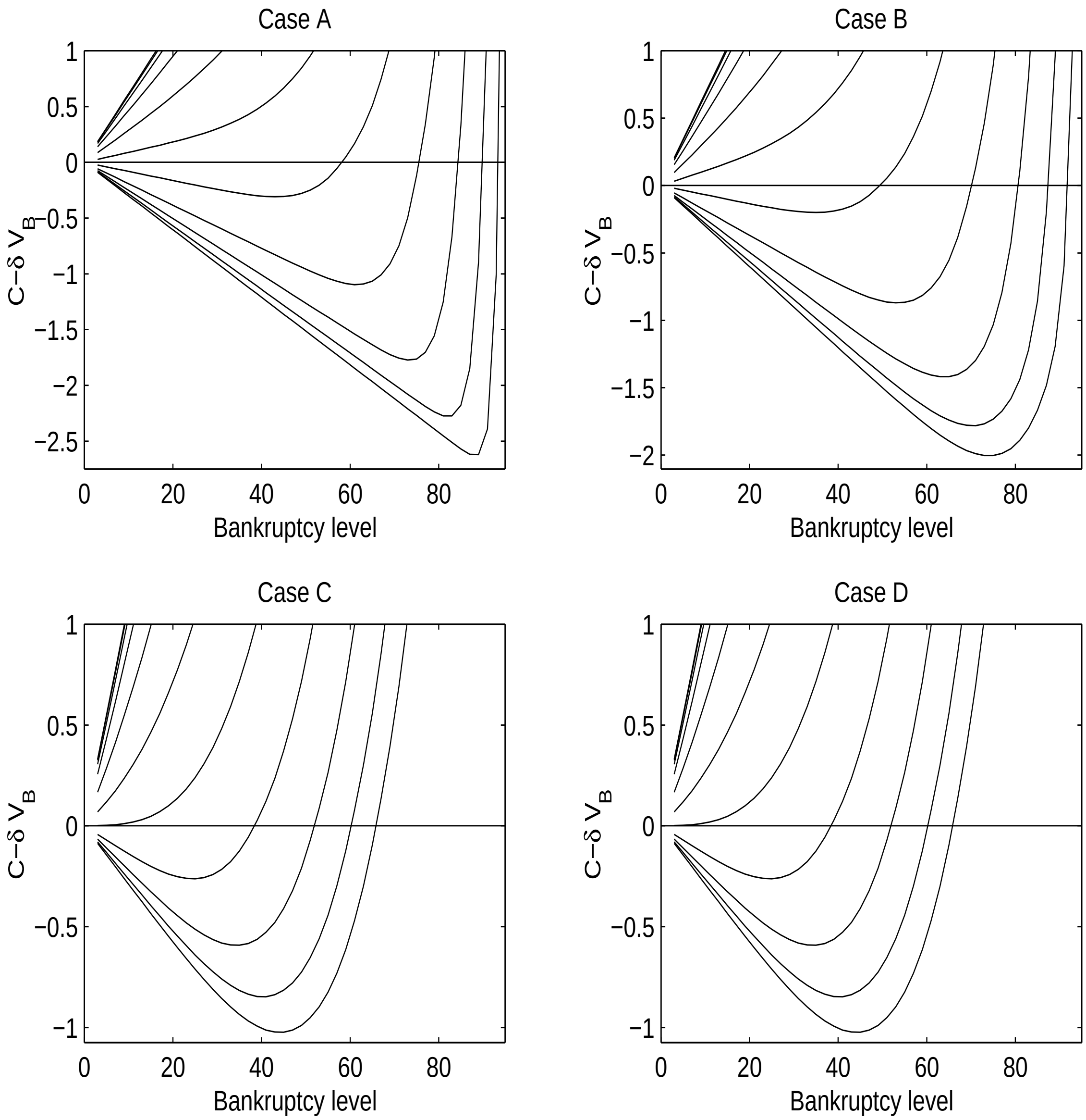

Figure 7: Plots of $C - \delta V_B$ against V_B for the four cases. The lowest curve in each case corresponds to the shortest mean maturity.

Table I: Comparison of the limiting spreads: Case B

Leverage	Limiting spread: extrapolated	Limiting spread: theoretical
5%	-0.0144	0.0000
10%	-0.0129	0.0000
15%	-0.0117	0.0000
20%	-0.0113	0.0001
25%	-0.0109	0.0006
30%	-0.0084	0.0031
35%	0.0002	0.0113
40%	0.0238	0.0331
45%	0.0823	0.0872
50%	0.2113	0.2076
55%	0.4528	0.4362
60%	0.8144	0.7858
65%	1.2445	1.2175
70%	1.7515	1.7976
75%	2.1171	2.5797

Table II: Comparison of the limiting spreads: Case C

Leverage	Limiting spread: extrapolated	Limiting spread: theoretical
5%	-0.0137	0.0005
10%	-0.0043	0.0080
15%	0.0326	0.0412
20%	0.1316	0.1320
25%	0.3434	0.3266
30%	0.7342	0.6866
35%	1.3819	1.2872
40%	2.3625	2.2065
45%	3.7159	3.4939
50%	5.3421	5.0668
55%	7.1592	6.8551
60%	9.3157	9.0140
65%	11.8176	11.5609
70%	14.6481	14.4931
75%	17.6454	17.7739

Table III: Comparison of the limiting spreads: Case D

Leverage	Limiting spread: extrapolated	Limiting spread: theoretical
5%	0.7831	0.7860
10%	1.6073	1.5976
15%	2.4529	2.4312
20%	3.3157	3.2829
25%	4.1921	4.1490
30%	5.0767	5.0237
35%	5.9588	5.8961
40%	6.8110	6.7391
45%	7.5748	7.4948
50%	8.2568	8.1696
55%	8.9044	8.8108
60%	9.5137	9.4143
65%	10.0731	9.9727
70%	10.5096	10.4770
75%	10.3578	10.9195

5 Conclusions and discussion.

We have taken the model of Leland (1994b) of a firm with constant debt structure, and extended it by incorporating downward jumps in the value of the firm's assets. The reason for doing this was to modify one undesirable feature of Leland's model, namely, the fact that yield spreads tend to zero as maturity tends to zero, which is at variance with observation. The problem was correctly diagnosed as arising from the diffusion assumption in Leland's model, and on introducing the possibility of downward jumps in the asset value, the zero limiting spreads vanish. Indeed, we are able to present an explicit expression for the limit of the spread, and confirm this by numerical examples. A further advantage of our approach is that it introduces the possibility of random loss on default, in a very simple way.

It is worth commenting on our choice of maturity profile, which we took to be exponential, as in Leland (1994b). Even for this simple profile, there are no closed-form solutions for most of the quantities of interest, and we have to resort to numerical methods. This being the case, it is now in principle and in practice no more difficult to let the maturity profile be some linear combination of exponentials of different rates. This then allows much more general maturity profiles, even to the point of approximating an arbitrary maturity profile to any desired degree of closeness. This is a project which might be pursued in the future (and would allow us to extend the work of Leland & Toft (1996) with fixed maturity of debt at issue to the jumping case); however, the assumption of the constant maturity profile, so necessary for the stationary nature of the optimal bankruptcy rule, is one which is unlikely to be even approximately true in practice, so detailed study of extensions which retain this assumption seems a lesser goal. We should regard the product of this present work more as insight into qualitative features of corporate bond yield structure than as a modelling tool for real data.

The different behaviour of the yield spreads at zero is the principal point of difference between Leland's conclusions and our own. In other respects, the results are qualitatively similar, as the various graphs show; firm value increases with maturity, and first increases and then ultimately decreases with leverage: the value of debt grows with mean maturity, and is initially increasing with leverage, though eventually (at least for large enough maturities) the curve turns down again and the value of debt falls (the 'junk bond' effect).

It should be possible to extend the current analysis to more than one class of debt, but this remains for the future. Another issue would be to attempt to characterise optimal bankruptcy policy for a firm which did not satisfy the constant maturity profile assumption, for example, for a firm which issued a tranche of fixed-dated bonds at time 0, and then at maturity redeemed the bonds and issued another tranche. Such an analysis for our modelling framework would require a study of the joint time and place of first crossing a moving barrier for a Lévy process, and can be expected to become a numerical task almost immediately.

Appendix A

As was explained earlier, in the expression (3.15) for the value of the firm, we want to be able to deal with the situation where there are tax rebates only when the value of the firm's assets exceed $V_T \equiv V_B e^b$. This requires us to replace the term $\tau C\varphi(x,r)/r$ in the right-hand side of (3.15) with $\tau C g(x)$, where

$$g(x) \equiv E^x \int_0^{H_0} e^{-rt} I_{\{X_t \geq b\}} dt. \tag{A.1}$$

We now derive a simpler expression for this. To begin with, let

$$B \equiv E^0 \int_0^{\infty} e^{-rt} I_{\{X_t \geq 0\}} dt. \tag{A.2}$$

We deal with two cases, assuming firstly that $0 < x \leq b$. If we let $H_b \equiv \inf\{t : X(t) > b\}$, we notice that

$$E^x \exp(-rH_b) = P^x[T_r > H_b] = P^x[\bar{X}(T_r) > b] = \exp(-(b-x)\beta^*(r)),$$

and so

$$E^x \int_0^{\infty} e^{-rt} I_{\{X_t \geq b\}} dt = B\exp(-(b-x)\beta^*(r)).$$

Thus for $0 < x \leq b$ we have (abbreviating $\beta^*(r)$ to β^*)

$$\begin{aligned} g(x) &= E^x \int_0^{\infty} e^{-rt} I_{\{X_t \geq b\}} dt - E^x \int_{H_0}^{\infty} e^{-rt} I_{\{X_t \geq b\}} dt && \text{(A.3)} \\ &= Be^{-\beta^*(b-x)} - BE^x \Big[\exp\{-\beta^*(b - X(H_0)) - rH_0\}\Big] \\ &\equiv Be^{-\beta^* b}[\ e^{\beta^* x} - 1 + \gamma(x, \beta^*, r)\ ]. && \text{(A.4)} \end{aligned}$$

To see this, note that the second integral in (A.3) does not start to grow until X first rises to level b (which it crosses continuously) *after* it first fell below level 0. If S denotes this time, then we have

$$E^x[e^{-rS}] = E^x\Big[\exp\{-\beta^*(b - X(H_0)) - rH_0\}\Big].$$

On the other hand, if $x > b$, then the integral in (A.1) grows until the first time H_b that X falls below b, and in this time it contributes $E^x[1 - \exp(-rH_b)]/r$ to $g(x)$. Thus

$$\begin{aligned} g(x) &= E^x\left[\frac{1 - \exp(-rH_b)}{r}\right] + E^x \int_{H_b}^{\infty} e^{-rt} I_{\{X_t \geq b\}} dt - E^x \int_{H_0}^{\infty} e^{-rt} I_{\{X_t \geq b\}} dt \\ &= \varphi(x-b, r)/r + B(E^x e^{-\beta^*(b - X(H_b)) - rH_b} - E^x e^{-\beta^*(b - X(H_0)) - rH_0}) \\ &= \varphi(x-b, r)/r + B(1 - \gamma(x-b, \beta^*, r) - e^{-\beta^* b}(1 - \gamma(x, \beta^*, r))) && \text{(A.5)} \end{aligned}$$

Notice that as we let $b \downarrow 0$, we recover the correct expression. Together, (A.4) and (A.5) determine g, but we shall now make the constant B more explicit. We have the celebrated Spitzer-Rogozin identity (see, for example, Bingham (1974)) that

$$\psi_\lambda^+(s) = \exp \int_0^\infty (e^{sx} - 1) \int_0^\infty \frac{e^{-\lambda t}}{t} P(X_t \in dx) dt,$$

and so

$$\begin{aligned} \frac{\partial}{\partial \lambda} \log \psi_\lambda^+(s) &= -\int_0^\infty (e^{sx} - 1) \int_0^\infty e^{-\lambda t} P(X_t \in dx) dt \\ &\to B \equiv \int_0^\infty e^{-\lambda t} P(X_t \geq 0) dt \end{aligned} \tag{A.6}$$

as $s \to \infty$. In this case, we have that $\psi_\lambda^+(s) = \beta^*(\lambda)/(\beta^*(\lambda) - s)$, and the derivative of β^* with respect to λ is just $1/\psi'(\beta^*(\lambda))$, so a few simple calculations from (A.6) give us that

$$B = \frac{1}{\beta^*(r)\psi'(\beta^*(r))} = \frac{\psi_r^-(\beta^*(r))}{r} \tag{A.7}$$

after a few more calculations.

We can now find the Laplace transform of g, using the identity (3.12) and the expression for g that has resulted from the preceding analysis. We find that

$$\int_0^\infty \mu e^{-\mu x} g(x) dx = \frac{\mu B}{\beta^* - \mu} . \left\{ e^{-\mu b} - e^{-\beta^* b} \right\} . \frac{\psi_r^-(\mu)}{\psi_r^-(\beta^*)} + e^{-\mu b} \frac{\psi_r^-(\mu)}{r} \tag{A.8}$$

after some calculations. Since $g(0) = 0$, we can deduce the derivative of g at 0 from (A.8) by multiplying by μ and letting $\mu \uparrow \infty$. Using (3.22), we conclude that the derivative of g at zero is

$$\frac{2rBe^{-\beta^* b}}{\sigma^2 \beta^* \psi_r^-(\beta^*)} = \frac{2e^{-\beta^* b}}{\sigma^2 \beta^*} \tag{A.9}$$

We could equally well have deduced this directly from (A.4) using (3.20).

Let us now see how this allows us to derive the alternative form (3.23) for the bankruptcy level V_B. We now know that for $0 < x < b$ the expression for the value of the firm is

$$v(V_B e^x, V_B) = V_B e^x - \alpha V_B \{1 - \gamma(x, 1, r)\} + \tau C g(x)$$

and the derivative of this at zero is easily seen to be

$$V_B + \alpha V_B \gamma'(0, 1, r) + \frac{2\tau C e^{-\beta^* b}}{\sigma^2 \beta^*} . \tag{A.10}$$

On the other hand, the equation (3.14) for the debt remains unaltered by the introduction of a tax threshold, and the derivative of $D(V_B e^x, V_B)$ at zero is as before

$$\frac{C + mP}{m + r} \varphi'(0, m + r) - (1 - \alpha) V_B \gamma'(0, 1, m + r).$$

Solving the smooth pasting condition for V_B gives us (3.23).

Appendix B

Here we explain how Figure 6 was computed. To calculate one point on the graph, we choose a value for m, $t > 0$ and leverage L and firstly calculate the values of V_B, P, and C for which (3.23) holds, $D(V_0, V_B) = P$, and

$$L = \frac{P}{v(V_0, V_B)} \ ,$$

where $v(V_0, V_B)$ is given by (4.1) and (4.2). We next return to (2.5) and for our fixed $t > 0$ compute the value ρ^* of ρ for which $d_0(V_0, V_B, t) = 1$. With $x = \log(V_0/V_B)$, the expression (2.5) for $d_0(V_0, V_B, t)$ can be written as

$$\rho E^x \Big[\int_0^{t \wedge H} e^{-rs} ds \Big] + E^x \Big[e^{-rt} : t < H \Big] + \frac{1-\alpha}{P} V_B E^x \Big[e^{X(H) - rH} : H \leq t \Big],$$

where $H = \inf\{t : X_t \leq 0\}$. Taking the Laplace transform in t gives us

$$\int_0^\infty e^{-\lambda t} d_0(V_0, V_B, t) dt = \frac{\lambda + \rho}{\lambda(\lambda + r)} \ E^x \Big[1 - e^{-(\lambda + r)H} \Big] + \frac{(1-\alpha)V_B}{\lambda P} \ E^x \Big[e^{X(H) - (\lambda + r)H} \Big]$$

after some calculations. Now if we take the Laplace transform in x we have (using (3.13))

$$\begin{aligned} \tilde{d}_0(\lambda, \mu) &\equiv \int_0^\infty e^{-\mu x} dx \int_0^\infty e^{-\lambda t} dt \ d_0(V_B e^x, V_B, t) \\ &= \frac{\lambda + \rho}{\lambda \mu (\lambda + r)} \ \psi^-_{\lambda + r}(\mu) \ + \ \frac{(1-\alpha)V_B}{\lambda P(\mu - 1)} \Big[\ 1 - \frac{\psi^-_{\lambda + r}(\mu)}{\psi^-_{\lambda + r}(1)} \ \Big] \\ &\equiv \rho \tilde{f}_1(\lambda + r, \mu) + \tilde{f}_2(\lambda + r, \mu), \end{aligned} \qquad \text{(B.1)}$$

and our task now is to invert the two double Laplace transforms $\tilde{f}_1$ and $\tilde{f}_2$. To do this, we use a variant of the method of Abate & Whitt (1992) developed by Rogers (2000). The inversion formula

$$d_0(t, y) = \int_{\Gamma_1} \frac{dz}{2\pi i} \int_{\Gamma_2} \frac{d\mu}{2\pi i} \ e^{t(z-r) + \mu y} \ \tilde{d}_0(z - r, \mu)$$

is the starting point, for suitably chosen contours Γ_1 and Γ_2. Conventionally, these would both be shifts of the imaginary axis, but here there is advantage in considering other contours. The reason is because the expression (3.10) for $\psi^-_\lambda(\mu)$ involves computation of $\beta^*(\lambda) = \psi^{-1}(\lambda)$, and this inversion of the function ψ is not going to be easy to do except in very simple special cases, such as the diffusion case where

$$\psi(z) = \psi_0(z) \equiv \frac{1}{2}\sigma^2 z^2 + bz,$$

and the inverse is explicitly given as

$$\psi_0^{-1}(\lambda) = \frac{\sqrt{b^2 + 2\lambda\sigma^2} - b}{\sigma^2};$$

the choice of root is determined by the fact that ψ^{-1} is increasing on $\mathbb{R}^+$. However, we avoid this problem *if the contour* Γ_1 *is itself the image under* ψ *of some other contour,* and this is the key to the method of Rogers (2000). In fact, by taking $\Gamma_1 = \psi \circ \psi_0^{-1}(\Gamma_0)$, where Γ_0 is some translation of the imaginary axis, we ensure that the contour Γ_1 lies very close to Γ_0 far out (since ψ and ψ_0 are very similar for large arguments), and that the integrand can be evaluated simply. Indeed, if we abbreviate $g \equiv \psi \circ \psi_0^{-1}$, we have

$$d_0(t, y) = \int_{\Gamma_0} \frac{d\zeta}{2\pi i} \int_{\Gamma_2} \frac{d\mu}{2\pi i}\; g'(\zeta) e^{t(g(\zeta)-r)+\mu y}\, \tilde{d}_0(g(\zeta - r, \mu).$$

The double integral is now approximated by a double sum, where the function evaluations are made at a lattice of well-chosen points, as explained in Choudhury, Lucantoni & Whitt (1994). See Rogers (2000) for further details.

Appendix C

We present here the form of the general results derived in Section (3) in the special case where there are no jumps, and so the Lévy process X is of the form

$$X_t = \sigma W_t + \mu t = \sigma(W_t + ct), \tag{C.1}$$

with $\mu \equiv \sigma c = r - \delta - \sigma^2/2$. Of course, the expressions we derive should agree with those of Leland (1994) in the cases where Leland gives an expression. However, we want also some expressions which are not given in Leland (1994), such as the value of debt (2.5) of a given maturity t with face value 1. From (2.5) we obtain

$$\begin{aligned} d_0(V, V_B, t) &= \frac{\rho}{r} E\Big[1 - e^{-r(t\wedge H)}\Big] + e^{-rt} P(t < H) + \frac{(1-\alpha)V_B}{P} E\Big[e^{-rH} : H \le t\Big] \\ &= \frac{\rho}{r} + \Big(1 - \frac{\rho}{r}\Big) e^{-rt} P(H > t) + \Big(\frac{(1-\alpha)V_B}{P} - \frac{\rho}{r}\Big) E\Big[e^{-rH} : H \le t\Big]. \end{aligned} \tag{C.2}$$

Now the first-passage-time density for a Brownian motion with drift c to a level $a < 0$ is well known:

$$P(H_a \in dt)/dt = |a| \exp(-(a - ct)^2/2t)/\sqrt{2\pi t^3};$$

see, for example, Borodin & Salminen (1996). Slightly less well known is the expression

$$P(H_a \le t) = \Phi(\frac{a - ct}{\sqrt{t}}) + e^{2ca}\Phi(\frac{a + ct}{\sqrt{t}}) \tag{C.3}$$

for the distribution function of H_a, but this can be confirmed by differentiating with respect to t. Here, Φ is the $N(0,1)$ distribution function. We can also deduce from (C.3) that

$$\begin{aligned} E\Big[e^{-rH_a} : H_a \leq t\Big] &= \int_0^t e^{-rs}|a|\exp(-(a-cs)^2/2s)/\sqrt{2\pi s^3}ds \\ &= \int_0^t e^{a(c-\tilde{c})}|a|\exp(-(a-\tilde{c}s)^2/2s)/\sqrt{2\pi s^3}ds \\ &= e^{a(c-\tilde{c})}\Big[\Phi(\frac{a-\tilde{c}t}{\sqrt{t}}) + e^{2\tilde{c}a}\Phi(\frac{a+\tilde{c}t}{\sqrt{t}})\Big], \end{aligned}$$

using (C.3), where $\tilde{c} \equiv \sqrt{c^2+2r}$. This now gives us an explicit expression

$$\begin{aligned} d_0(V,V_B,t) &= \frac{\rho}{r} + \Big(1-\frac{\rho}{r}\Big)e^{-rt}\Big[1-\Phi(\frac{a-ct}{\sqrt{t}}) - e^{2ca}\Phi(\frac{a+ct}{\sqrt{t}})\Big] \\ &\quad + \Big(\frac{(1-\alpha)V_B}{P} - \frac{\rho}{r}\Big)e^{a(c-\tilde{c})}\Big[\Phi(\frac{a-\tilde{c}t}{\sqrt{t}}) + e^{2\tilde{c}a}\Phi(\frac{a+\tilde{c}t}{\sqrt{t}})\Big] \end{aligned} \tag{C.4}$$

where $a = \sigma^{-1}\log(V_B/V)$, and $c = \sigma^{-1}(r-\delta-\sigma^2/2)$. Notice that if we return to the expression (3.24) for the spread, we see a ratio whose numerator is $o(T)$, and whose denominator is $O(T)$, so in this case we see that the limit as $T \downarrow 0$ of the spread will be 0.

If we introduce the functions

$$\theta_-(z) \equiv \frac{\sqrt{\mu^2+2\sigma^2 z}+\mu}{\sigma^2}, \qquad \theta_+(z) \equiv \frac{\sqrt{\mu^2+2\sigma^2 z}-\mu}{\sigma^2}, \tag{C.5}$$

we have for $X_0 = 0$ and $x > 0$

$$E\exp(-\lambda H_{-x}) = \exp(-\theta_-(\lambda)x), \qquad E\exp(-\lambda H_x) = \exp(-\theta_+(\lambda)x). \tag{C.6}$$

These allow us to make explicit the expressions for the value of debt (2.7) and the firm value (2.8) (and therefore also the value of equity (2.9)) proved in Section 2; we obtain

$$D(V,V_B) = \frac{C+mP}{m+r}\Big[1-(V_B/V)^{\theta_-(m+r)}\Big] + (1-\alpha)V_B(V_B/V)^{\theta_-(m+r)} \tag{C.7}$$

$$v(V,V_B) = V + \frac{\tau C}{r}\Big[1-(V_B/V)^{\theta_-(r)}\Big] - \alpha V_B(V_B/V)^{\theta_-(r)} \tag{C.8}$$

in agreement with Leland (1994). For this special case, the Wiener-Hopf factorisation (3.8) takes the simple explicit form

$$\frac{\lambda}{\lambda-\psi(z)} = \frac{\theta_+(\lambda)}{\theta_+(\lambda)-z}\cdot\frac{\theta_-(\lambda)}{\theta_-(\lambda)+z}, \tag{C.9}$$

and $\beta^* = \theta_+$. The functions φ and γ are the same: $\varphi(x,\lambda) = \gamma(x,\beta,\lambda)$. Thus the expression (3.16) for V_B becomes

$$V_B = \frac{\frac{C+mP}{m+r}\,\theta_-(m+r) - \frac{\tau C}{r}\,\theta_-(r)}{1+\alpha\theta_-(r)+(1-\alpha)\theta_-(m+r)}\,, \tag{C.10}$$

again in agreement with Leland (1994). If we introduce a tax threshold, then the expression (3.23) for V_B simplifies to

$$V_B = \frac{\frac{C+mP}{m+r}\,\theta_-(m+r) - \frac{\tau C}{r}\,e^{-\beta^*(r)b}\theta_-(r)}{1+\alpha\theta_-(r)+(1-\alpha)\theta_-(m+r)}\,, \tag{C.11}$$

where $b = \log(V_T/V_B)$. This is not quite the same as the expression in Leland (1994), but the discrepancy is explained by an error in the analysis of Appendix B of that paper.

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