A new approach to the modelling and pricing of correlation credit derivatives

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Abstract

The modelling of credit events is in effect the modelling of the times to default of various names. The distribution of individual times to default can be calibrated from CDS quotes, but for more complicated instruments, such as CDOs, the joint law is needed. Industry practice is to model this correlation through a copula, an approach with significant deficiencies. We present a new approach to default correlation modelling, where defaults of different names are driven by a common continuous-time Markov process. Individual default probabilities and default correlations can be calculated in closed form. As illustrations, the prices of K^{th}-to-default baskets and CDO tranches with name-dependent random losses are computed using Laplace transform techniques.

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1 Introduction

The current industry approach to the pricing of multi-name credit derivatives makes use of copula functions to model the dependence between issuers in a given portfolio of defaultable securities. This approach is problematic for two main reasons: there is no dynamic consistency, and there is no theoretical basis for the choice of any particular dependence structure. The root cause of the problems is bad modelling - the dependence is forced into the model at the very last stage, rather than growing organically from the modelling assumptions.

What we propose here is a new approach to the problem based on the use of a Markov process within the reduced-form framework. This completely deals with the main problems of the current copula-based approach. Default correlation is determined from market data by fitting the model to CDS and portfolio derivative data.

We start by assuming there exists a process $(\xi_t)_{t\geq 0}$ which drives the common dynamics of the credits in the portfolio. We then model the survival probabilities up to time t of a given obligor, say i, conditional on the filtration generated by the process F_t^ξ as

$$P\left(\tau^i \geq t | F_t^\xi\right) = \exp\left(-C_t^i\right), \tag{1}$$

where τ^i indicates the default time of the i^{th} reference entity, and C^i is an additive functional of the process. The simplest thing[1] for this is to take

$$C_t^i = \int_0^t \lambda^i(\xi_u) du$$

where $\lambda^i(\xi)$ is a deterministic function of the chain, which we will refer to as the *(default) intensity (function)* of entity i. For simplicity, we shall limit our discussion to the case where $(\xi_t)_{t\geq 0}$ is a continuous-time finite-state Markov chain. This framework is already sufficiently flexible for practical purposes, and is simple enough to allow explicit computation using fast linear algebra routines.

Taking (1) as a starting point, it is easy to derive the individual conditional default probabilities in closed form. It is also straightforward to compute default correlations. Moreover, we show how to obtain a fast and reasonably accurate approximation to the price of CDO tranches and K^{th}-to-default baskets based on a Poisson approximation. Exact solutions can be obtained by computing the Laplace transform of the portfolio loss distribution and related quantities and then resorting to numerical inversion techniques.

[1]... but as we shall see, not the only thing ...

2 Model specification and basic results.

In this section we introduce the main modelling ideas of the paper which will form the basic building blocks for the pricing of multi-name credit derivatives.

Consider a portfolio of N defaultable securities and assume that there exist a continuous-time finite-state irreducible Markov chain $(\xi_t)_{t\geq 0}$ with Q-matrix Q, generating a filtration F_t^ξ. Assume that conditional on the path of the chain, defaults of the N names will be independent, the survival probability of the i^{th} reference entity being given by

$$q_t^i = P\left(\tau^i \geq t | F_t^\xi\right) = \exp\left(-C_t^i\right), \tag{2}$$

where C_t^i is some additive functional of the chain of the form

$$C_t^i = \int_0^t \lambda^i(\xi_u)du + \sum_{j\neq k} w_{jk}^i J_{jk}(t). \tag{3}$$

Here, τ^i is the default time of the i^{th} name in the portfolio, λ^i is a deterministic function of the chain, $J_{jk}(t)$ denotes the number of jumps by time t from state j to state k, and the w_{jk}^i are non-negative weights.

In order to gain some intuition, one could think of the chain as representing the *state of health* of the economy. If the chain jumps from a state of economic growth to a state of recession, this may cause the conditional default intensity of some of the reference entities to go up, increasing the chances of observing a larger number of defaults in the portfolio. The jump itself may also trigger defaults. Note that the information about how the various credits in the portfolio are correlated is contained in the λ^i, the w_{jk}^i, and Q. An expression for the dynamic default correlation will be derived in section 3.

Throughout this paper we will also assume that the money market account takes the following form

$$B_t = \exp\left(\int_0^t r(\xi_u)du\right), \tag{4}$$

where again r is a deterministic function of the chain. Routine calculations allow us to recover the discounted survival probability of the i^{th} reference entity. In particular, for any function g we have that

$$\begin{aligned} \tilde{q}_t^i(\xi_0) &\equiv E\left[1_{\{\tau^i\geq t\}} B_t^{-1} g(\xi_t) \mid \xi_0\right] \\ &= E\left[\exp\left(-\int_0^t (\lambda^i + r)(\xi_u)du - \sum_{j\neq k} w_{jk}^i J_{jk}(t)\right) g(\xi_t)\right] \\ &= \exp(t\tilde{Q}^i)g(\xi_0), \end{aligned} \tag{5}$$

where

$$\begin{aligned}\tilde{Q}^i_{jk} &= q_{jj} - (\lambda^i_j + r_j) \qquad\qquad (j = k);\\ &= \exp(-w^i_{jk}) q_{jk} \qquad (j \neq k).\end{aligned}$$

Taking $g = \mathbf{1}$, the vector all of whose entries are 1, gives the discounted survival probabilities $E\left[1_{\{\tau^i \geq t\}} B_t^{-1} \mid \xi_0\right]$. Note that survival probabilities depend on the current state of the chain ξ_0.

When we move to a portfolio of N defaultable securities, we need to be able to find the distributions of more complicated random variables. For example, if $\ell_i \equiv A_i(1 - R_i)$ denotes the loss on the i^{th} name, in terms of the notional A_i and the (possibly random) recovery rate R_i, then the portfolio *cumulative loss process*

$$L_t \equiv \sum_{i=1}^{N} \ell_i I_{\{\tau_i \leq t\}} \tag{6}$$

is an object of interest. Apart from Monte Carlo, the only tools available to find the law of L_t are based on transforms. By conditioning firstly on the path of the chain, it is easy to see that the (discounted) Laplace transform of L_t is given by

$$E \exp(-\int_0^t r(\xi_s) ds - \alpha L_t) = E\left[\exp(-\int_0^t r(\xi_s) ds) \prod_{i=1}^{N} \left((1 - q^i_t)\zeta_i(\alpha) + q^i_t\right) \right], \tag{7}$$

where q^i_t is given by (2) and $\zeta_i(\alpha) = E[e^{-\alpha \ell_i}]$. This is the key relation linking our modelling approach at an abstract level to the kinds of calculation needed to price credit derivatives of various sorts. The (numerical) inversion of the Laplace transform (7) is the common first step; the method of Hosono [Ho], popularised by Abate & Whitt [AW], [AW1], is a fast and accurate solution. We discuss numerical approaches in Section 4, but before that we record the form of default correlation given by this approach.

3 Default correlation

In the previous Section, we obtained an expression for the survival probability of a single obligor. The argument used extends easily to more than one obligor, so we present here the corresponding results, and derive the correlation of defaults from that.

Assume for example we want to compute the joint survival probability of obligors i and j. Using the independence of default times given ξ, we obtain

$$\begin{aligned}\tilde{q}_t^{ij}(\xi_0) &\equiv P(\tau^i \geq t, \tau^j \geq t \mid \xi_0) \\ &= \exp(t\tilde{Q}^{ij})\mathbf{1}(\xi_0),\end{aligned}$$

where

$$\begin{aligned}\tilde{Q}_{kl}^{ij} &= q_{kk} - \lambda_k^i - \lambda_k^j & (k = l); \\ &= \exp(-w_{kl}^i - w_{kl}^j)q_{kl} & (k \neq l).\end{aligned}$$

Elementary algebraic calculations allow us to recover the default correlation $\rho_T(\xi_t)$ of i and j from the joint survival probability function and the individual survival probabilities:

$$\rho_t(\xi_0) = \frac{\tilde{q}_t^{ij}(\xi_0) - \tilde{q}_t^i(\xi_0)\tilde{q}_t^j(\xi_0)}{\sqrt{\tilde{q}_t^i(\xi_0)(1 - \tilde{q}_t^i(\xi_0))}\sqrt{\tilde{q}_t^j(\xi_0)(1 - \tilde{q}_t^j(\xi_0))}} \tag{8}$$

where

$$\tilde{q}_t^i(\xi_0) = \exp(t\tilde{Q}^i)\mathbf{1}(\xi_0) \tag{9}$$

as at (5).

Note that the correlation of defaults is obtained endogenously from the model, rather than being exogenously imposed as in the copula-based industry approach to default correlation.

4 Computational approaches

Our attention now focuses on the exact expression (7) for the discounted Laplace transform of the cumulative loss at time t. We have good techniques for inverting the transform, but first we have to be able to calculate it (or some approximation), and in this Section we discuss three possible approaches.

4.1 Exact method

The approach here is to multiply out the product on the right-hand side of (7). The individual terms are all quite easy to deal with, because each is the exponential of some additive functional of the Markov chain, and we are able to compute these expectations using fast linear algebra routines. The problem with this approach comes when the number N of names gets too big; with N names there are 2^N terms in the product when multiplied out, and each of these needs to be evaluated and inverted separately. When $N = 10$ there are 1024 such calculations, and typically we need to be able to handle values of N that are an order of magnitude bigger. Thus the 'exact' calculation method will be too cumbersome for general use.

4.2 Poisson approximation

The expression (2) for the survival probability of name i can be understood in terms of a standard Poisson process ν independent of the chain ξ. If the jump times of ν are denoted $S_1 < S_2 < \dots$, then we may set

$$\tau^i \equiv \inf\{t : C_t^i > S_1\},$$

and then the relation (2) holds. The Poisson approximation we propose here is to allow name i to default *more than once*, at times

$$\tau_m^i \equiv \inf\{t : C_t^i > S_m\}, \qquad m = 0, 1, \dots.$$

By doing this, we arrive at an expression $\bar{L}_t$ for the portfolio cumulative loss which overestimates L_t, because it includes (non-existent) second and subsequent losses of each of the names. The error we are committing by this is of the same order as the default probabilities themselves; typically this would be of the order of a few percent, which would be comparable to the error we could expect from a Monte Carlo approach. Some more sophisticated method for the estimation and control of this error is needed, but the virtues of this simplified approach are evident on inspection of the expression for the Laplace transform of the cumulative loss $\bar{L}_t$:

$$E \exp(-\int_0^t r(\xi_s)ds - \alpha \bar{L}_t) = E \exp\left[\exp(-\int_0^t r(\xi_s)ds) \sum_{i=1}^N (\zeta_i(\alpha) - 1)C_t^i\right]. \tag{10}$$

For each α, we are computing the mean of the exponential of an additive functional of the chain, and this is a simple and rapid calculation.

4.3 Monte Carlo

Another approach to calculating (7) is to use Monte Carlo simulation to evaluate the right-hand side, and then invert the transform. We do not discuss this in detail, but restrict ourselves to a few remarks concerning pitfalls to be avoided.

Firstly, we do not generate paths by discretising the time interval into a large number of subintervals and then simulating the (many) individual steps of the chain; rather, we use the jump-hold construction of the Markov chain, starting from the embedded discrete-time jump chain with exponentially-distributed residence times in the states passed through. This is far more efficient, and makes the calculation of additive functionals of the path a triviality.

Secondly, inversion of the Laplace transform will require evaluation of the transform at many different values of α; we do not of course simulate a different chain for each value of α, but keep the same chain for all evaluations.

Thirdly, if the Monte Carlo approach is to be used for calibration, we will also be varying the parameters of the model, including the parameters of the chain; our simulation must be done in such as way as to keep the dependence on the parameters as smooth as possible.

5 Example: K^{th}-to-default basket

To illustrate the methods presented, we will show how to price a K^{th}-to-default basket under the simplifying assumption that there is a homogeneous recovery rate R.

The cumulative loss process $L_t = \sum_{i=1}^{N} 1_{\{\tau^i \leq t\}}$ is now just the number of defaults incurred by time t, with τ_K denoting the time of the K^{th} default. Let $\Delta_i = T_i - T_{i-1}$ be the year fraction between dates T_{i-1} and T_i. As with a plain vanilla CDS, a default basket consists of a premium leg (a protection fee stream paid up til the K^{th} default) and a default leg (the loss induced by the K^{th} default).

5.1 Premium leg

If for simplicity we ignore accrued payments at default, the risky PV01 of the premium leg of a K^{th}-to-default basket (that is, the present value of a contract paying 1bp at each payment date prior to default) maturing at time T, and paying at dates $T_1, \ldots, T_M$, is given by

$$
\begin{aligned}
PV01 &= \sum_{i=1}^{M} \Delta_i E\left[\exp\left(-\int_0^{T_i} r(\xi_u)du\right); \tau^K \geq T_i\right] && (11)\\
&= \sum_{i=1}^{M} \Delta_i E\left[\exp\left(-\int_0^{T_i} r(\xi_u)du\right); L_{T_i} \leq K-1\right] && (12)
\end{aligned}
$$

and calculating such expressions amounts to inverting the discounted Laplace transform (7). Since L_t has a discrete distribution, we can get a better level of accuracy by using the methodology developed by Abate-Whitt for probability generating functions (see [AW1]).

5.2 Default leg

The default leg of a K^{th}-to-default basket is the present value DL of the expected losses caused by the K^{th} default. Since we have assumed all names in the portfolio are characterized the same (random) recovery, we do not need to know the identity of the K^{th} firm-to-default, which simplifies the calculation significantly. Standard calculations show that

$$
\begin{aligned}
\frac{DL}{E(1-R)} &= E\left[\exp(-\int_0^{\tau_K} r(\xi_s)ds); \tau_K < T\right]\\
&= E\left[\exp(-\int_0^{\tau_K \wedge T} r(\xi_s)ds)\right] - E\left[\exp(-\int_0^{T} r(\xi_s)ds); \tau_K > T\right]\\
&= 1 - E\left[\int_0^{\tau_K \wedge T} r(\xi_s)B_s^{-1}ds\right] - E\left[B_T^{-1}; L_T < K\right]\\
&= 1 - E\left[\int_0^{T} r(\xi_s)B_s^{-1}I\{L_s < K\}ds\right] - E\left[B_T^{-1}; L_T \leq K-1\right].
\end{aligned}
$$

All of the terms appearing in this expression can in principle be computed from inverting the discounted Laplace transform (7), though we will need to do an integral with respect to s just to finish things off.

Finally the par-spread S of a K^{th}-to-default basket is then given by the ratio of the default leg and the risky PV01, i.e.

$$
S = \frac{DL}{PV01}. \qquad (13)
$$

6 Example: synthetic CDOs

We turn now our attention to the problem of pricing a CDO tranche, and find the techniques developed so far will again serve. As before, we derive first the value of the premium leg and then the value of default leg.

6.1 Premium leg

Let L^+ and L^- be the upper and lower attachment points of the tranche respectively. At each payment date, investors receive a coupon which is proportional to the notional of the tranche, net of the losses suffered by the credit portfolio up to that point. The tranche PV01 is equal to

$$PV01 = \sum_{j=1}^{M} \Delta_i E\left[\exp\left(-\int_0^t r(\xi_u)du\right)\Phi(L_{T_j})\right], \tag{14}$$

where

$$\Phi(x) = \frac{1}{L^+ - L^-}\left[\left(L^+ - x\right)^+ - \left(L^- - x\right)^+\right], \tag{15}$$

and M is the number of total payments occurring at dates $T_1, \ldots, T_M$. In order to evaluate the PV01, we need to calculate the price of a portfolio of put options on the portfolio cumulative losses at each payment date T_j. In particular, $\Phi(x)$ is the difference of two put options with strike L^+ and L^-. Elementary manipulations show that,

$$\begin{aligned}
P_t(K) &= E\left[B_T^{-1}(K - L_t)^+\right] \\
&= KE\left[\exp\left(-\int_0^t r(\xi_u)du\right); L_t \leq K\right] - \\
&\qquad -E\left[\exp\left(-\int_0^t r(\xi_u)du\right)L_t;\ L_t \leq K\right]
\end{aligned}$$

As before, once we can evaluate the discounted Laplace transform of the cumulative loss L_t (7), we are able to compute both of these terms.

6.2 Default leg

The value of the default leg of a CDO tranche is the expected present value of the tranche's losses. More precisely, define

$$\Xi(x) = \frac{1}{L^+ - L^-}\left[\left(x - L^-\right)^+ - \left(x - L^+\right)^+\right].$$

The value of the default leg of the tranche, is then given by

$$DL = E\left[\int_0^T \exp\left(-\int_0^t r(\xi_u)du\right) d\Xi(L_u)\right].$$

Integrating by parts and noting that $\Xi(x) = 1 - \Phi(x)$, we can simplify the previous expression to

$$DL = 1 - E\left[\exp\left(-\int_0^T r(\xi_u)du\right)\Phi(L_T)\right] - E\left[\int_0^T r(\xi_u)\exp\left(-\int_0^t r(\xi_u)du\right)\Phi(L_u)du\right]. \tag{16}$$

Again the integral in (16) can be calculated numerically. Note that the basic elements needed to calculate the default leg are the same as the ones we derived when calculating the premium leg, with some minor modification to account for the term $r(\xi)$ appearing in the second expectation of (16).

The tranche spread is recovered as usual by dividing the default leg by the PV01 of the premium leg.

Remark. All calculations simplify if we assume interest rates the Markov chain ξ are independent. Then all we need to do is substitute the relevant discount factor B_t^{-1}, with the corresponding riskless zero-coupon bond $B(0,t)$ which we can observe in the market. This assumption, albeit crude, can be useful to simplify the calibration.

7 Numerical results

We shall now present some numerical result to provide the reader with intuition about the behavior of the model. As it should be clear from the previous sections, the distribution of the number of defaults and the cumulative portfolio losses plays a key role in the pricing of multi-name credit derivatives. As a consequence most of the results in this section will be concerned with those two distributions.

In what follows we will assume that $(\xi_t)_{t\geq 0}$ is a three state Markov chain with Q-matrix

$$Q = \left(\begin{array}{ccc} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{array} \right) \tag{17}$$

Figures 1 to 3 give a flavor of the accuracy of the Poisson approximation in the case of homogeneous losses at default of unit size, as a function of the number of credits in the portfolio. The conditional default intensities were taken to be flat at 200 bps across chain states and obligors. The three graphs on the left picture the exact distribution function and its Poisson approximation for portfolio of increasing sizes ($N = 10$, 20 and 40 respectively). Graphs on the right show the absolute error one incurs when using the Poisson approximation in the three cases above mentioned. As one would expect, the accuracy of the approximation decreases as the size of the portfolio increases. However in all the cases considered, the absolute error is at most 36 basis points.

Figure 4 to 6 show how the shape of the cumulative number of default density function changes with the level of the conditional intensities λ^i. Again we assumed λ^i were constant across states and issuers. Figure 4 refers to a relatively safe portfolio, characterized by low conditional intensities. As intensities go up (see Figure 5 and 6) the density moves to the right (inducing a higher number of expected defaults) and the peak of the density decreases (i.e. extreme events become more likely). This is due to the combined effect of higher individual default probabilities and change in default correlation induced by increasing levels of λ^i.

So far we have not exploited the main feature of the model, i.e. the possibility of having different conditional intensities for different states of the chain. Figure 7 shows the density function of a portfolio of 40 names as a function of the state of the chain. We choose $\lambda^i(\xi_1) = 500$ bps, $\lambda^i(\xi_2) = 20000$ bps, $\lambda^i(\xi_3) = 50000$ bps for all is. Result were computed using the Laplace transform method. Recovery rates were assumed to be uniform $(0, 1)$ independent random variables. Intuitively, ξ_1 is the *safest* state and ξ_3 the *riskiest*. As we move from ξ_1 (black line) to ξ_2 (red line) the density function shifts on the right because the individual default probabilities are now higher and the expected loss increases. However, the tails of the distribution shrink, because the move is accompanied by a significant drop in default correlation ρ from 3.63 to 2.85. Note that the two effects tend to compensate each other. When moving from state ξ_2 to state ξ_3 (blue line), correlation remains pretty much unchanged but the portfolio becomes more risky and the density function shifts on the right and it peak decreases as expected.

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Figure 1: **Accuracy of the Poisson approximation**, N=10, T=1, $\lambda^i(\xi) = 200$ bps for all i and ξ

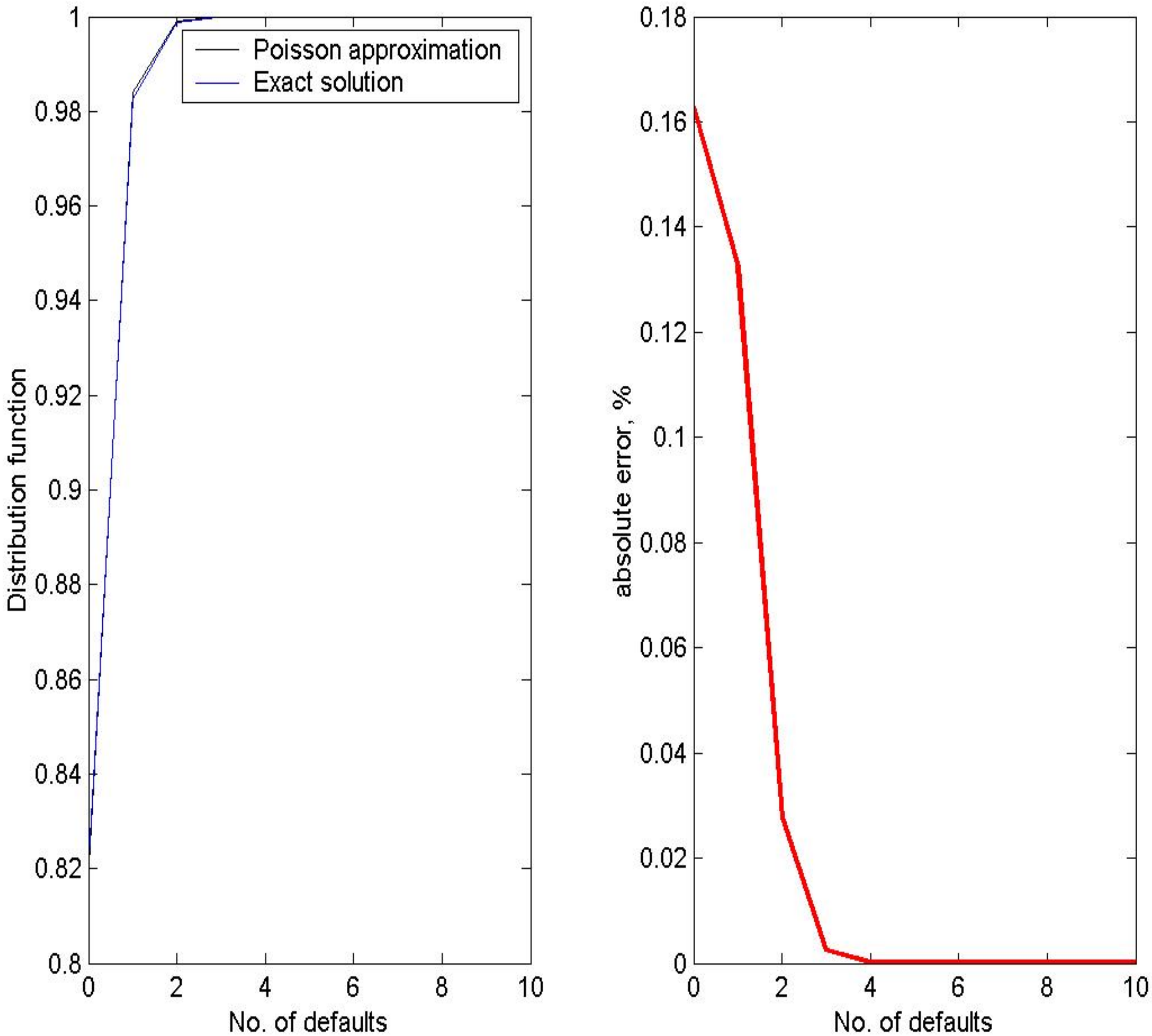

Figure 2: **Accuracy of the Poisson approximation**, N=20, T=1, $\lambda^i(\xi) = 200$ bps for all i and ξ

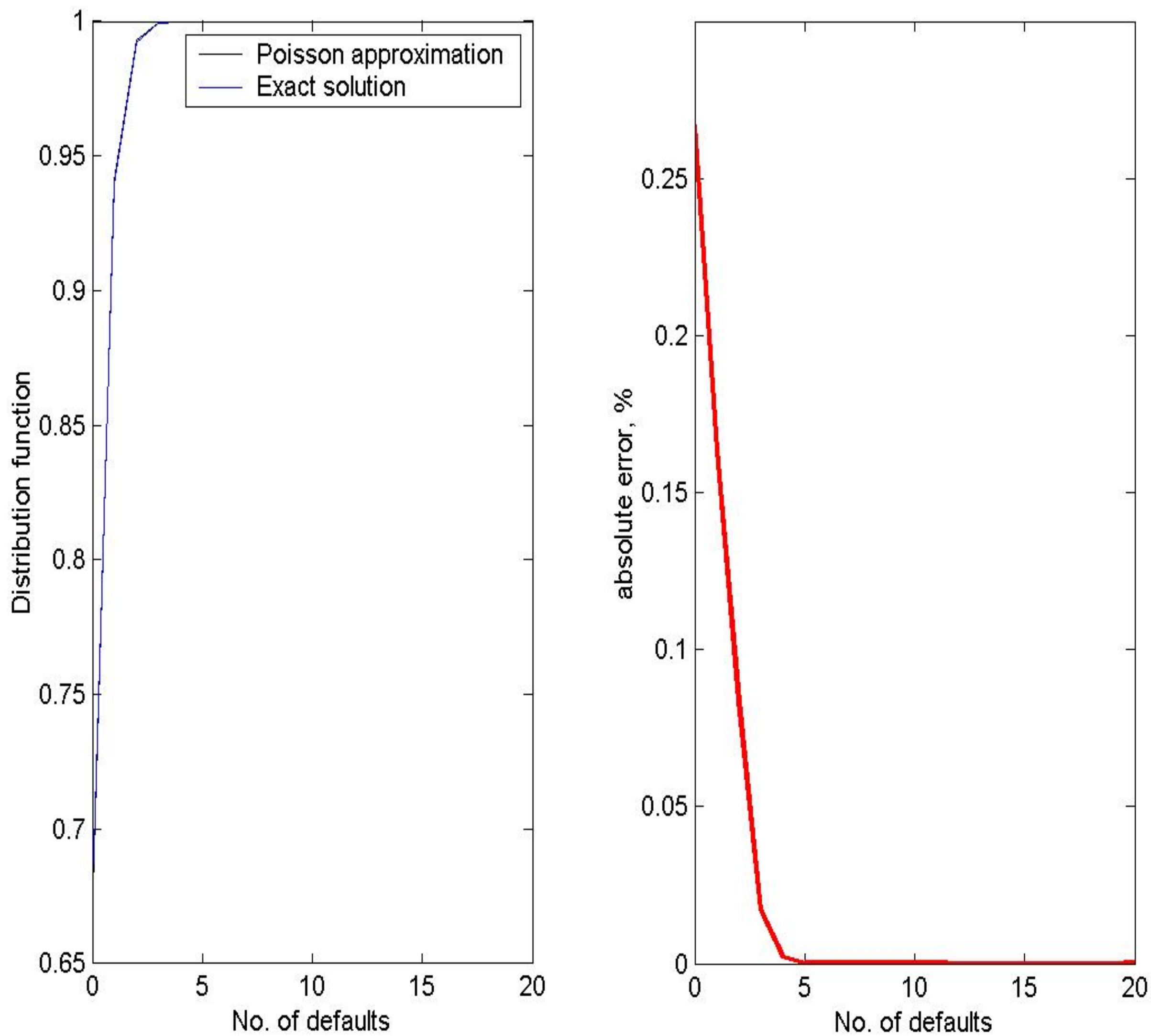

Figure 3: **Accuracy of the Poisson approximation**, N=40, T=1, $\lambda^i(\xi) = 200$ bps for all i and ξ

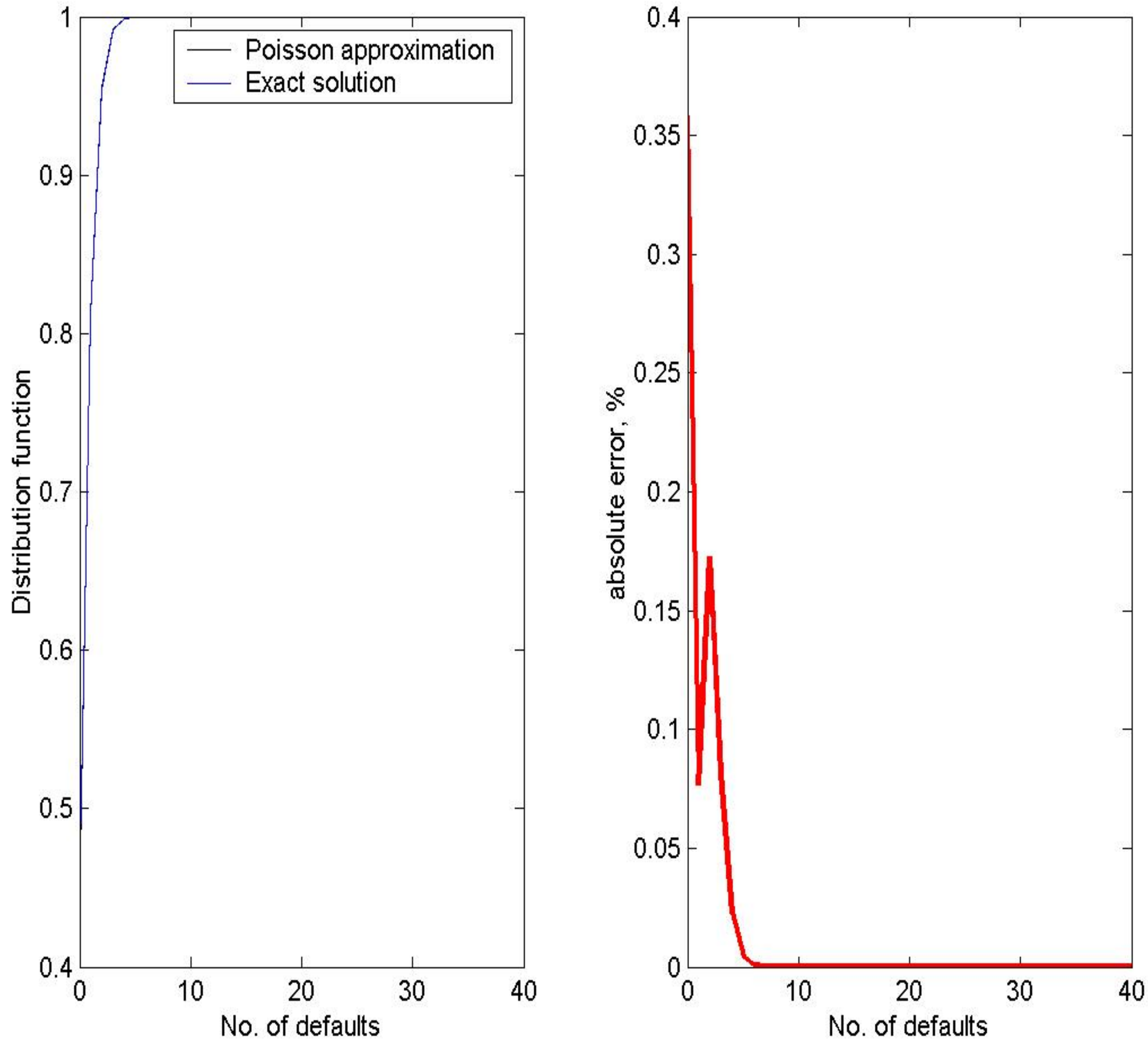

Figure 4: **Number of defaults density**, N=40, T=1, low λ^i

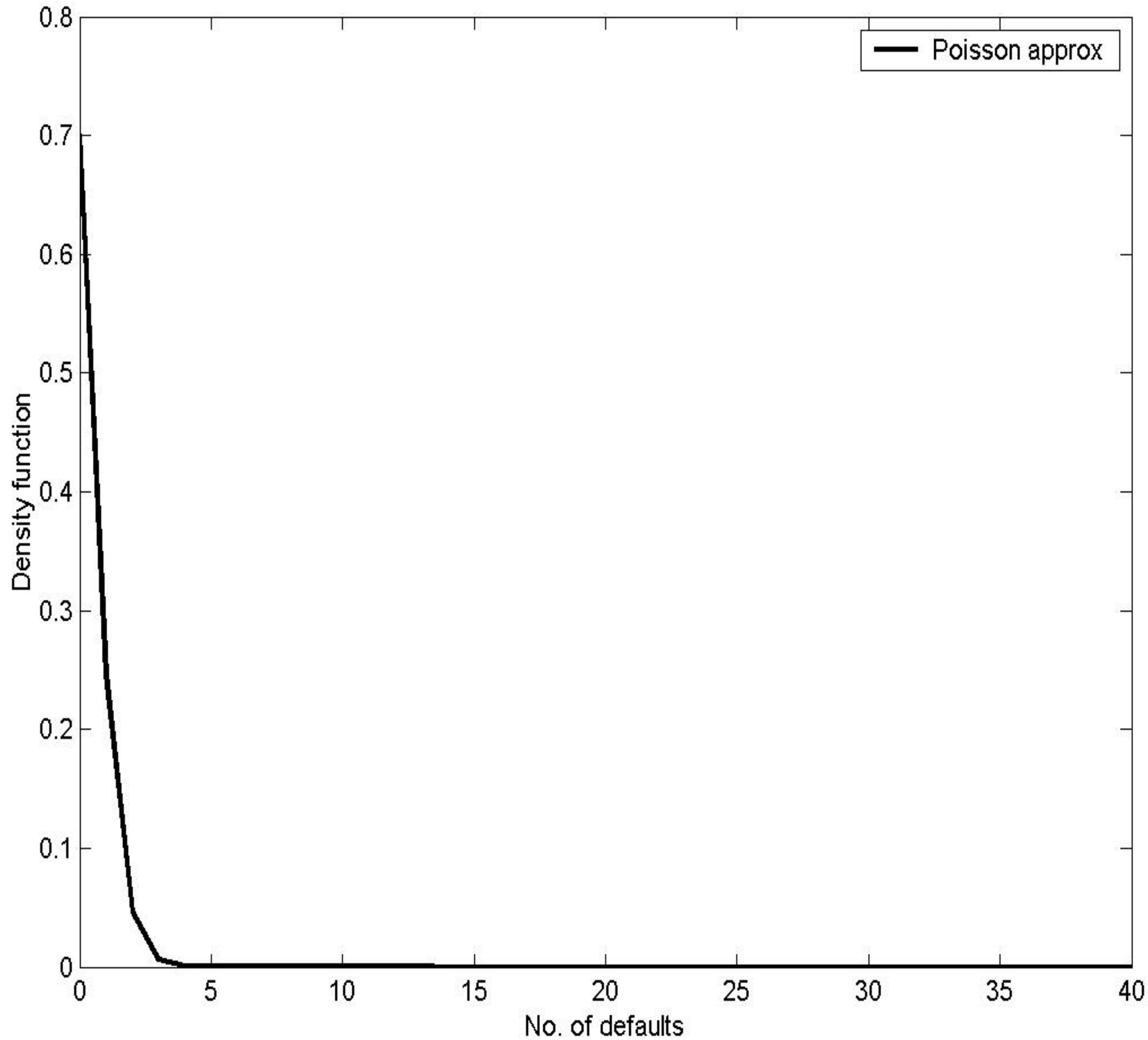

Figure 5: **Number of defaults density**, N=40, T=1, mid λ^i

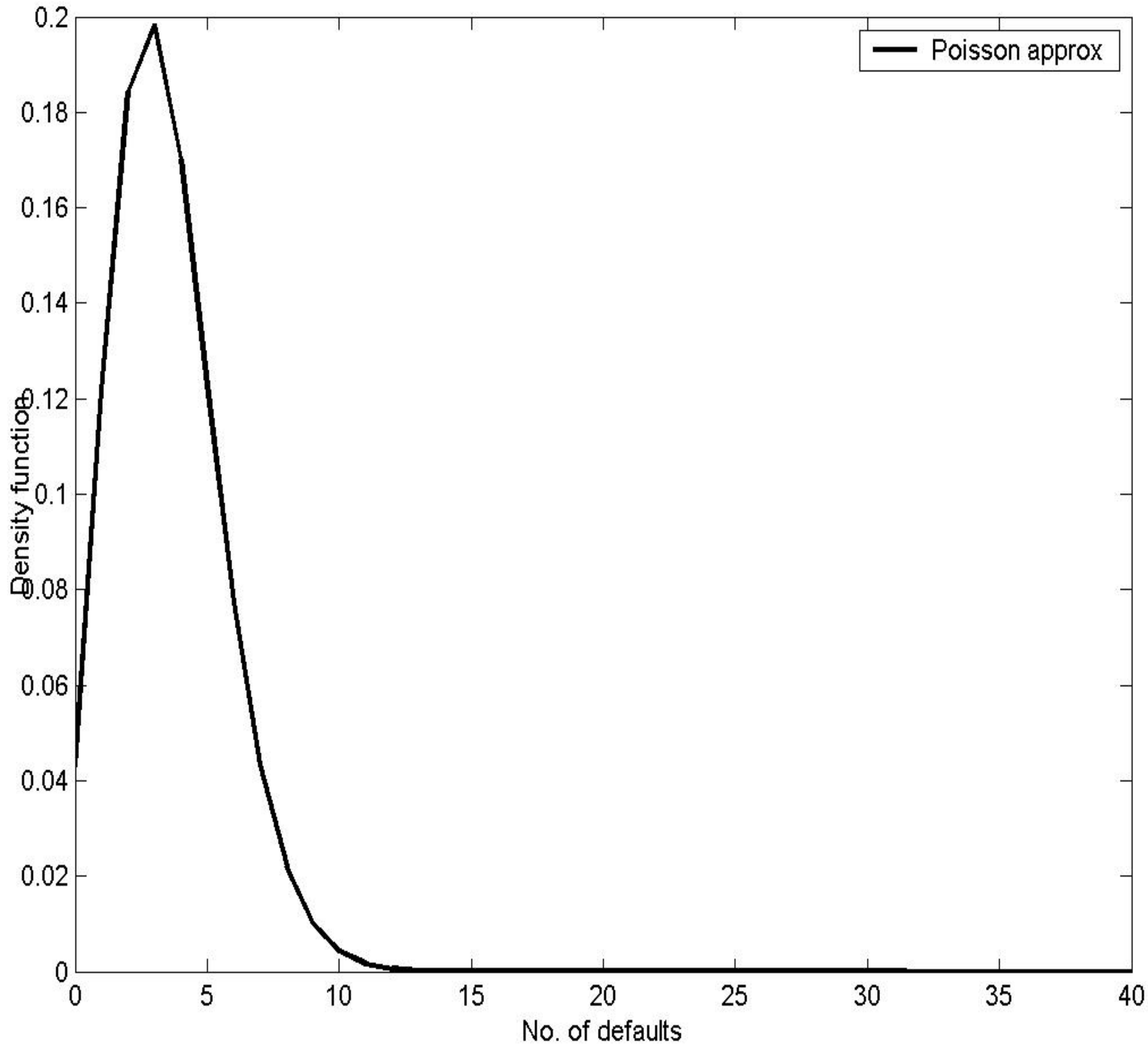

Figure 6: **Number of defaults density**, N=40, T=1, high λ^i

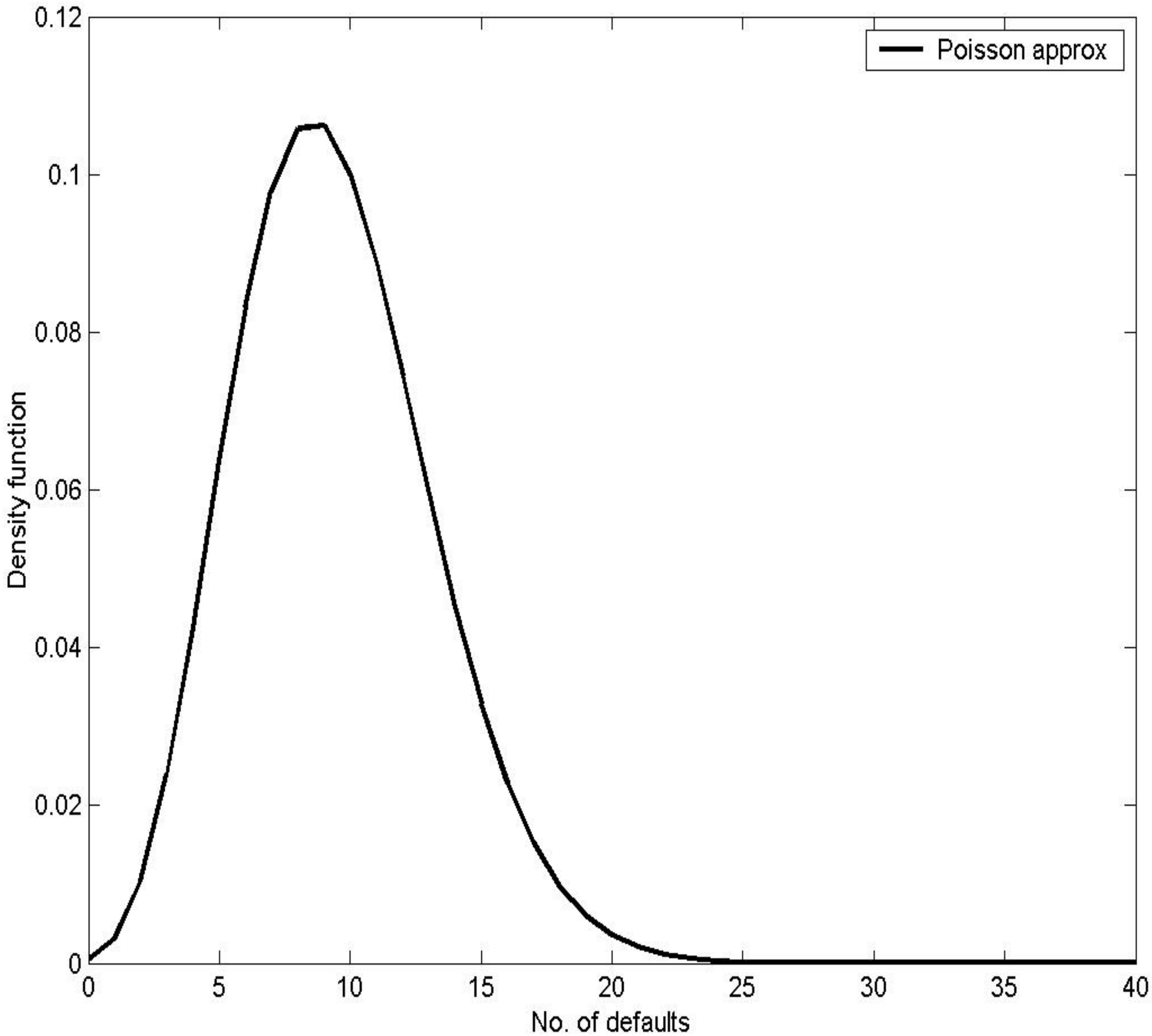

Figure 7: **Portfolio default density as a function of the chain's initial state**, N=40, T=1