FAST ACCURATE BINOMIAL PRICING

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Abstract. We discuss here an alternative interpretation of the familiar binomial lattice approach to option pricing, illustrating it with reference to pricing of barrier options, one- and two-sided, with fixed, moving or partial barriers, and also the pricing of American put options. It has often been observed that if one tries to price a barrier option using a binomial lattice, then one can find slow convergence to the true price unless care is taken over the placing of the grid points in the lattice; see, for example, the work of Boyle & Lau [2]. The placing of grid points is critical whether one uses a dynamic programming approach, or a Monte Carlo approach, and this can make it difficult to compute hedge ratios, for example. The problems arise from translating a crossing of the barrier for the continuous diffusion process into an event for the binomial approximation. In this article, we show that it is not necessary to make clever choices of the grid positioning, and by interpreting the nature of the binomial approximation appropriately, we are able to derive very quick and accurate pricings of barrier options. The interpretation we give here is applicable much more widely, and helps to smooth out the 'odd-even' ripples in the option price as a function of time-to-go which are a common feature of binomial lattice pricing.

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1 Introduction

We shall give in this paper a different interpretation of the standard binomial lattice approach to pricing of contingent claims written on an asset whose log price X is a Brownian motion with constant variance and constant drift, in the presence of a constant interest rate. This interpretation leads to effective methods for pricing barrier and other options; for now, we shall concentrate on the pricing of double barrier European call options, but it is important to realise that the approach described can be applied to a range of other path-dependent options involving the supremum or infimum of the path.

The pricing of barrier options has been widely studied; see, for example, Boyle & Lau [2], Broadie, Glasserman & Kou [3] and [4], Carr [5], Chance [6], Cheuk & Vorst [7], Heynen & Kat [12] and [13], Kunitomo & Ikeda [16], Li & Lu [17], Rich [21] and [22], Ritchken [23], Rogers & Zane [25], and Rubinstein & Reiner [26].

It is well known that the price of the option can be computed by solving a second-order partial differential equation (PDE). The binomial pricing method can be interpreted as a finite-difference approximation to this PDE, but it is actually more helpful to think of the binomial pricing method as being an *exact* calculation relative to a discrete-time discrete-state Markov process which approximates the log-price process. This approximation is a random walk which jumps at the times $\Delta t, 2\Delta t, \ldots$. At each jump, the random walk moves either up by Δx or down by Δx; the probabilities of these two alternatives, and the size of the jump, are chosen to make the local drift and variance of the random walk match those of the diffusion, so that as Δt tends to zero the random walk converges weakly to the diffusion.

This is the conventional interpretation of the approximation of the log Brownian motion by the random walk, but we can usefully think of another approximation, namely, we fix some $\Delta x > 0$, and view the diffusion only at the discrete set of times at which it has moved by Δx from where it was when we last observed it; formally,

$$\tau_0 = 0, \quad \tau_{n+1} \equiv \inf\{t > \tau_n : |X(t) - X(\tau_n)| > \Delta x\}, \;\; n \geq 0.$$

The (discrete-time) process that arises is now a random walk approximating the underlying diffusion X uniformly closely (at least after the appropriate adjustment of the time scale). *What we shall do in this article is to approximate* $(X_t)_{0\leq t \leq T}$ *by the random walk* $(\xi_n)_{0\leq n\leq \nu}$, where $\nu \equiv \sup\{n : \tau_n < T\}$, and T is the expiry of the option.

Routine scale function calculations lead us to the probability of an upward step. When we think of the random walk approximation in this form, it is easy to see how we should handle the barrier condition. It is not necessary to place the upper barrier b^* at some grid point; all we need is that if x is the grid point immediately below b^*, then we modify the jump probabilities from x. More precisely, the probability of a down step from x to $x - \Delta x$ will be the probability that the diffusion X, starting from x, reaches $x - \Delta x$ before b^*, and with the complementary probability the barrier will be crossed and the option knocked out.

This explains the dynamics of our random walk with the barrier condition, but how should we account for the number of steps taken? In contrast to the usual approximation, the number ν of steps taken is random. However, as we argue in Section 2, *the path followed by the random walk is independent of* ν. The price of the option depends on the number of time-steps to go, and we simply compute the price for all (or enough) values of time-steps to go, and mix over the distribution of the number ν of time-steps. To be more precise, the law of ν is not easy to describe in closed form, but we use approximations based on classical renewal theory. In Section 2, we compute the price of the standard European call, the down-and-out European call, and a double barrier European call by various different methods:

(i) exact (Black-Scholes-like) formula;

(ii) numerical integration;

(iii) the conventional binomial approximation ;

(iv) the conventional binomial approximation with averaging correction;

(v) the binomial approximation developed here;

Of course, method 1 is not available for the double barrier option. We report on the speed and accuracy of the numerical methods.

One virtue of the approach proposed here is that we are actually considering the *true* price process, viewed at a discrete (random) set of time points. Thus there are essentially only three places where errors could arise:

(a) on the last time interval, during which T falls;

(b) in the approximation of the distribution of the number of time steps to be used;

(c) when the random walk visits a grid point next to a barrier, the time taken to reach the next grid point given that the barrier is not breached will be stochastically smaller than for a grid point in the middle of the 'live' region.

We have corrections to deal with (a) and (b); the correction for (a) is established in an appendix, and the procedure for (b) is given in 2. We investigated a correction for (c), but this was cumbersome and did not improve the results significantly, so we omit further discussion of it. The results reported are sufficiently good that corrections are not crucial.

Section 3 presents and discusses the numerical results for fixed one- and two-sided knockout barriers on a European call option. It turns out that the modified binomial method which is the subject of this paper outperforms the standard binomial method easily. It is not as good as numerical integration, though. This is not surprising, in that exact or quasi-exact analytic methods are most likely to win when they are available. But the approach given here is *robust*, as we demonstrate in Section 4, where we apply the method to moving barrier options, to partial barrier options, and finally to American options. There are some rather crude assumptions linking the random timescale of the approximation we consider and the actual timescale, but it turns out that, using some interpolation where necessary, we can obtain quite good results. Section 5 concludes the paper.

2 Modifying the binomial random walk approximation.

The asset price S is a log Brownian motion, so that

$$X_t \equiv \log S_t = \sigma W_t + (r - \sigma^2/2)t \equiv \sigma W_t + \mu t, \tag{1}$$

where W is a standard (one-dimensional) Brownian motion, and σ is the volatility of S, and r is the riskless rate of return, both constants.

We consider a European call option with expiry T and strike K, knocked out if ever the log-price rises above b^*, or falls below b_*; arbitrage pricing theory gives its price as

$$E\left[e^{-rT}(S_T-K)^+;\zeta>T\right], \tag{2}$$

where $\zeta \equiv \inf\{t>0 : X_t>b^* \text{ or } X_t<b_*\}$. It is not hard to show that the price function

$$\varphi(t,x) \equiv E\left[e^{-r(T-t)}(S_T-K)^+;\zeta>T \mid S_t=e^x\right] \tag{3}$$

satisfies a second-order partial differential equation

$$\frac{\partial\varphi}{\partial t}+\mathcal{G}\varphi-r\varphi=0,\ \varphi(T,x)=(e^x-K)^+,\ \varphi(t,b_*)=\varphi(t,b^*)=0,$$

where

$$\mathcal{G}\equiv \tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}+\mu\frac{\partial}{\partial x}.$$

We do this by fixing some $\Delta x>0$, and considering the times

$$\tau_0=0,\ \tau_{n+1}\equiv\inf\{t>\tau_n : |X(t)-X(\tau_n)|>\Delta x\},\ \ n\geq 0.$$

The process $(\xi_n)_{n\geq 0}\equiv(X(\tau_n))_{n\geq 0}$ is now a random walk with values in the lattice $\Lambda\equiv X_0+(\Delta x)\,\mathbf{Z}$, and it approximates the underlying diffusion X uniformly closely (at least after the appropriate adjustment of the time scale). We approximate $(X_t)_{0\leq t\leq T}$ by the random walk $(\xi_n)_{0\leq n\leq\nu}$, where $\nu\equiv\sup\{n:\tau_n<T\}$. Routine scale function calculations lead us to the probability p of an upward step:

$$p=\frac{s(0)-s(-\Delta x)}{s(\Delta x)-s(-\Delta x)}=\frac{e^{2c\Delta x}-1}{e^{2c\Delta x}-e^{-2c\Delta x}} \tag{4}$$

where $s(x)=-\exp(-2\mu x/\sigma^2)\equiv-\exp(-2cx)$.

When we think of the random walk approximation in this form, it is easy to see how we should handle the barrier conditions. It is not necessary to place the barriers b_* and b^* at grid points; all we need is that if x^* is the grid point immediately below b^*, and x_* is the grid point immediately above b_*, then we modify the jump probabilities to

$$P(\xi_{n+1}=x^*-\Delta x|\xi_n=x^*) = \frac{s(b^*)-s(x^*)}{s(b^*)-s(x^*-\Delta x)}, \tag{5}$$

$$P(\xi_{n+1}=x_*+\Delta x|\xi_n=x_*) = \frac{s(x_*)-s(b_*)}{s(x_*+\Delta x)-s(b_*)}, \tag{6}$$

and with the complementary probability the barrier will be crossed and the option knocked out.

The price of the barrier option for this approximating process is computed by solving the dynamic-programming equation

$$\begin{aligned} \psi(0,x) &= (e^x - K)^+ \\ \psi(n+1,x) &= p(x)\psi(n,x+\Delta x) + q(x)\psi(n,x-\Delta x) \end{aligned} \tag{7}$$

for all $b_* < x < b^*$ in the grid Λ and all $n \geq 0$. The probability $p(x)$ is given by (4) for all x except x^* and x_* (for which $p(x^*) = 0$, and $q(x_*) = 0$), and $q(x) = 1 - p(x)$ for all $x_* < x < x^*$, with finally $q(x^*)$ and $p(x_*)$ being given by (5) and (6).

Solving (7) does not answer our original problem of pricing the barrier option, but it is very close. The missing piece is provided by this little result.

PROPOSITION 1. *The random variables $(\tau_{n+1} - \tau_n)_{n\geq 0}$ are independent and identically distributed, with distribution given by*

$$\varphi(\lambda) \equiv E\left[\exp(-\lambda\tau_1)\right] = \frac{\cosh \mu\sigma^{-2}\Delta x}{\cosh \gamma \Delta x} = \frac{\cosh(c\Delta x)}{\cosh(\gamma\Delta x)}, \tag{8}$$

where $\gamma \equiv \sqrt{\mu^2 + 2\lambda\sigma^2}/\sigma^2$. The common mean is

$$E\left[\tau_1\right] \equiv \frac{\Delta x}{\mu} \tanh c\Delta x \tag{9}$$

and common second moment is

$$E\left[\tau_1^2\right] = 2\left(E\left[\tau_1\right]\right)^2 + \frac{\sigma^2\Delta x}{\mu^3} \tanh c\Delta x - \left(\frac{\Delta x}{\mu}\right)^2. \tag{10}$$

Moreover, they are independent of the random walk ξ.

Proof. To obtain (8), we have to solve

$$\mathcal{G}f - \lambda f = 0 \tag{11}$$

in the interval $[-\Delta x, \Delta x]$ with boundary values 1 at each end of the interval, and then take $\varphi(\lambda) = f(0)$. The routine calculations leading to (8) do not need to be spelled out, and two differentiations get from (8) to (9) and (10).

What may be a little less obvious is the final statement, namely, the independence of the times τ_i and the random walk ξ. But if we compute $E\left[\exp(-\lambda\tau_1) : X(\tau_1) = -\Delta x\right]$ by solving (11) with boundary conditions 0 at Δx and 1 at $-\Delta x$, we obtain

$$E\left[\exp(-\lambda\tau_1) : X(\tau_1) = -\Delta x\right] = \frac{1}{2e^{c\Delta x}\cosh\gamma\Delta x} \; .$$

Now it is easy to calculate from this that

$$E\left[\exp(-\lambda\tau_1) | X(\tau_1) = -\Delta x\right] = \frac{\cosh c\Delta x}{\cosh\gamma\Delta x} \; ,$$

which is unchanged if we replace c by $-c$. However, a moment's thought shows that when we replace c by $-c$, we are actually calculating $E\left[\exp(-\lambda\tau_1)|X(\tau_1) = \Delta x\right]$, and we conclude that the law of τ_1 conditional on $X(\tau_1) = -\Delta x$ is the same as the law of τ_1, and is the same as the law of τ_1 conditional on $X(\tau_1) = \Delta x$. In other words, τ_1 is independent of $X(\tau_1)$, as was claimed.

It now follows from Proposition 1 that the price of the barrier option can approximated by

$$\sum_{n\geq 0} P(\nu = n)\, \psi(n, x_0), \tag{12}$$

where $x_0 \in \Lambda$ is the starting value of X. The dynamic programming recursion (7) allows us to compute ψ, so we only need to calculate $P(\nu = n)$ for all n. It amounts to the same thing to compute $P(\nu \geq n) = P(\tau_n < T)$, and this is made much easier by the fact that the increments of the sequence (τ_n) are independent with the same law, characterised by (8). If we abbreviate $m(\Delta) \equiv E[\tau_1]$ and $\sigma(\Delta)^2 \equiv \text{var}\,(\tau_1)$, then by the Central Limit Theorem we shall have that approximately

$$\frac{(\tau_n - nm(\Delta))}{\sigma(\Delta)\sqrt{n}} \;\sim\; N(0,1).$$

However, this approximation turns out to be rather too crude, and a refinement of the Central Limit Theorem is required. Fortunately, such refinements are well developed; Petrov [20], Chapter 5, gives a good account of expansions of the Central Limit Theorem. In particular, Theorem 5.22 of Petrov states the following:

$$P\left(\frac{(\tau_n - nm(\Delta))}{\sigma(\Delta)\sqrt{n}} \leq x\right) = \Phi(x) + \frac{\alpha_3(1-x^2)e^{-x^2/2}}{\sqrt{72\pi n}} + o(n^{-1/2}),$$

where

$$\alpha_3 \equiv E\left[\left(\frac{\tau_1 - m(\Delta)}{\sigma(\Delta)}\right)^3\right]$$

is the third moment of the centred and scaled random time-step. Higher order terms in the expansion are available, but we found that they made no appreciable difference to the accuracy of the results, so have omitted them entirely. A more sophisticated approach to the approximation of the probabilities $\mathrm{P}(\tau_n < T)$ would be to use a saddlepoint approximation (see, for example, Daniels [10] or Wood, Booth & Butler [27] for clear accounts of the main results, and Jensen [15] for a thorough treatment). Nonetheless, the expansion used here was accurate enough for all practical purposes. Clearly, we can compute the value of α_3 explicitly from (8); we obtain

$$\frac{\Delta x[A+B-C]}{c^5\sigma^6(s(\Delta x)-1)^3}$$

where

$$\begin{aligned} A &= 12c\Delta x[s(2\Delta x)+s(\Delta x)] \\ B &= 8c^2(\Delta x)^2[s(\Delta x)-s(2\Delta x)] \\ C &= 3[1+s(\Delta x)-s(2\Delta x)+s(3\Delta x)] \end{aligned}$$

This was how we computed the values $P(\nu \geq n) = P(\tau_n < T)$ to get to (12).

3 Numerical results.

The computations reported here were performed on a Sun Sparcserver 1000E. We computed the values of a European call option with expiry 1 year, volatility 25 %, interest rate r at 10 %, strike of 100 and initial price of 95. The parameter values chosen allow comparison with the results of Boyle & Lau [2]. First we computed the values for the standard European call, then for the down-and-out barrier option with barrier at 90. Finally for the double barrier knockout option we computed prices for three sets of parameter values, corresponding to those used in Geman & Yor [11]. We report the results for the three methods: standard binomial, averaged binomial (where the price is taken to be $(V(n-1)+2V(n)+V(n+1))/4$, with $V(n)$ being the price of the option with n steps to go), and the modified binomial method of this paper.

The speed and accuracy depend on n, the number of timesteps taken; for the modified binomial method, this is interpreted as the average number of time steps taken, which is determined by the choice of grid spacing. We also calculated the prices using the Black-Scholes formula, and numerical integration. The speed and accuracy of the modified method is greatly superior to the other binomial methods presented; for the standard call, the price is accurate to one part in 1000 with n=50, requiring 0.02 seconds, as opposed to n=3200 for the standard binomial method, requiring 4.9 seconds. With the down-and-out call, the modified method gets within one part in 1000 using 75 timesteps on average, which requires a time of 0.08 seconds; even with 3200 timesteps, the other binomial methods are out by three parts in 100, an order of magnitude worse. For the double-barrier option, the modified method achieves accuracy of one part in 1000 when n=800, taking time 0.043 seconds; the binomial method with 3200 steps takes 0.2 seconds, and even then is out by 3 %, thirty times as bad!

In all these cases, the numerical integration method appears to be faster and more accurate, but for the double barrier situation we see that it is only about 5 times faster.

Table 1: Standard European Call

n	Standard Binomial	Averaged Binomial	Modified Binomial
25	13.0265	12.9446	11.5994
	(0.0083)		(0.01)
50	12.3168	12.3208	11.6497
	(0.01)		(0.02)
75	12.0791	12.0955	11.6480
	(0.014)		(0.03)
100	12.0003	11.9803	11.6443
	(0.02)		(0.073)
150	11.8783	11.8787	11.6549
	(0.03)		(0.077)
200	11.8109	11.8198	11.6519
	(0.048)		(0.12)
400	11.7431	11.7386	11.6546
	(0.14)		(0.38)
800	11.7001	11.6984	11.6564
	(0.77)		(1.05)
1600	11.6787	11.6779	11.6569
	(1.83)		(3.46)
3200	11.6679	11.6677	11.6572
	(4.9)		(11.0)

Accurate values:

Method	Price	Time taken
Black-Scholes	11.65735	0.001
Numerical Integration	11.65737	0.001

Table 2: Down and Out Barrier Option

With lower barrier set at: 90

n	Standard Binomial	Averaged Binomial	Modified Binomial
25	9.6114	9.5830	5.9810
	(0.006)		(0.03)
50	7.5028	7.5023	5.9852
	(0.007)		(0.05)
75	6.4382	6.4415	5.9921
	(0.0087)		(0.08)
100	7.6536	7.6485	5.9932
	(0.012)		(0.13)
150	6.6374	6.6377	5.9945
	(0.018)		(0.24)
200	7.2923	7.2945	5.9957
	(0.025)		(0.34)
400	6.6817	6.6808	5.9960
	(0.06)		(0.96)
800	6.6193	6.6189	5.9966
	(0.196)		(2.64)
1600	6.1794	6.1792	5.9967
	(0.53)		(3.63)
3200	6.2705	6.2705	5.9968
	(1.54)		(19.21)

Accurate values:

Method	Price	Time taken
Black-Scholes type formula	5.99684	0.002
Numerical Integration	5.99684	0.02

Table 3: Double Barrier Option (a)

With parameters:

σ	0.5
r	0.05
T	1.0
S_0	100.0
K	100.0
Lower barrier	75.0
Upper barrier	150.0

n	Standard Binomial	Averaged Binomial	Modified Binomial
25	1.7040	1.6322	1.0836
	(0.005)		(0.009)
50	1.3097	1.2132	0.9253
	(0.005)		(0.01)
75	1.3299	1.3908	0.9512
	(0.006)		(0.01)
100	1.3392	1.3566	0.9365
	(0.008)		(0.015)
150	1.0746	1.0454	0.8921
	(0.007)		(0.019)
200	1.2289	1.2056	0.9089
	(0.008)		(0.024)
400	1.1391	1.1430	0.9030
	(0.015)		(0.048)
800	0.9550	0.9567	0.8927
	(0.03)		(0.043)
1600	1.0227	1.0236	0.8949
	(0.08)		(0.24)
3200	0.9227	0.9215	0.8930
	(0.2)		(0.6)

Accurate value:

Method	Price	Time taken
Numerical Integration	0.8929	0.03
Geman and Yor	0.89	–
MC	0.955	–

Table 4: Double Barrier Option (b)

With parameters:

σ	0.5
r	0.05
T	1.0
S_0	100.0
K	87.5
Lower barrier	50.0
Upper barrier	150.0

n	Standard Binomial	Averaged Binomial	Modified Binomial
25	5.8405	5.5312	4.2776
	(0.005)		(0.009)
50	4.0824	3.9173	3.8872
	(0.005)		(0.013)
75	4.8993	5.0056	3.9605
	(0.007)		(0.013)
100	4.6836	4.7600	3.9208
	(0.007)		(0.017)
150	3.7897	3.7605	3.8132
	(0.008)		(0.02)
200	4.2326	4.1926	3.8502
	(0.01)		(0.028)
400	4.2344	4.2529	3.8354
	(0.02)		(0.06)
800	3.8182	3.8252	3.8098
	(0.044)		(0.13)
1600	3.9700	3.9745	3.8142
	(0.1)		(0.35)
3200	3.8353	3.8342	3.8090
	(0.3)		(0.81)

Accurate value:

Method	Price	Time taken
Numerical Integration	3.8086	0.03
Geman and Yor	3.8075	–
MC	3.86	–

3.1 Table 5: Double Barrier Option (c)

With parameters:

σ	0.2
r	0.02
T	1.0
S_0	100.0
K	100.0
Lower barrier	75.0
Upper barrier	125.0

n	Standard Binomial	Averaged Binomial	Modified Binomial
25	2.0222	2.2028	2.1472
		(0.006)	(0.007)
50	2.0836	1.9935	2.0518
		(0.005)	(0.0095)
75	2.0957	2.1578	2.0788
		(0.006)	(0.01)
100	2.4693	2.4213	2.0984
		(0.014)	(0.012)
150	2.1774	2.1461	2.0670
		(0.019)	(0.02)
200	2.1228	2.0994	2.0591
		(0.027)	(0.025)
400	2.2094	2.2124	2.0642
		(0.021)	(0.047)
800	2.1315	2.1256	2.0578
		(0.05)	(0.11)
1600	2.0970	2.0978	2.0558
		(0.13)	(0.3)
3200	2.1342	2.1328	2.0558
		(0.38)	(0.82)

Accurate value:

Method	Price	Time taken
Numerical Integration	2.0544	0.03
Geman and Yor	2.055	–
MC	2.125	–

4 Other types of options

Having seen that this method works well on the standard single and double barrier cases, we next tested it on more complicated barriers. Firstly, we took the case of *moving barriers*, both one- and two-sided, where the barrier moved linearly in the log-price scale. This choice of barriers was made to allow comparison with the results of Kunitomo & Ikeda [16]; for a completely different approach, see Rogers & Zane [25]. Secondly, we considered the case of *partial barriers*, where the barrier is effective only for part of the interval. We took just a one-sided example here, where the barrier was effective only for the first half of the interval. This allows us to compare with the quasi-analytic result expressing the price as a one-dimensional integral. And finally we priced the *American put*, to demonstrate the use of the method on a non-barrier example; we were able to compare with results of Ait-Sahalia & Carr [1].

An added difficulty is involved in each of the above cases since the time points of our lattice correspond to random real times, so we do not know exactly where the barrier is at that lattice time point. Our treatment of this is primitive, but seems to be effective. For the moving barrier, we had a barrier $\alpha + \beta t$ so we assume the position of the barrier at time step i is $\alpha + \beta i \mathrm{E}[\tau]$. Analogously, for the partial barrier we suppose that at time step i we are actually at time $i\mathrm{E}[\tau]$ for the purposes of deciding knockout probabilities.

The numerical results which follow show that these approximations worked well.

To price the American put we changed the problem slightly. The classical American put is an optimal stopping problem, where the decision whether or not to stop is based on the current time and the current share price. Instead, we supposed that the controller can at each decision time see the share price, and the number of moves the random walk ξ has remaining before the expiry of the option. This leads to the dynamic programming equation for the value function:

$$\begin{aligned} V_n(x;\Delta t) &= \max\{(K-\mathrm{e}^x)^+, \mathrm{E}[\mathrm{e}^{-r\tau}\{pV_{n+1}(x+\Delta x) + qV_{n+1}(x-\Delta x)\}]\} \\ &= \max\{(K-\mathrm{e}^x), 0, p\mathrm{E}[\mathrm{e}^{-r\tau}]V_{n+1}(x+\Delta x) + q\mathrm{E}[\mathrm{e}^{-r\tau}]V_{n+1}(x-\Delta x)\} \end{aligned}$$

which involved using our previous calculation of $\mathrm{E}[\mathrm{e}^{-r\tau}]$.

The alteration of the problem makes one suspect that the solutions obtained may not be very accurate, and to correct for possible errors we used a variant of Richardson extrapolation, where we take a linear combination of three approximate values computed using different values of n, the number of time steps. The coefficients of the linear combination were picked to eliminate errors of the form $a_0 n^{-1/2} + a_1 n^{-1}$. Again, the results for this method were very satisfactory.

Table 6: Barrier Option Results

	Sgle lin	Dble lin	Partial
25	4.9977	5.2504	6.1391
	(0.01)	(0.01)	(0.02)
50	4.9508	5.3107	6.1200
	(0.02)	(0.015)	(0.03)
75	4.9482	5.3345	6.1266
	(0.03)	(0.018)	(0.05)
100	4.9436	5.3270	6.1296
	(0.04)	(0.023)	(0.07)
150	4.9364	5.3539	6.1292
	(0.06)	(0.033)	(0.1)
200	4.9357	5.3598	6.1322
	(0.086)	(0.043)	(0.17)
400	4.9314	5.3602	6.1323
	(0.22)	(0.093)	(0.53)
800	4.9296	5.3660	6.1330
	(0.6)	(0.21)	(1.89)
1600	4.9286	5.3668	6.1329
	(1.65)	(0.53)	(7.4)
3200	4.9281	5.3672	6.1332
	(4.7)	(1.31)	(29.1)
Accurate	4.9277	5.3679	6.1332
	(0.02)	(0.05)	(0.02)

The accurate prices at the bottom of the table were calculated using closed form solutions. The formula for the double-sided moving barrier option was obtained from Kunitomo and Ikeda (1992).

Table 7: American Put

The option valued had strike price 100, time to maturity 0.5 years, volatility 0.40 and risk-free rate 0.06. As an example, three different start prices are shown for an in-the-money, at-the-money and out-of-the-money option. The values our method is tested against are the average standard binomial with 1000 steps, the method of lines, and Broadie and Detemple's LUBA. The results of all three methods can be found in a paper by Ait-Sahalia & Carr [1].

Initial Stock Price		Modified Binomial	Richardson Extrapolation	Time
85	25	18.0438	0.0	0.09
	50	18.0409	0.0	0.19
	100	18.0395	18.0380	0.49
	200	18.0382	18.0339	1.47
	400	18.0375	18.0370	4.76
	800	18.0371	18.0364	16.66
	Av Bin	18.0374		
	Lines	18.0402		
	BD LUBA	18.0346		
100	25	11.7469	0.0	0.09
	50	11.4224	0.0	0.19
	100	10.8907	8.5750	0.5
	200	10.5526	9.8656	1.48
	400	10.3432	9.9390	4.81
	800	10.2106	9.9431	16.76
	Av Bin	9.9458		
	Lines	9.9417		
	BD LUBA	9.9466		
115	25	5.1275	0.0	0.08
	50	5.1272	0.0	0.2
	100	5.1264	5.1226	0.49
	200	5.1260	5.1254	1.42
	400	5.1257	5.1248	4.78
	800	5.1255	5.1250	16.89
	Av Bin	5.1265		
	Lines	5.1047		
	BD LUBA	5.1261		

For the barrier options, we used the 1 year call option, volatility 25%, interest rate 10%, strike 100 and initial stock price 95. The upper barrier function f_{hi} and lower barrier function f_{lo} used (referred to the log-price scale) were chosen as follows. For the single moving (lower) barrier, we had $f_{\mathrm{lo}}(t) = \log(90) + 0.1\,t$. For the two-sided moving barrier, we took $f_{\mathrm{lo}}(t) = \log(90) - 0.1\,t$, and $f_{\mathrm{hi}}(t) = \log(160) + 0.1\,t$. For the partial (lower) barrier, we used $f_{\mathrm{lo}}(t) = \log(90)$ for $0 \leq t \leq T/2$; $= -\infty$ otherwise.

Despite the crude treatment of the issue of the random times, the computed values away from the money agree with the quasi-analytic ones to one part in a thousand with computing time of about 0.5 s. At the money, it needs about 10 times as long to achieve this accuracy, consistent with the results of Rogers & Zane [25].

For the American put, we compared our method with three methods from [1]. It is worth noting that the accuracy of our method compares well with the three methods. It is also of interest that for American puts which are at the money, the improvement gained by Richardson extrapolation is considerable, whereas away from the money it appears to be less important. This is borne out by calculations for other values of the parameters as well. The speed of our calculations is entirely acceptable for real-time use, though not as good as problem-specific methods, such as the Broadie-Detemple LUBA method.

5 Conclusions.

We have developed a variant of the binomial pricing algorithm based on interpreting the random walk on the binomial lattice in terms of the Brownian motion crossing equally-spaced levels in the log-price variable, rather than equally-spaced intervals in time. The speed and accuracy of the method for pricing standard European call options, with or without barriers, is superior to the usual binomial lattice. For these options, analytic solutions, or quasi-analytic solutions based on numerical integration of the transition density, perform best of all. For more complex options for which no quasi-analytic solutions are known, the new method gives good accuracy in acceptable time.

Appendix.

When we embed the random walk in the drifting Brownian motion as described in the Introduction, and then do the calculation of the option price as in Section 2, we are in fact computing the value of

$$e^{-rT} E\left[(S(\tau_{N(T)}) - K)^+; \zeta > \tau_{N(T)} \mid S_t = e^x\right]$$

where $N(T) \equiv \sup\{n : \tau_n \leq T\}$. How can we correct for the difference between $S(\tau_{N(T)})$ and S_T? Observe that the piece of the path of X between $\tau_{N(T)}$ and T starts at some point z in Λ and gets stopped before it reaches either $z + \Delta x$ or $z - \Delta x$. For simplicity of exposition, assume that $z = 0$. We may approximate the distribution of the particle in $(-\Delta x, \Delta x)$ when stopped by the long-run distribution of a process $\tilde{X}$ which diffuses like X in $(-\Delta x, \Delta x)$, but once it reaches either end of the interval is immediately returned to 0 and starts again. Clearly, this is a positive recurrent Markov process, and we can compute its invariant law by finding the law of the process at an independent exponential time η with very large mean. Indeed, for a smooth test function $f \geq 0$ we have

$$E^0[f(\tilde{X}_\eta)] = E^0[\int_0^\rho \beta e^{-\beta t} f(X_t) dt] + E^0[e^{-\beta\rho}] E^0[f(\tilde{X}_\eta)]$$

where $\rho \equiv \inf\{t : |\tilde{X}_{t-}| = \Delta x\}$. Rearranging this gives

$$E^0[f(\tilde{X}_\eta)] = \frac{E^0[\int_0^\rho \beta e^{-\beta t} f(X_t) dt]}{1 - E^0[e^{-\beta\rho}]},$$

and letting $\beta \downarrow 0$ gives in the limit

$$\frac{E^0[\int_0^\rho f(X_t) dt]}{E^0[\rho]}.$$

The Green function for the drifting Brownian motion killed on exit from $(-\Delta x, \Delta x)$ can be computed quite simply; see, for example, Mandl [18], Itô & McKean [14], or Rogers & Williams [24] for accounts of the methodology. When we finish, we get the distribution to have density

$$h(x) = \frac{e^{2c\Delta x} - e^{2cx}}{\Delta x(e^{2c\Delta x} - 1)}, \qquad x \geq 0, \tag{13}$$

$$= \frac{e^{2c(x+\Delta x)} - 1}{\Delta x(e^{2c\Delta x} - 1)}, \qquad x \leq 0. \tag{14}$$

Thus to correct for the last interval, we just replace the terminal value $(e^x - K)^+$ for $x \in \Lambda$ by the expectation $\int_{-\Delta x}^{\Delta x} h(y)(e^{x+y} - K)^+ dy$.

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