Two-sector stochastic growth models

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Abstract. This paper develops the study of two-sector growth models of the form introduced by Arrow and Kurz (1970).

Being purely deterministic, the original model of Arrow & Kurz was unable to distinguish between open-loop and closed-loop control of the economy; by allowing stochastic terms into the model, we are able to resolve this difficulty of interpretation. Moreover, we also find that in some important cases the model can be solved explicitly in closed form, to the extent that we can write down expressions for tax rates and interest rates. This leads to new one-factor interest rate models, with related taxation policies; numerical examples show very reasonable behaviour.

Key words: Two-sector growth model, stochastic, fiscal policy, debt policy.

JEL classification system: O41, E43, E63.

1 Introduction

The history of growth models is long and illustrious, stretching back at least to Ramsey (1928). Throughout this development, much attention has been devoted to single-sector models, where there is just one type of capital or good, which is produced at a rate depending on current capital levels, labour force and technology levels, and is then either consumed or reinvested into capital. One analogy is a farm producing corn which can either be eaten or used to produce more corn. There are two basic types of continuous time single-sector growth model appearing in the economic literature. Firstly the Solow model as developed by Solow (1956) and Swan (1956). This is a growth model with an exogenously given savings rate which determines the proportion of capital reinvested (and hence also the proportion consumed). Denison (1961) showed that this model was able to explain trends in empirical growth data for the United States. Secondly there is the Ramsey model. This was originally conceived by Ramsey (1928) but the term is now used to refer

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to the version as refined[3] by Cass (1965) and Koopmans (1965). This is a growth model with consumer optimization - households choose their rates of consumption over time to maximise a utility functional. See, for example, the books of Romer (2001) and Barro and Sala-I-Martin (1995) for a more complete description of these models and their variants.

The first two-sector model was developed by Uzawa (1961), (1963) who considered an economy with two produced goods, a consumer good and an investment good, produced by investment capital and labour. Again using the farm analogy, this is using labour and tractors to make corn and tractors. Uzawa (1965) then refined this model to one where the two goods are physical capital and human capital, both of which are required for production of further physical capital (by manufacturing) and human capital (by education). Arrow and Kurz (1970) chose public capital rather than human capital and our work in this paper develops this model.

Arrow & Kurz proposed a deterministic model where there were two types of capital, government capital and private capital, which were both needed in the production of the single consumption good. They first set about solving the government's optimization problem, where the government's objective was to maximise the integrated discounted felicity from *per capita* consumption, where the felicity also depends on the *per capita* level of government capital. This feature of the model recognises that the felicity of the population is improved if the provision of education, healthcare, transportation, *etc.* is improved, and that such infrastructure is provided largely (if not exclusively) by government capital. Since Arrow & Kurz assume that private and government capital can be freely switched at any time, it is clear that the state of the optimally-controlled system at any time is completely described by the total amount of capital, the split between government and private sectors being dictated by optimality.

The problem gets more interesting when it comes to the behaviour of the private sector, which is viewed as very large collection of identical non-collaborating small households, each individually optimising its common objective, which is again an integrated discounted felicity of *per capita* consumption and government capital, but not of course the same as the government's objective. History and fashion have overwhelmed the centrally-planned economy, so we suppose that the government's control of the economy is exercised only through levying various proportional taxes, and issuing and retiring riskless debt from time to time. The central question studied by Arrow & Kurz is: *can the government manipulate taxes and debt in such as way as to induce the private sector to follow the government's optimal policy?*

The analysis of Arrow & Kurz is quite involved, but they are able to conclude that, under certain conditions, various combinations of taxes and debt can steer the economy onto the government's desired trajectory. However, the solutions they

[3]Although Ramsey's original model was actually more subtle than Cass/Koopmans in some respects, for example it included a disutility function to reflect the amount of labour supplied (i.e. the longer the hours worked the less the utility). We will adopt a similar approach.

find are in terms of deterministic trajectories for the various tax rates for all future times, and this leaves undecided the interpretation of the solution: *is this open-loop or closed-loop control?* That is, do we think of the solution for the income tax rate (which will be an explicit function of time) as something that the government commits to at time 0, or do we rather think of the income tax rate as being a function of the underlying state variable (the total amount of capital)? The former interpretation seems viable only if we assume that the world really is deterministic, and that the government can predict with perfect foresight for all time. Casual observation suggests that this is very unlikely to be the case, so we would prefer to have a solution where tax rates would be expressed in terms of the current state of the economy, rather than being set according to a centuries-old plan[4]. In the deterministic model of Arrow & Kurz, these two cannot be distinguished.

Another feature of Arrow & Kurz's solution is that we have little insight into the properties of the tax regimes the government would need to follow: in particular, are the tax rates always between 0 and 1? If not, are the suggested values actually credible?

To address these issues, we plan in this paper to take the model of Arrow & Kurz, and modify it in two respects:

(i) introduce random fluctuations in output and population size;

(ii) allow the population to choose their level of effort.

The first modification allows us to distinguish between solutions which are functions of the underlying state of the economy, and solutions which are pre-determined processes. Without the second modification, we find that the effects of income tax are unrealistic. Once again, it turns out that the optimal solution of the government's problem can be expressed in terms of a single underlying state variable, the technology-adjusted *per capita* capital in the economy, which now follows a stochastic differential equation, and is thus a diffusion. We are then able to solve the private sector's problem, and deduce relations which must be satisfied by the various tax rates and by government debt in order to induce the private sector to follow the trajectory desired by government. In particular, we look for (and find) solutions for the tax rates which are functions only of the state process.

As yet, these expressions for tax rates are still quite opaque, so we are no better placed to decide whether they will always be between 0 and 1, for example. Our response to that has been to find explicit examples which can be *solved in closed form*, and where it is possible to find the range of any of the tax rates, as these are expressed now as explicit functions of the state variable. A collection of such examples helps us to build up a library of possible behaviours, may lead us to other

[4]See Christiaans (2001) for further discussion on this point. He concludes that open-loop solutions of dynamic optimization problems are unstable and therefore provide no reasonable basis for a positive theory of economic growth.

interesting questions, and allows us to check further analytical and numerical work. The approach we use is simply to take the inverse problem; write down the solution we would like, and then see whether we can find a model to which that is the solution! So we obtain a simple solution to a possibly slightly complicated model, rather than no solution to a simple model. This approach applies even to the basic one-sector model, and we show in an appendix some of the solutions which can be obtained for that. Our consideration of explicit examples is similar to the so called "inverse optimal" problem first studied by Kurz (1968) of constructing the class of objective functions that could give rise to given specified consumption-investment functions. Chang (1988) solves a similar stochastic inverse optimal problem.

Shortly after the work of Arrow & Kurz growth theory fell out of favour, not making a return until the mid-1980s. Lucas (1988) extended the work of Denison (1961) by showing that a two-sector model can explain not only the trends in growth data, but also diversity between countries in the data. Consequently much of the recent growth literature deals with economies with two capital goods, the first usually being physical capital and examples of the second including human capital, public capital, financial capital, quality of products and embodied and disembodied knowledge (Mulligan and Sala-I-Martin 1993).

Models considering directly the effects of public investment come in two formulations. The first considers how the *rate* of government expenditure on public services effects the productivity of the economy; see Aschauer (1988) for a discrete example or Barro (1990) for a continuous time model. The second type of formulation considers the total *stock* of public capital, invested in such things as roads and hospitals, as the key input to the production rate. This was the problem first studied by Arrow & Kurz, with the stock of government capital appearing in the utility function as well as the production function. This second approach is arguably more realistic but has not been widely adopted, although Futagami, Morita, and Shibata (1993) have extended the model of Barro (1990) to include government capital, and Fisher and Turnovsky (1998) have adopted a Ramsey style framework, although in both these models the public capital only appears in the production function and not also the utility[5]. Baxter and King (1993) considered a discrete time model very similar to that of Arrow and Kurz.

Use of continuous time stochastic calculus in economic growth models first appeared in the papers of Bourguignon (1974), Merton (1975) and Bismut (1975). These extend the Solow growth model to a random setting by addition of a Brownian element to the labour supply (Bourguignon, Merton) or to the production process (Bourguignon, Bismut). Merton also considers a stochastic version of the Ramsey problem, again with Brownian motion appearing in the dynamics of the labour supply. Chapter 3 of Malliaris and Brock (1982) contains a good overview of these and similar models. More recent contributions building on Merton's 'Stochastic Ramsey Model' include Foldes (1978), (2001) who adds Brownian motions to further

[5]This is true for other two-sector models too. Usually the utility function is restricted to being a function of consumption and not of levels of capital or rates of investment.

parameters of the model, and Amilon and Bermin (2001) who allow the government to control the expected growth rate of the labour supply. We have been unable to find any continuous time stochastic two-sector (private sector and government capital) models anywhere in the literature.

One of the possible uses of a stochastic growth model is to study interest rate dynamics. Merton (1975) does this for the stochastic Solow model using a Cobb-Douglas production function and a constant savings ratio. Amilon and Bermin (2001) use a stochastic Ramsey model and generate a variety of interest rate processes by considering different production and utility functions. Cox, Ingersoll, and Ross (1985a), (1985b) develop a simple stochastic model of capital growth which they use to determine the behaviour of asset prices including the term structure of interest rates. Sundaresan (1984) builds on this work and that of Merton by considering multiple consumption goods with a Cobb-Douglas production function.

The layout of the remainder of the paper is as follows. In Section 2 we describe our model and consider the *central-planning problem* where the government has total control over the economy and wishes to maximise its own utility functional. We give conditions that the government's optimal choices must obey. Section 3 introduces taxation and a private sector independently optimising its own utility functional subject to taxation constraints. We find expressions that the tax rates must satisfy in order to force the private sector to follow the *government's* optimal choices. In Section 4 we look at ways to find explicit solutions to the problems considered in the previous sections and give an example. We conclude in Section 5, which is followed by four appendices. Appendix A has proofs of statements made earlier in the paper. Appendix B is a (technical) discussion of the behaviour of the level of government debt. Appendix C shows how our results simplify to the one-sector Ramsey model. Finally Appendix D contains a useful summary of the notation used in the paper.

2 The government's problem

The dynamics of the total capital K_t in the economy at time t evolves according to the equation[6]

$$dK(t) = K(t)dZ_t^0 + \Big[\ F(K_p(t), K_g(t), T(t)L(t)\pi(t)) - \delta K_t - C_t\ \Big]dt, \qquad (2.1)$$

where Z^0 is some positive multiple of a standard Brownian motion, L_t is the size of the population at time t, $\pi(t) \in [0,1]$ is the proportion of the population's effort devoted to production, and $K_p(t)$ is the amount of private capital in existence at time t, $K_g(t) \equiv K(t) - K_p(t)$ the amount of government capital at time t. The parameter δ is the rate of depreciation, a positive constant, the process C is the aggregate consumption rate, and the process T is the labour-augmenting effect of

[6] As a notational convenience, we use subscript and argument notations $K_t \equiv K(t)$ interchangeably throughout, and will omit appearance of the time argument where there is no risk of confusion.

improvements in technology. We shall assume always that K, K_g, K_p and C are non-negative. Concerning the production function F, we shall make the usual assumption of homogeneity of degree 1, which is to say that

$$F(\lambda K_p, \lambda K_g, \lambda L) = \lambda F(K_p, K_g, L) \tag{2.2}$$

for any $\lambda > 0$. We shall also suppose that

$$dL_t = L_t(dZ_t^L + \mu_L dt), \tag{2.3}$$
$$dT_t = \mu_T T_t dt, \qquad T_0 = 1, \tag{2.4}$$

where μ_L and $\mu_T \geq 0$ are constants and where now Z^L is again a multiple of a standard Brownian motion, with

$$\langle dZ^i, dZ^j \rangle = v_{ij} t, \qquad i, j \in \{0, L\}, \tag{2.5}$$

where $\langle M, N \rangle$ denotes the quadratic covariation between continuous martingales M and N; see, for example, Rogers & Williams (2000) for definitions and basic properties.

The objective of the government is to maximise

$$E \int_0^\infty e^{-\rho_g t} L_t \, U\left(\frac{C_t}{L_t}, \frac{K_g(t)}{L_t}, \pi_t \right) dt, \tag{2.6}$$

where U is strictly concave, and increasing in the first two arguments, decreasing in the last. The objective (2.6) depends on *per capita* consumption and per capita government capital, and the felicity is weighted according to the current population size. In order to have the prospect of a time-homogeneous solution, we require that U is also homogeneous of degree $(1 - R_g)$ for some $R_g > 0$[7]; this means that U can be represented as

$$U(C, K_g, \pi) = K_g^{1-R_g} h(\xi, \pi), \qquad \xi \equiv C/K_g \tag{2.7}$$

for some C^2 function h strictly concave and increasing in its first argument and decreasing in its second[8]. Our main results will be proved only subject to the

ASSUMPTION: h is either non-negative or non-positive.

This restriction seems to be satisfied in many interesting cases, and is probably not really necessary; we require it to save us from over-clumsy statements of results, and leave to the reader the trouble of checking our results still hold in any particular example where h is not of constant sign.

[7] We also assume that $R_g \neq 1$, not because the case of logarithmic utility is in any way difficult, but rather because some of the expressions to be developed have a different appearance in this special case.

[8] In fact for U to have the required properties we will also need that $(1 - R_g)h > \xi h_\xi$, $\xi^2 h_{\xi\xi} + 2R_g \xi h_\xi - R_g(1 - R_g)h < 0$ and $R_g h_\xi^2 < -(1 - R_g) h h_{\xi\xi}$.

As a consequence of the assumptions so far, it turns out to be advantageous to work with *per capita* technology-adjusted variables, rather than their aggregated equivalents. So if we define

$$\eta_t \equiv L_t T_t = L_0 \exp\left\{ Z_t^L + (\mu_L - \frac{1}{2} v_{LL} + \mu_T)t \right\}, \tag{2.8}$$

and then define

$$k_t \equiv K_t/\eta_t, \quad k_g(t) \equiv K_g(t)/\eta_t, \quad k_p(t) \equiv K_p(t)/\eta_t, \quad c_t \equiv C_t/\eta_t, \tag{2.9}$$

and so forth, we find that the dynamics of k follow from the dynamics (2.1) of K:

$$dk_t = k_t(dZ_t^0 - dZ_t^L) + \left[F(k_p(t), k_g(t), \pi_t) - \gamma k_t - c_t \right] dt, \tag{2.10}$$

where

$$\gamma \equiv \delta + \mu_L + \mu_T + v_{0L} - v_{LL}.$$

It is now necessary to re-express the government objective (2.6) in terms of *per capita* technology-adjusted variables, and here the assumption that U is homogeneous of degree $(1 - R_g)$ enters in an essential way. We find that the objective of the government can be expressed as

$$\begin{aligned} E \int_0^\infty e^{-\rho_g t} L_t \, U\left(\frac{C_t}{L_t}, \frac{K_g(t)}{L_t}, \pi_t \right) dt &= E \int_0^\infty e^{-\rho_g t} L_t \, U(c_t T_t, k_g(t) T_t, \pi_t) \, dt \\ &= E \int_0^\infty e^{-\rho_g t} L_t T_t^{1-R_g} U(c_t, k_g(t), \pi_t) \, dt \\ &= L_0 E_g \int_0^\infty e^{-\lambda_g t} U(c_t, k_g(t), \pi_t) \, dt \quad (2.11) \end{aligned}$$

where

$$\lambda_g \equiv \rho_g - (1 - R_g)\mu_T - \mu_L,$$

and the final expectation is with respect to the measure P_g which is absolutely continuous with respect to P on every $\mathcal{F}_t$[9], and has density

$$\left. \frac{dP_g}{dP} \right|_{\mathcal{F}_t} = \exp(Z_t^L - \frac{t}{2} v_{LL}).$$

The effect of changing measure from P to P_g is to introduce additional drift[10] into the Brownian motions Z^0 and Z^L; precisely, we have

$$\begin{aligned} Z_t^0 &= z_t^0 + v_{0L} t, \\ Z_t^L &= z_t^L + v_{LL} t, \end{aligned}$$

[9] The filtration $(\mathcal{F}_t)_{t \geq 0}$ denotes the working filtration, with respect to which all processes are adapted.

[10] This is the famous Cameron-Martin-Girsanov Theorem; see, for example, Rogers and Williams (2000) for an account

where (z^0, z^L) are two P_g-martingales possessing the same covariance structure as (Z^0, Z^L). This therefore transforms the dynamics (2.10) into

$$dk_t = k_t(dz_t^0 - dz_t^L) + \Big[\, F(k_p(t), k_g(t), \pi_t) - \gamma_g k_t - c_t \,\Big] dt, \tag{2.12}$$

where the constant γ_g is given by

$$\gamma_g = \gamma - v_{0L} + v_{LL} = \delta + \mu_L + \mu_T.$$

In order to maximise (2.11) with the dynamics (2.12), we can proceed to find the Hamilton-Jacobi-Bellman equation for the value function

$$V(k) \equiv \sup E_g\Big[\int_0^\infty e^{-\lambda_g t} U(c_t, k_g(t), \pi_t)\, dt \,\Big|\, k_0 = k \Big]. \tag{2.13}$$

The HJB equation satisfied by V is

$$\sup_{c, k_g, 0 \le \pi \le 1} U(c, k_g, \pi) - \lambda_g V(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + [F(k - k_g, k_g, \pi) - \gamma_g k - c] V'(k) = 0, \tag{2.14}$$

where

$$\sigma^2 \equiv v_{00} - 2v_{0L} + v_{LL}.$$

From this, we deduce the necessary first-order conditions for optimality:

$$U_c(c, k_g, \pi) = V'(k) \tag{2.15}$$
$$U_g(c, k_g, \pi) = V'(k)(F_p(k_p, k_g, \pi) - F_g(k_p, k_g, \pi)), \tag{2.16}$$
$$U_\pi(c, k_g, \pi) = -V'(k) F_L(k_p, k_g, \pi) \tag{2.17}$$

where we use subscripts to denote differentiation, as in the abbreviations:

$$U_c \equiv \frac{\partial U}{\partial c}, \quad U_g \equiv \frac{\partial U}{\partial k_g}, \quad f_p \equiv \frac{\partial f}{\partial k_p}, \quad f_g \equiv \frac{\partial f}{\partial k_g}.$$

The conditions (2.15), (2.16) and (2.17) arise from considering the optimization problem

$$\sup_{c, k_g, 0 \le \pi \le 1} U(c, k_g, \pi) + p[\, F(k - k_g, k_g, \pi) - c\,]; \tag{2.18}$$

implicit in the statements (2.15) and (2.16) is the following assumption:

For every $p, k > 0$, the problem (2.18) has an interior solution which depends in a C^1 fashion on (p, k) (2.19)

(In fact, the assumed strict concavity of U makes an interior solution unique.) This assumption does not always hold, but we shall make it for the sake of the simplifications in the statements and proofs of results; no doubt similar conclusions can be reached without it, but we leave that as an issue for further research.

The observation that the optimising values (c, k_g, π) are uniquely determined as functions of (p, k) reduces the HJB equation (2.14) to a non-linear differential equation for V; once the solution is found, we are able to express the optimal values of (c, k_g, π) as functions of $(V(k), k)$, or, more simply put, functions of k. We shall henceforth use the notation c^*,k_g^* and π^* for these optimal functions[11] of the underlying state variable k, and also we shall introduce the notation

$$\Phi(k) = F(k_p^*(k), k_g^*(k), \pi^*(k)) - \gamma_g k - c^*(k) \tag{2.20}$$

for the drift in the dynamics (2.12), which therefore are more compactly expressed as

$$\begin{aligned} dk_t &= k_t(dz_t^0 - dz_t^L) + \Phi(k_t)dt & (2.21) \\ &= \sigma k_t dw_t + \Phi(k_t)dt, & (2.22) \end{aligned}$$

where the P_g-Brownian motion w is defined by $w \equiv (z^0 - z^L)/\sigma$. Under the original measure P the dynamics (2.10) can be written as

$$\begin{aligned} dk_t &= k_t(dZ_t^0 - dZ_t^L) + \tilde{\Phi}(k_t)dt, & (2.23) \\ &= \sigma k_t dW_t + \tilde{\Phi}(k_t)dt, & (2.24) \end{aligned}$$

with the identifications $\tilde{\Phi}(k) \equiv \Phi(k) + (\gamma_g - \gamma)k$, and $W \equiv (Z^0 - Z^L)/\sigma$. Under mild conditions[12] on Φ, (2.21) has a pathwise-unique strong solution, and the value function V will satisfy the equation

$$U(c^*(k), k_g^*(k), \pi^*(k)) - \lambda_g V(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + \Phi(k)V'(k) = 0. \tag{2.25}$$

Although there may be some issues concerning smoothness of the (c, k_g) optimizing in (2.19), the following result is the starting point of our investigations.

Theorem 1 *(i) Assuming that the value function* (2.13) *is finite valued and* C^3*, and that assumption* (2.19) *holds then there exist differentiable functions* Φ, c^*, k_g^*, π^* *and twice-differentiable* $\Psi \equiv V'$ *such that the equalities*

$$\begin{aligned} 0 &= \Psi(F_p - \gamma_g - \lambda_g) + \Psi'(\Phi + \sigma^2 k) + \frac{1}{2}\sigma^2 k^2 \Psi'' & \text{(G1)} \\ U_c &= \Psi & \text{(G2)} \\ U_\pi &= -F_L \Psi & \text{(G3)} \\ U_g &= (F_p - F_g)\Psi & \text{(G4)} \\ \Phi &= F - \gamma_g k - c & \text{(G5)} \end{aligned}$$

hold along the path given by $(c^*(k), k_g^*(k), \pi^*(k))$[13].

[11] The notation k_p^* will also be used, with the obvious interpretation $k_p^*(k) = k - k_g^*(k)$.

[12] Global Lipschitz will certainly be enough: Rogers & Williams (2000) again, Theorem V.11.2.

[13] This means, for example, that $U_g(c^*(k), k_g^*(k), \pi^*(k)) = (F_p - F_g)(k_p^*(k), k_g^*(k), \pi^*(k))\Psi(k)$ in the case of (G4).

(ii) Conversely suppose that there exist differentiable functions Φ, c^*, k_g^*, π^* and twice-differentiable Ψ such that the equalities (G1)–(G5) *hold along the path given by $(c^*(k), k_g^*(k), \pi^*(k))$. If k^* is the solution to the SDE* (2.21) *then provided the transversality condition*

$$\sup_t e^{-\lambda_g t} k_t^* \Psi(k_t^*) \in L^1, \qquad \lim_{t\to\infty} e^{-\lambda_g t} k_t^* \Psi(k_t^*) = 0 \tag{GT}$$

holds, the policy given by (c^*, k_g^*, π^*) is optimal for the government, the optimally-controlled economy follows the dynamics (2.21) *and there is a value function $V(k)$ given by*

$$V(k) \equiv \int_1^k \Psi(y)\, dy + V_1$$

which satisfies the HJB equation

$$0 = -\lambda_g V + V'\Phi + \frac{1}{2}\sigma^2 k^2 V'' + U \tag{G6}$$

along the optimal path, where V_1 is a constant that can be determined explicitly.

Proof. (i) follows from the discussion above; (G1) is obtained by differentiating the HJB equation (2.14) with respect to k and then making use of conditions (2.15)–(2.17). (ii) - see Appendix A

Theorem 1 characterises the optimal solution to the government's problem, but what can we do with it? Are there examples where the solution can be expressed in closed form? In view of the complicated way in which the optimising values c^*, k_g^* and π^* were defined, it appears at first sight unlikely, but we shall later see that it *is* possible to exhibit explicit solutions.

3 Government borrowing and taxation

The government's optimal policy has been determined, but the issue now is how to implement that policy when the government cannot directly control the private sector, but can only shape its choices through taxation and the issuing of government debt. Since the optimal policy of the previous Section was Markovian, in the sense that the total technology-adjusted *per capita* capital k was a Markov process (even a diffusion), we shall now seek Markovian taxation policies, which are defined by the property that the rates of tax are functions only of k.

Before we can understand the effects of government fiscal policy, we have to understand the behaviour of the private sector on which it acts, and we turn to that now. We think of the private sector as made up of a very large number L_0 of identical households; if one of these households receives a cashflow process of $(\Delta C_t)_{t\geq 0}$, then it values this cashflow as

$$E\int_0^\infty e^{-\rho_p t} u\Big(\frac{L_0 \Delta C_t}{L_t}, \frac{K_g(t)}{L_t}, \pi_t\Big)\, dt, \tag{3.1}$$

and it wishes to maximise this. Here, u is strictly concave, increasing in its first two arguments, and decreasing in the third, and $\rho_p > 0$ is constant. The felicity u depends on the *per capita* level of government capital, and on the *per capita* rate of consumption for the household, which is assumed to be subject to the same size fluctuations as the entire population; it also varies inversely with the proportion of effort devoted to production. As with the government objective, we assume that u is homogeneous, of degree $(1 - R_p)$, where $R_p > 0$ is different from 1, and typically different from R_g.

We suppose that the objectives of the government and private sector are different, and that the government aims to set taxes and to borrow in such a way as to induce the private sector to follow the government's desired path. We need now to decompose the dynamics (2.1) of the economy so as to understand the effects of the taxes. Homogeneity of order 1 of F implies[14] that we may express the output as the sum of three terms,

$$F(K_p, K_g, L) = K_p F_p(K_p, K_g, L) + K_g F_g(K_p, K_g, L) + L F_L(K_p, K_g, L),$$

which are interpreted as the return on private capital, the return on government capital, and the return on labour, respectively. Including the random effects term (dZ^0) then, the returns on private capital, government capital and labour are (respectively)

$$K_p dZ_t^0 + K_p F_p dt, \quad K_g dZ_t^0 + K_g F_g dt, \quad \pi LTF_L dt. \tag{3.2}$$

We shall suppose that the government is able to appropriate some fixed proportion $1 - \theta_p - \theta_L$ of the returns to its capital by direct charging for services such as toll roads, university tuition fees, subsidised rail fares, and some healthcare costs, but it is in the nature of government expenditure that much of the return on government capital cannot be directly appropriated, so in practice this proportion may be near to zero. A proportion θ_p of the returns to government capital are included in the returns to private capital, and the remaining proportion θ_L is included in returns to labour, so that from an accounting point of view we suppose that the returns on private capital and labour are (respectively)

$$K_p dZ_t^0 + K_p F_p dt + \theta_p(K_g dZ_t^0 + K_g F_g dt), \quad \theta_L(K_g dZ_t^0 + K_g F_g dt) + \pi LTF_L dt, \tag{3.3}$$

with the remaining $(1 - \theta_p - \theta_L)(K_g dZ_t^0 + K_g F_g dt)$ going directly to government.

The evolution of the levels of private and government capital are determined by the equations

$$dK_p = dI_p - \delta K_p dt \tag{3.4}$$

$$dK_g = dI_g - \delta K_g dt, \tag{3.5}$$

where $I_p(t)$ is the amount invested in private capital by time t.

[14]Differentiate the identity (2.2) with respect to λ.

The government will issue debt and levy taxes; returns on private capital will be taxed at rate $1-\beta_k$, income at rate $1-\beta_w$, consumption at rate $1-\beta_c$, and interest on government debt at rate $1-\beta_r$, so that the private sector's aggregate budget equation[15] is therefore

$$\begin{aligned} dI_p \;&+\; dD + \beta_c^{-1} C dt \\ &= \beta_k \big[\, K_p(dZ_t^0 + F_p dt) + \theta_p(K_g dZ_t^0 + K_g F_g dt)\,\big] + r\beta_r D dt \\ &\quad + \beta_w \big[\, \theta_L K_g dZ_t^0 + (F - K_p F_p - (1-\theta_L)K_g F_g)dt\,\big] \end{aligned} \tag{3.6}$$

where D_t denotes the amount of government debt at time t. The interpretation of the left-hand side is that this is the total outgoings of the private sector: the investment in private capital, the investment in government debt, and the cost of consumption. The right-hand side (3.6) is the after-tax income of the private sector: return on private capital plus interest on government debt plus wage income.

The relation (3.4) can be used to eliminate dI_p and rewrite the private-sector budget equation as

$$\begin{aligned} dK_p + dD \;=\; & K_p\big[\,\beta_k dZ^0 + (\beta_k F_p - \delta)dt\,\big] + r\beta_r D dt - \beta_c^{-1} C dt \\ & + \beta_w \pi \eta F_L dt + (\beta_k \theta_p + \beta_w \theta_L)(K_g dZ^0 + K_g F_g dt) \end{aligned} \tag{3.7}$$

which bears the simple interpretation that the change in private-sector wealth is accounted for by the return on private capital (adjusted for depreciation) plus the return on government debt, less consumption plus the wage income.

Recall that we seek tax rates as functions of k which will cause the private sector to follow the government's optimal trajectory. So we shall suppose that such tax rates have been set, the economy as a whole is following the government's optimal policy as discussed in Section 2, and shall consider the optimisation problem faced by a single household. *If any deviation from the government's optimal path is suboptimal for the individual household, then we have an equilibrium in which all households follow the government's optimal path;* we shall suppose that this is what happens, and deduce the implications for the tax rates and borrowing policy. These are summarised in the following result.

Theorem 2 *Suppose that the government sets proportional taxes $1-\beta_c$ on consumption, $1-\beta_w$ on income, $1-\beta_k$ on returns on private capital, and $1-\beta_r$ on returns on government debt, all functions only of the total technology-adjusted* per capita *capital k in the economy at the time. If there exists a C^2 function ψ, and a*

[15] Arrow & Kurz have also a tax on savings, which alters the term $dI_p + dD$ in (3.6) to $\beta_s^{-1}(dI_p + dD)$. Since this could be absorbed into our formulation simply by reinterpreting the other $\beta_\cdot$, we lose no generality by studying the equations as given.

function r such that the equations

$$0 = \psi(\beta_k F_p - \gamma - \lambda_p + v_{0L}(1-\beta_k)) \qquad \text{(PS1)}$$
$$+ \psi'(\tilde{\Phi} + \beta_k \sigma^2 k + (1-\beta_k)(\gamma_g - \gamma)k) + \frac{1}{2}\sigma^2 k^2 \psi''$$
$$u_c = \beta_c^{-1}\psi \qquad \text{(PS2)}$$
$$u_\pi = -\beta_w F_L \psi \qquad \text{(PS3)}$$
$$0 = \psi(r\beta_r + \mu_0 - \lambda_p) + \psi'(\tilde{\Phi} + (\gamma_g - \gamma)k) + \frac{1}{2}\sigma^2 k^2 \psi'' \qquad \text{(PS4)}$$

all hold along the government's optimal path[16]*, where $\lambda_p = \rho_p - (1-R_p)\mu_T$, and $\mu_0 = v_{LL} - \mu_L - \mu_T$, then the private sector faced with these tax rates will choose to follow the government's optimal path, provided the transversality condition*

$$\sup_t e^{-\lambda_p t}|x_t|\psi(k_t^*) \in L^1, \qquad \lim_{t\to\infty} e^{-\lambda_p t} x_t \psi(k_t^*) = 0 \qquad \text{(PST)}$$

is satisfied, where $x \equiv k_p + \Delta_p$ is the total technology-adjusted per capita wealth of the private sector, split between private capital k_p and government debt Δ_p.

Proof. See Appendix A

Remarks. (i) Of course, the way we plan to use Theorem 2 is to enable us to *find* the tax regimes which will persuade the private sector to follow the government optimal path. So if we suppose that the government's optimal path has been determined, as in Section 2, we want now to know whether it is possible to have the conditions (PS1), (PS2), (PS3) and (PS4) all holding at the same time. But this is in fact quite easy: for example, if we *choose* the functional form of β_c and β_r, then (PS2) determines the function ψ and then β_k, β_w and r are determined from (PS1), (PS3) and (PS4) respectively.

(ii) Note the similarities between conditions (PS1), (PS2) and (PS3) and the corresponding conditions (G1), (G2) and (G3) of Theorem 1. If we set the tax rates to zero (so $\beta_k = 1$ etc.) then these conditions of Theorem 2 are identical in form to those of Theorem 1; however they depend on the private sector parameters λ_p and γ and on the private sector utility function u rather than the corresponding government quantities. Only if the private sector and government share identical values $\lambda_p = \lambda_g$, $\gamma = \gamma_p$ and $u \equiv U$ will the private sector follow the government's optimal path under a no-tax regime.

(iii) We do not claim (nor is it true in general) that the solution is Markovian in the sense defined above, because the process Δ_p may fail to be a function only of k^*. However, under certain conditions we can characterize the long-term behaviour

[16] For example, in full (PS3) says $u_\pi(c^*(k), k_g^*(k), \pi^*(k)) = -\beta_w(k) F_L(k_p^*(k), k_g^*(k), \pi^*(k))\psi(k)$.

of the debt, writing it in the form

$$\Delta_p(t) = G_1(k_t)k_t^{-a/\sigma} + \int_{-\infty}^{t} \left(\frac{k_u}{k_t}\right)^{a/\sigma} e^{-b(W_t'-W_u')-\frac{1}{2}b^2(t-u)+\int_u^t G_0(k_v)dv} \left\{G_2(k_u)dW_u' + G_3(k_u)du\right\},$$

where G_0, G_1, G_2, G_3 are given functions of k and W' is a Brownian motion that is *completely independent* of the Brownian motion $W \equiv (Z^0 - Z^L)/\sigma$ that drives the dynamics of k. See Appendix B for the details.

In a model with tax rates as proposed above we are unlikely to be able to find a solution with a Markovian debt process. If we require that Δ_p is a function of k then $x \equiv \Delta_p + k_p$ must also be a function of k and hence

$$dx = x'(\tilde{\Phi}dt + k(dZ^0 - dZ^L)) + \frac{1}{2}\sigma^2 k^2 x'' dt. \tag{3.8}$$

Equating the dZ^L term in the above with that in the other expression we have for the private sector dynamics (A.5) we find that

$$x'k = x$$

and hence $x \equiv \Gamma k$ for some constant Γ. With this identification we can now equate the dZ^0 and dt terms in (3.8) and (A.5) giving (after some rearrangement)

$$(\Gamma - \beta_k)k = \{\beta_w \theta_L - (1 - \theta_p)\beta_k\}k_g \tag{3.9}$$

and

$$\Gamma\tilde{\Phi} + \beta_c^{-1}c - \beta_w \pi F_L = k_p\left[\beta_k(F_p - F_g) - \delta - r\beta_r\right] + \Gamma k\left[\mu_0 - v_{0L} + r\beta_r + F_g\right] \tag{3.10}$$

We now effectively have six equations (PS1)–(PS4), (3.9), (3.10) in five unknowns ψ, β_c, β_k, $r\beta_r$ and β_w so we are unlikely to be able to find a consistent solution to these, and even if we can the solution is likely to be highly dependent on exact choice of parameters.

(iv) If we subtract equation (PS4) from (PS1) then, after some rearrangement, we obtain

$$\beta_k\left(F_p - v_{0L} + \frac{\psi'}{\psi}(v_{00} - v_{0L})k\right) = r\beta_r + \delta. \tag{3.11}$$

Thus the net return on private capital $\beta_k F_p$ is equal to the net return on debt $r\beta_r$ plus depreciation δ and some 'price of risk' terms.

4 Explicit solutions

Theorem 1 tells us that provided there exist functions Ψ, Φ, c^*, k_g^* and π^* satisfying the equations (G1)–(G5) and the transversality condition (GT), then we have a solution to the original problem. In general, it will be hard to find explicit solutions for a given problem; nonetheless, we shall show in this Section that explicit solutions abound, and can be manufactured readily by considering the *inverse problem*, where we postulate a form for the solution and seek a problem whose solution is as postulated.

The homogeneity of degree $1 - R_g$ assumed for U gives the expression

$$U(c, k_g, \pi) = k_g^{1-R_g} h(\xi, \pi) \tag{4.1}$$

where $h(x, \pi) \equiv U(x, 1, \pi)$, and $\xi \equiv c/k_g$. Differentiation gives

$$\begin{aligned} U_c(c, k_g, \pi) &= k_g^{-R_g} h_\xi(\xi, \pi), & (4.2)\\ U_g(c, k_g, \pi) &= k_g^{-R_g} \left[\, (1 - R_g) h(\xi, \pi) - \xi h_\xi(\xi, \pi) \,\right], & (4.3)\\ U_\pi(c, k_g, \pi) &= k_g^{1-R_g} h_\pi(\xi, \pi). & (4.4) \end{aligned}$$

To find an explicit solution[17], we first *make a choice of the functions* Ψ *(equivalently* V*),* U *(equivalently,* h *and* R_g*), and* k_g. Not all such choices will result in soluble problems; for example, we will have to have that V is concave. Moreover, we shall require of our proposed solution that

$$0 \le k_g \le k \tag{4.5}$$

to avoid the possibility that either of k_p, k_g should be negative[18]. However, we can from these choices deduce what the solution (if it exists) must be, by solving the equations (G1)–(G5) and (4.1)–(4.4). To see how this is done, first note that as F is required to be homeogenous of order 1, we have a consistency condition

$$F = k_p F_p + k_g F_g + \pi F_L\,, \tag{4.6}$$

hence

$$\begin{aligned} \Phi + \gamma_g k + c &= k_p F_p + k_g F_g + \pi F_L \\ &= k F_p - (F_p - F_g) k_g + \pi F_L\,, & (4.7) \end{aligned}$$

by (G5). Since (G1) can be written as

$$0 = V'(-\Phi' + F_p - \gamma_g) - U' \tag{4.8}$$

where U' denotes the derivative with respect to k of the function

$$U(k) \equiv U(c(k), k_g(k), \pi(k)), \tag{4.9}$$

[17]Please excuse us if we do not use superscript asterisks in this discussion.

[18]Depending on the form of the production function and the felicities, negative values may be mathematically possible, but we shall restrict attention to more realistic situations where this does not happen.

we can use conditions (G1), (G3) and (G4) to rewrite (4.7) as

$$V'(\Phi + \gamma_g k + c) = k(U' + V'\gamma_g + V'\Phi') - k_g U_g - \pi U_\pi.$$

Now as U is homogeneous of order $(1-R_g)$ and from (G2) we know that $(1-R_g)U = cU_c + k_g U_g = cV' + k_g U_g$ it follows that

$$V'\Phi = k(U' + V'\Phi') - (1 - R_g)U - \pi U_\pi$$

and so

$$(V' + kV'')\Phi = kU' + k(V'\Phi)' - (1 - R_g)U - \pi U_\pi.$$

We can use equation (G6) to get an expression for Φ and $V'\Phi$, giving

$$\left(1 + \frac{kV''}{V'}\right)(\lambda_g V - U - \frac{1}{2}\sigma^2 k^2 V'') = kU' + k(\lambda_g V - U - \frac{1}{2}\sigma^2 k^2 V'')' - (1 - R_g)U - \pi U_\pi,$$

which can be rearranged to give

$$\begin{aligned} \left(\frac{\lambda_g V - \frac{1}{2}\sigma^2 k^2 V''}{kV'}\right)' &= \frac{1}{k^2 V'}\left[\pi U_\pi - \left(R_g + \frac{kV''}{V'}\right)U\right] \\ &= \frac{k_g^{1-R_g}}{k^2 V'}\left[\pi h_\pi - \left(R_g + \frac{kV''}{V'}\right)h\right]. \end{aligned} \tag{4.10}$$

(G2) gives us that

$$V' = U_c = k_g^{-R_g} h_\xi(\xi, \pi). \tag{4.11}$$

We will require that this equation along with (4.10) *allows us to determine* $\xi(k)$ *and* $0 \leq \pi(k) \leq 1$ *for all* k. As we originally chose a specific $k_g(k)$ we now also know $c(k) \equiv k_g(k)\xi(k)$, and hence the form of the function $U(k) \equiv U(c(k), k_g(k), \pi(k))$. We can recover $\Phi(k)$ from (G6) and we can express F_p (evaluated along the path $(k_p(k), k_g(k), \pi(k))$) explicitly using (4.8). Similarly, combining (4.3) with (G4) gives the relation

$$F_p - F_g = \frac{(1 - R_g)h(\xi, \pi) - \xi h_\xi(\xi, \pi)}{k_g^{R_g} V'(k)}, \tag{4.12}$$

expressing the difference $F_p - F_g$ as a known function of k, and combined with our knowledge of F_p we get F_g as a function of k. Finally we obtain F_L from (4.4) combined with (G3). How near are we to a solution? Equations (G1)–(G4), (4.1), (4.2) and (4.3) hold along the trajectory by construction; equation (G5) could be used to define the value of F along the trajectory as a function of k, but is this consistent with the forms of F_p, F_g and F_L which we have just found? We have to check that if F_p, F_g and F_L are obtained as above then

$$\begin{aligned} (\Phi + \gamma_g k + c)' &= \frac{d}{dk} F(k_p(k), k_g(k), \pi(k)) \\ &= F_p - (F_p - F_g)k_g' + F_L \pi'. \end{aligned}$$

Multiplying throughout by V', what we have to show is

$$\begin{aligned} V'\Phi' + V'c' &= V'(F_p - \gamma_g) - k_g' U_g - U_\pi \pi' \\ &= U' + V'\Phi' - k_g' U_g - U_\pi \pi' \end{aligned}$$

which is equivalent to proving

$$U' = c' U_c + k_g' U_g + U_\pi \pi',$$

and this is immediate.

By this inverse approach we have constructed a trajectory $(c(k), k_g(k), \pi(k))_{k\geq 0}$, and have found the values of the production function F along this path. What we still need to check is that the function F can be extended off the path where it known to some concave function $\tilde{F}(K_p, K_g, L)$ increasing in all its arguments, homogeneous of degree 1, that agrees with F along the path $(k_p(k), k_g(k), \pi(k))_{k\geq 0}$. Let us abbreviate $F(k_p(k), k_g(k), \pi(k))$ to $F(k)$, with similar interpretations of $F_p(k)$, $F_g(k)$ and $F_L(k)$. Clearly, if there *is* such a concave increasing function $\tilde{F}$, we shall have to have at very least the conditions

$$F_p(k) \geq 0, \quad F_g(k) \geq 0, \quad F_L(k) \geq 0 \qquad \forall k \geq 0, \tag{4.13}$$

along with the homogeneity condition (4.6), which holds by construction, and the 'tangent inequality'

$$F(k) \leq \Lambda(k_p(k), k_g(k), \pi(k); w), \quad \forall k, w \geq 0 \tag{4.14}$$

where

$$\begin{aligned} \Lambda(x, y, z; w) &\equiv F(w) + (x - k_p(w))F_p(w) + (y - k_g(w))F_g(w) + (z - \pi(w))F_L(w) \\ &= xF_p(w) + yF_g(w) + zF_L(w) \end{aligned}$$

is the equation of the tangent plane to F at $(k_p(w), k_g(w), \pi(w))$. However, these three conditions (4.13), (4.6) and (4.14) are already almost enough. Defining

$$\tilde{F}(x, y, z) \equiv \inf_{w\geq 0} \Lambda(x, y, z; w), \tag{4.15}$$

it is clear that $\tilde{F}$ is concave and increasing in all its arguments. If we assume also

$$\text{the infimum in (4.15) is attained uniquely,} \tag{4.16}$$

then for a general (x, y, z) there exists a unique $w_0 = w_0(x, y, z)$ such that

$$\begin{aligned} \tilde{F}(x, y, z) &= \Lambda(x, y, z; w_0), \\ \tilde{F}_p(x, y, z) &= F_p(w_0), \\ \tilde{F}_g(x, y, z) &= F_g(w_0), \\ \tilde{F}_L(x, y, z) &= F_L(w_0) \end{aligned}$$

and so

$$\begin{aligned}&\tilde{F}(x,y,z) - x\tilde{F}_p(x,y,z) - y\tilde{F}_g(x,y,z) - z\tilde{F}_L(x,y,z)\\&= F(w_0) - k_p(w_0)F_p(w_0) - k_g(w_0)F_g(w_0) - \pi(w_0)F_L(w_0) = 0\end{aligned}$$

by (4.6), as required. The non-negativity of $\tilde{F}$ still needs to be checked, but because of the non-negativity of F_p, F_g and F_L, it is immediate that $\Lambda(x,y,z;w)$ is non-negative for any $x,y,z \geq 0$, and so $\tilde{F}$ is non-negative.

Thus we see that in general if we propose k_g, concave U homogeneous of degree $1-R_g$, and concave V, we can construct a candidate solution: provided we can check (4.13), (4.14), (4.16), and (GT), then we have a solution. It may well be, of course, that the production function defined by (4.15) cannot be expressed more simply; in this sense, then, we will have built an explicit solution to a problem whose statement is somewhat implicit, which is arguably more use than an implicit solution to an explicit problem.

Taking the right-hand side of the tangent inequality (4.14) less the left-hand side and differentiating with respect to k gives us

$$k_p'(k)\,[F_p(w) - F_p(k)] + k_g'(k)\,[F_g(w) - F_g(k)] + \pi'(k)\,[F_L(w) - F_L(k)] \qquad (4.17)$$

as $F' = k_p'F_p + k_g'F_g + \pi'F_L$. If we can show that this expression is non-negative for $k \geq w$ and non-positive for $k \leq w$ then the tangent inequality (4.14) follows. We want a solution where k_p and k_g are increasing functions of k and π decreases with k so that means that a sufficient condition for the tangent equality to hold is that F_p and F_g are decreasing functions of k and F_L is an increasing function of k. In practice the following reworking will prove more useful. Using the abbreviation

$$[F_p]_k^w = F_p(w) - F_p(k)$$

and similar, (4.17) above can be written as

$$\begin{aligned}&k_p'(k)\,[F_p]_k^w + k_g'(k)\,[F_g]_k^w + \pi'(k)\,[F_L]_k^w\\&\quad= \left[k_p'F_p + k_g'F_g + \pi'F_L\right]_k^w - F_p(w)\,\left[k_p'\right]_k^w - F_g(w)\,\left[k_g'\right]_k^w - F_L(w)\,[\pi']_k^w\\&\quad= [F']_k^w + (F_p(w) - F_g(w))\,\left[k_g'\right]_k^w + F_L(w)\,[-\pi']_k^w\,. \qquad (4.18)\end{aligned}$$

4.1 Specialising: V is CRRA

If we now suppose that

$$V(k) = \frac{A_g k^{1-S}}{(1-S)}$$

for some $S > 0$ different from 1, and A_g a positive constant, it turns out that the form of the candidate solution simplifies considerably. (G6) is now

$$\begin{aligned}U &= V'\left[\left(\frac{\lambda_g}{1-S} + \frac{1}{2}\sigma^2 S\right)k - \Phi\right]\\&\equiv V'[\,Qk - \Phi\,],\end{aligned}$$

where $Q \equiv \lambda_g/(1-S) + \frac{1}{2}\sigma^2 S$ and (4.8) gives

$$F_p = \gamma_g + Q - \frac{S}{k}(Qk - \Phi). \tag{4.19}$$

With this form for V the left hand side of (4.10) is identically zero, hence we require simply that

$$\pi h_\pi = (R_g - S)h. \tag{4.20}$$

Equation (4.11) becomes

$$A_g k^{-S} = k_g^{-R_g} h_\xi. \tag{4.21}$$

With these simplifications it is now possible to follow the steps of Section 4 and obtain the following relations:

$$F_p = \gamma_g + Q - \frac{S}{k}\frac{U}{V'} \tag{4.22}$$

$$F_p - F_g = \frac{(1-R_g)}{k_g}\frac{U}{V'} - \xi \tag{4.23}$$

$$F_L = -\frac{(R_g - S)}{\pi}\frac{U}{V'} \tag{4.24}$$

$$F = (\gamma_g + Q)k + c - \frac{U}{V'} \tag{4.25}$$

$$\Phi = Qk - \frac{U}{V'} \tag{4.26}$$

where

$$\frac{U}{V'} = k_g \frac{h}{h_\xi} = c\frac{h}{\xi h_\xi}.$$

As V' is positive we will find it easier to ensure $F_p \geq 0$ if we consider $U < 0$. We have also

$$F_g = \gamma_g + Q + \xi - \frac{U}{V'}\left(\frac{S}{k} - \frac{(R_g - 1)}{k_g}\right);$$

we will require this to be non-negative too.

We will now consider the effect of different assumptions for the form of $h(\xi, \pi)$. Equations (4.20) and (4.21) will then be used to determine k_g, π and ξ as functions of k (we will usually assume the form of k_g and sometimes also π). The consumption rate c is given by $c = \xi k_g$ and equations (4.22)–(4.26) above determine the other quantities we require. We need to show that the conditions (4.13) and (4.16) are satisfied along with the transversality condition (GT). All that then remains is to check the tangent inequality holds which we will do by considering (4.17) or (4.18).

4.2 Specialising further A: $h(\xi, \pi) \equiv h_1(\xi)h_2(\pi)$

We will assume that h is of product form, so that

$$h(\xi, \pi) \equiv h_1(\xi)h_2(\pi)$$

where we assume that we know the form of the (monotone) functions h_1 and h_2 and also of k_g. Equations (4.20) and (4.21) become

$$\pi h_2'(\pi) = (R_g - S)h_2(\pi) \tag{4.27}$$
$$A_g k^{-S} = k_g^{-R_g} h_1'(\xi) h_2(\pi). \tag{4.28}$$

The first of these determines π as a *constant value for all* k (there may be ambiguity if h_2 solves the ODE on an interval). The second equation then determines $\xi(k)$ and hence c. Finally

$$\frac{U}{V'} = c\frac{h_1(\xi)}{\xi h_1'(\xi)}. \tag{4.29}$$

Let's assume that we know k_g and

$$h_1(\xi) = -\frac{\xi^{-\nu}}{\nu} \qquad\qquad h_2(\pi) = (1-\pi)^{-\kappa}$$

so that

$$\begin{aligned} U(c, k_g, \pi) &= -\frac{k_g^{-(R_g-1-\nu)} c^{-\nu} h_2(\pi)}{\nu} \\ &\equiv -\frac{k_g^{-\omega} c^{-\nu}(1-\pi)^{-\kappa}}{\nu} \end{aligned} \tag{4.30}$$

where $\omega \equiv R_g - 1 - \nu$. We will assume that

$$\nu > 0, \quad \omega > 0, \quad \kappa > 0, \quad R_g > S > 1,$$

so that U has all the required properties (concave and increasing in k_g and c, decreasing in π and so on). U is negative so the derived utility V must also be negative, therefore $S > 1$, and the condition (4.27) then means that $R_g > S$. From (4.27) the optimal π is given by

$$\pi = \frac{R_g - S}{\kappa + R_g - S} \in (0,1)$$

and the value of h_2 at the optimal π is thus

$$\Theta = \left(\frac{\kappa}{\kappa + R_g - S}\right)^{-\kappa}.$$

Note that the specific choice of the function $h_2(\pi)$ doesn't really matter; the solution depends only on the parameter Θ which we can choose arbitrarily, e.g. by multiplying the choice of h_2 above by a constant.

From (4.28) we obtain

$$\xi = \left(A_g \Theta^{-1} k^{-S} k_g^{R_g}\right)^{-1/(1+\nu)} \tag{4.31}$$

and hence

$$c = \left(A_g \Theta^{-1} k^{-S} k_g^{\omega}\right)^{-1/(1+\nu)}. \tag{4.32}$$

We also find that

$$\frac{U}{V'} = -\frac{c}{\nu}$$

and so from equations (4.22)–(4.26) we obtain

$$\begin{aligned} F_p &= \gamma_g + Q + \frac{S}{\nu}\frac{c}{k} && (4.33) \\ F_p - F_g &= \frac{\omega}{\nu}\xi && (4.34) \\ F_L &= \frac{(R_g - S)}{\nu}\frac{c}{\pi} && (4.35) \\ F &= (\gamma_g + Q)k + (1 + \frac{1}{\nu})c && (4.36) \\ \Phi &= Qk + \frac{c}{\nu}. && (4.37) \end{aligned}$$

As we will choose k_g to be non-negative, c is also non-negative and thus F_L, F_p and F are also non-negative for suitably large $\gamma_g + Q$.

The remaining problem is to make a good choice of k_g. We may think of the problem as one of choosing a non-negative function

$$\varphi(k) \equiv \left(\frac{k_g(k)}{k} \right)^{-\omega/(1+\nu)} \qquad (4.38)$$

in such a way as to guarantee non-negativity of F_g together with the tangent inequality, and the inequality (4.5): $0 \le k_g \le k$. The final inequality (4.5) can be equivalently expressed by saying that we need to have $\varphi(k) \ge 1$. In terms of φ, we have more simply

$$\begin{aligned} c &= Ak^B\varphi && (4.39) \\ \xi &= Ak^{B-1}\varphi^{R_g/\omega} && (4.40) \\ F_g &= \gamma_g + Q + Ak^{B-1}(S\varphi - \omega\varphi^{R_g/\omega})/\nu && (4.41) \\ \Phi &= Qk + Ak^B\varphi/\nu && (4.42) \end{aligned}$$

where the parameters A and B are related to the other parameters by

$$\Theta A^{-(1+\nu)} = A_g, \qquad (4.43)$$

$$(1+\nu)B = S - \omega. \qquad (4.44)$$

Non-negativity of F_p will be guaranteed by

$$\gamma_g + Q \equiv \gamma_g + \frac{\lambda_g}{1-S} + \frac{1}{2}\sigma^2 S \ge 0. \qquad (4.45)$$

Non-negativity of F_g needs to be checked case by case. Note that equation (4.44) implies that

$$(1-B)(1+\nu) = R_g - S$$

and hence $B < 1$. If we consider the limit as $k \downarrow 0$ of (4.41), we see that we must have

$$S\varphi(0) - \omega\varphi(0)^{R_g/\omega} \geq 0,$$

and since $\varphi \geq 1$ we conclude from this that a necessary condition to be satisfied is

$$S \geq \omega,$$

from which it follows from (4.44) that $B > 0$.

The derivative (4.18) of the tangent inequality (bearing in mind we have $\pi'(k) = 0$) is

$$\left(1 + \frac{1}{\nu}\right)(c'(w) - c'(k)) + \frac{\omega}{\nu}\xi(w)(k_g'(w) - k_g'(k))$$

so if $c'(k)$ and $k_g'(k)$ are decreasing functions of k then the tangent inequality will hold.

Alternatively observe from (4.32) that

$$k_g' = \frac{S}{\omega}\frac{k_g}{k} - \frac{(1+\nu)}{\omega}\frac{c'}{\xi}$$

and so the derivative (4.18) can be written as

$$\left(1 + \frac{1}{\nu}\right)c'(k)\left(\frac{\xi(w)}{\xi(k)} - 1\right) + \frac{S}{\nu}\xi(w)\left(\frac{k_g(w)}{w} - \frac{k_g(k)}{k}\right)$$

so that if c is increasing and k_g/k and ξ are decreasing functions then the tangent inequality will be satisfied. For the example we consider below this turns out to be a *more restrictive condition* on the range of parameters we can use than the previous condition.

We also need to check condition (4.16), that $\Lambda(x, y, z; w)$ has a unique infimum over w for any fixed (x, y, z). In this case

$$\begin{aligned}\Lambda(x, y, z; w) &= (x + y)F_p(w) - y(F_p(w) - F_g(w)) + zF_L(w)\\ &= (x + y)(\gamma_g + Q + \frac{S}{\nu}\frac{c(w)}{w}) - y(\frac{\omega}{\nu}\xi(w)) + z(\frac{(R_g - S)}{\nu}\frac{c(w)}{\pi}).\end{aligned}$$

We summarize the discussion above in the following lemma.

Lemma 1 *Suppose that we have suitable positive constants σ, A, ν, λ_g, κ and further constants satisfying the relations*

$$\begin{aligned} &R_g > S > 1, && S > \omega = R_g - 1 - \nu > 0,\\ &B = \frac{S - \omega}{1 + \nu}, && \gamma_g + Q = \gamma_g + \frac{\lambda_g}{1 - S} + \frac{1}{2}\sigma^2 S \geq 0. \end{aligned}$$

Take a function $\varphi(k) \geq 1$ *satisfying*

$$\varphi(0) \leq \left(\frac{S}{\omega}\right)^{\omega/(1+\nu)},$$

define

$$c(k) = Ak^B\varphi(k), \qquad \xi(k) = Ak^{B-1}\varphi(k)^{R_g/\omega}, \qquad k_g(k) = c(k)/\xi(k),$$

and k^* *solving*

$$\begin{aligned} dk^* &= \sigma k^* dw + \Phi(k^*)dt \\ &= \sigma k^* dw + (Qk^* + \varphi(k_t^*){k^*}^B A/\nu)dt. \end{aligned} \tag{4.46}$$

Assume also that the following following four conditions hold:

(L1) $F_g = \gamma_g + Q + Ak^{B-1}(S\varphi - \omega\varphi(k)^{R_g/\omega})/\nu \geq 0$ *for all* k.

(L2) Either

$$c'' < 0, \qquad k_g'' < 0$$

or

$$c' > 0, \qquad \left(\frac{k_g}{k}\right)' < 0, \qquad \xi' < 0.$$

(L3) The expression

$$(x + Sy)\,\frac{c}{k} - \omega y\,\xi + z\,c \tag{4.47}$$

attains its infimum over $k \geq 0$ *uniquely for all non-negative* x, y, z.

(L4) k^* *satisfies the transversality condition* (GT)

$$\sup_t e^{-\lambda_g t} {k_t^*}^{1-S} \in L^1, \qquad \lim_{t\to\infty} e^{-\lambda_g t} {k_t^*}^{1-S} = 0.$$

Then we have constructed an explicit solution to the problem of Theorem 1 with

$$V(k) = \frac{\Theta A^{-(1+\nu)}k^{1-S}}{(1-S)}, \qquad U(c, k_g, \pi) = -\frac{k_g^{-\omega}c^{-\nu}(1-\pi)^{-\kappa}}{\nu},$$

$$F(x,y,z) = \inf_{w\geq 0}\left\{(x+y)(\gamma_g + Q + \frac{S}{\nu}\,\frac{c(w)}{w}) - \frac{\omega y}{\nu}\,\xi(w) + \frac{z(R_g - S)}{\nu}\,\frac{c(w)}{\pi^*}\right\}$$

and constants π^* *and* Θ *given by*

$$\pi^* = \frac{R_g - S}{\kappa + R_g - S}, \qquad \Theta = \left(\frac{\kappa}{\kappa + R_g - S}\right)^{-\kappa}.$$

Example. We now consider choices of φ of the form

$$\varphi(k) = \varphi_0(1+ak)^{\varepsilon}, \tag{4.48}$$

and check the conditions of Lemma 1. We will take $\varepsilon \geq 0$ and $a \geq 0$ (so that $\varphi \geq 1$), and $\varphi_0 \geq 1$ will be chosen small enough so that

$$S\varphi_0 > \omega\varphi_0^{R_g/\omega}; \tag{4.49}$$

this *can* always be done, since $S > \omega$. We will also choose ε so that

$$\varepsilon \leq \frac{\omega(1-B)}{R_g + S}. \tag{4.50}$$

The function φ is increasing and hence k_g/k is decreasing. In order that we have $F_g \geq 0$ we demand that

$$\xi = A\varphi_0^{R_g/\omega}k^{B-1}(1+ak)^{\varepsilon R_g/\omega}$$

be decreasing (and for A to be small enough in relation to $\gamma_g + Q$). This will be true provided

$$\varepsilon \leq \frac{\omega(1-B)}{R_g},$$

which follows from equation (4.50).

For the tangent inequality to hold we need $c''(k)$ and $k_g''(k)$ negative. In order for $c''(k)$ to be negative, it is sufficient that $\varepsilon \leq 1-B$ which is guaranteed by inequality (4.50). This condition also ensures that c/k is decreasing and hence, from equation (4.33), F_p is decreasing. For $k_g''(k)$ to be negative, it is sufficient that

$$\varepsilon \leq \frac{\omega}{1+\nu} = \frac{\omega(1-B)}{R_g - S}$$

which again follows from equation (4.50).

Under the condition (4.50) we can show that (L3) holds - see Appendix A for the proof.

All that remains now is to check the transversality condition (L4) with the dynamics of k^* as given by (4.46) and bearing in mind that $(1-S)$ is negative. If we now introduce the process $x \equiv \log(k^*)$, we see that x satisfies the SDE

$$dx = \sigma dw + (Q - \frac{1}{2}\sigma^2 + h(x))dt,$$

where $h(x) \equiv A\varphi(e^x)e^{(B-1)x}/\nu$, and our task is therefore to establish lower bounds on the process

$$\tilde{x}_t \equiv x_t + \frac{\lambda_g t}{S-1}.$$

Now the process $\tilde{x}$ itself is not a diffusion; however, because $\varepsilon < 1 - B$ (from (4.50)), it is readily seen that h is decreasing, and so $h(x_t) \geq h(\tilde{x}_t)$. Using this, we can apply the Yamada-Watanabe stochastic comparison theorem (see, for example, Rogers & Williams (2000), V.43); the process $\tilde{x}$ dominates the process y which starts at the same value, but solves instead the SDE

$$\begin{aligned} dy &= \sigma dw + (Q - \frac{1}{2}\sigma^2 + \frac{\lambda_g}{S-1} + h(y))dt \\ &= \sigma dw + (\frac{1}{2}\sigma^2(S-1) + h(y))dt \end{aligned}$$

The process y is a diffusion, with scale function s which satisfies

$$s'(y) = \exp(-(S-1)y + \int_y^0 \frac{2h(v)}{\sigma^2}\,dv)$$

and (without loss of generality) the additional properties $s(-\infty) = -\infty$, $s(\infty) = 0$. If we now denote $Y \equiv \inf_t y_t$, we have

$$P^0(Y < a) = \frac{s(0)}{s(a)}$$

for all $a < 0$, and from this and the behaviour of h at large negative values it is not hard to conclude that

$$E\exp(-(S-1)Y) < \infty,$$

and (GT) follows easily from this.

4.3 Some numerics

In this section we will give graphs to show typical optimal trajectories for the above explicit example. We will compare these with a non-explicit example where we specify the functions U and F and then solve for the optimal (c, k_g, π) using the numerical methods described in Hartley (2003). For this non-explicit example we will take the government's utility function as in (4.30) with the government's coefficient of relative risk aversion given by[19] $R_g = 4$ and the parameters of the utility function given by $\nu = 1.2$, $\omega = 1.8$ and $\kappa = 0.2$. For the production function F we take

$$F(K_p, K_g, L) = K_p^{0.4} K_g^{0.3} L^{0.4}$$

and the various other constants needed for a full specification of the government's problem will be taken to be $\mu_L = 0.01$, $\mu_T = 0.05$, $\delta = 0.10$, $v_{LL} = 0.01$, $v_{00} = 0.2$, $v_{0L} = 0.005$ and $\rho_g = 0.10$. We will use these same constants as the basis for the

[19] Romer (2001) states that a value of 4 is 'towards the high end of values that are viewed as plausible', although he also shows that methods for calculating R based on equity-premiums can give values as high as 240!

explicit solution taking also $A = 0.2$ and $B = 0.5$ (so that $S = 2.9$). These constants satisfy all the relations needed for Lemma 1. Finally we take $\varphi_0 = 1.25$, $a = 1$ and $\varepsilon = 0.1$ to specify $\varphi(k)$ as in equation (4.48). Relations (4.49) and (4.50) are both satisfied as required.

Figure 1 shows the optimal π, k_g/k and c against total capital k for the explicit example of Section 4.2 - note that π is constant by construction. Figure 2 shows the same optimal values for the numerical example. The extra line shows the stationary distribution of k, scaled so that its maximum is 1. This can be easily calculated in the numerical case - again see Hartley (2003) for the details. In this example π is *not constant*, but it is close to the constant value we find in the explicit example.

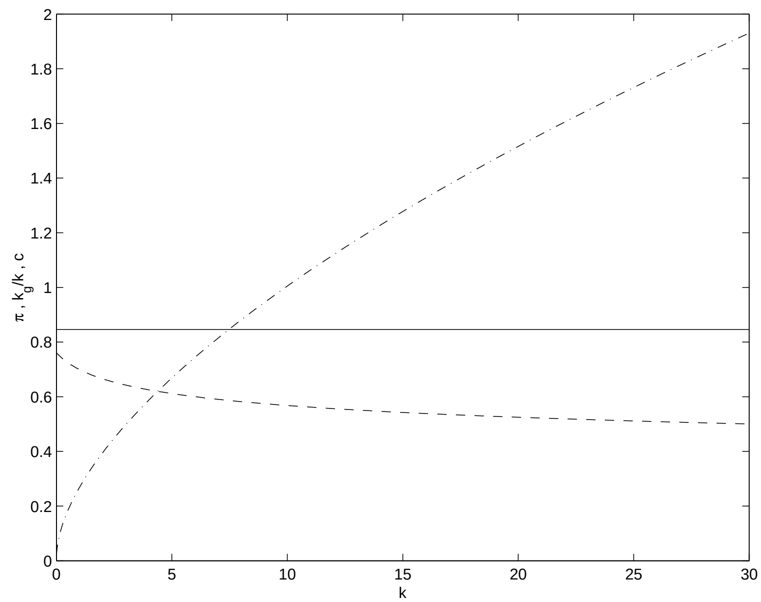

Figure 1: Plots of the optimal π (solid line), k_g/k (dashed line), c (dash-dot line) against total capital k for the explicit example.

4.4 Introducing taxes.

The government's choice of taxes will depend on the private sector's preferences, which we here will assume are of the form

$$u(c, k_g, \pi) = -\frac{k_g{}^{-\omega_p} c^{-\nu_p} (1-\pi)^{-\kappa_p}}{\nu_p}\,, \tag{4.51}$$

where $\nu_p > 0$, $\omega_p \equiv R_p - 1 - \nu_p > 0$ and $\kappa_p > 0$. We modify the notation of the previous subsections by writing ω_g in place of ω, ν_g in place of ν and so on, to emphasise the distinction between government and private-sector parameters in what is an otherwise similar specification. With the private sector's felicity function

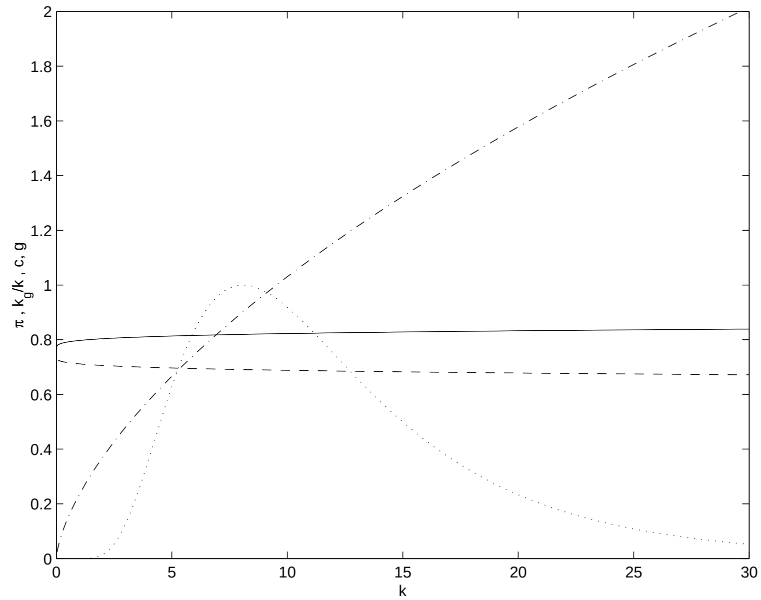

Figure 2: Plots of the optimal π (solid line), k_g/k (dashed line), c (dash-dot line) and the (scaled) stationary distribution of k (dotted line) against total capital k for the numerical example.

specified as above conditions (PS2) and (PS3) from Theorem 2 combined with the very similar conditions (G2) and (G3) from Theorem 1 tell us that

$$\begin{aligned}\beta_c\beta_w &= \frac{u_\pi U_c}{u_c U_\pi}\\ &= \frac{\kappa_p\nu_g}{\kappa_g\nu_p}.\end{aligned} \tag{4.52}$$

Combining the expressions (4.38) for k_g and (4.39) for c in terms of k and φ, and using condition (PS2) of Theorem 2, we have

$$\begin{aligned}\beta_c^{-1}\psi &= A^{-(1+\nu_p)}\Theta_p k^{-R_p+(1-B)(1+\nu_p)}\varphi^{-(1+\nu_p)+\omega_p(1+\nu_g)/\omega_g}\\ &\equiv A^{-(1+\nu_p)}\Theta_p k^{-Sp}\varphi^{-\alpha}\end{aligned} \tag{4.53}$$

where $\Theta_p \equiv (1-\pi^*)^{-\kappa_p}$, $S_p \equiv R_p-(1-B)(1+\nu_p)$ is defined in an analogous manner to S_g (4.44) and $\alpha \equiv 1+\nu_p-\omega_p(1+\nu_g)/\omega_g$. *We shall assume the inequality*

$$\frac{R_p}{1+\nu_p} \le \frac{R_g}{1+\nu_g}, \tag{4.54}$$

which is easily seen to be equivalent to

$$\alpha \equiv 1+\nu_p-\omega_p(1+\nu_g)/\omega_g \ge 0.$$

There seem to be two approaches we can take to taxation, depending on whether we take equation (4.52) or (4.53) as our starting point.

Approach 1 : Given the form of (4.53) and looking back at our choice of government $\Psi \equiv V'$ of

$$\Psi = A_g k^{-S_g} \equiv A_g k^{-R_g+(1-B)(1+\nu_g)},$$

it seems natural to pick an analogous function for the private sector's function $\psi(k)$, i.e.

$$\psi \equiv A_p k^{-R_p+(1-B)(1+\nu_p)} \equiv A_p k^{-S_p},$$

where A_p is a constant which we shall choose as follows. We will pick some $\beta_w(0) \in [0,1]$ and then equation (4.52) determines $\beta_c(0)$. The consumption tax is then

$$\beta_c = \beta_c(0)\left(\frac{\varphi}{\varphi_0}\right)^{\alpha},$$

and hence $A_p \equiv \beta_c(0)\varphi_0^{-\alpha}A^{-(1+\nu_p)}\Theta_p$. Similarly

$$\beta_w = \beta_w(0)\left(\frac{\varphi}{\varphi_0}\right)^{-\alpha}.$$

As $\alpha > 0$ we have automatically ensured the desirable property $0 \leq \beta_w \leq 1$ where the tax rate on wages $1-\beta_w$ increases as k increases. We have also constructed a consumption tax $1-\beta_c$ that decreases as k increases, eventually becoming a subsidy at high k (if it wasn't already a subsidy to begin with). This is intuitively correct - in a poor economy the population should be encouraged to work more and consume less, whilst in a very rich economy the population should be taxed highly on their income to pay for better public services and should also be consuming more of their capital. Taking

$$\nu_p = 1.4, \qquad \omega_p = 0.6, \qquad \kappa_p = 0.1, \tag{4.55}$$

in the private sector's felicity function (4.51) and other parameters as in Section 4.3 and taking $\beta_w(0) = 1$ gives wage and consumption[20] tax rates as shown in Figure 3.

Now solving (PS4) for $r\beta_r$ gives us

$$r\beta_r = A\frac{S_p}{\nu_g}k^{B-1}\varphi(k) + A_r, \tag{4.56}$$

where

$$A_r = \lambda_p - \mu_0 + S_p(Q + 2(\gamma_g - \gamma) - \frac{1}{2}\sigma^2(1+S_p)).$$

For large enough values of ρ_p the constant A_r will be non-negative, and thus $r\beta_r$ will be non-negative and decreasing. We can get β_k from condition (PS1) in the same way and find that

$$\beta_k(F_p - S_p v_{00} + (S_p - 1)v_{0L}) = r\beta_r + \delta. \tag{4.57}$$

[20] The consumption tax is given by $\beta_c^{-1} - 1$ as $\beta_c c$ is the amount actually consumed if the private sector tries to consume c, whereas conventional consumption taxes (e.g. VAT in the UK) add a charge onto the amount that the private sector actually consumes.

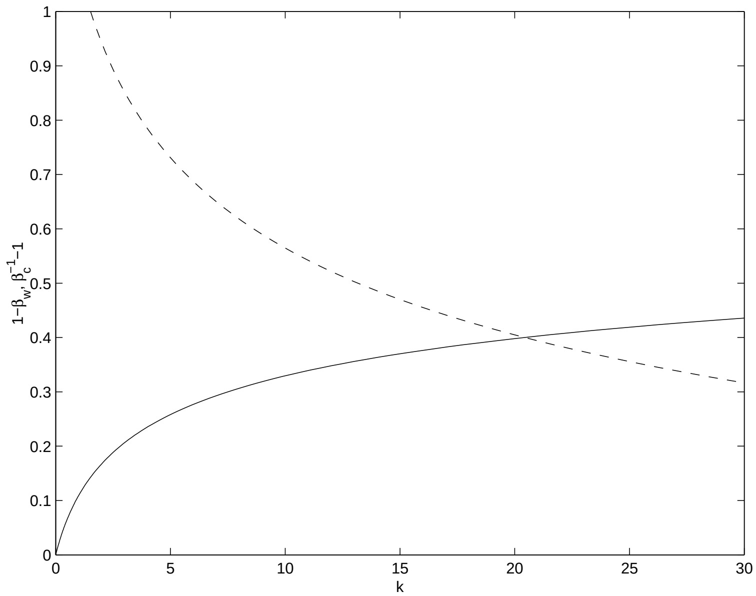

Figure 3: Plot of the wage tax rate (solid line) and the consumption tax rate (dashed line) against total capital k for the explicit example.

Thus the return on private capital $\beta_k F_p$ is equal to the return on debt $r\beta_r$ plus depreciation δ and some 'price of risk' terms. Substituting in the expression (4.33) for F_p we have that

$$\begin{aligned} \beta_k &= \frac{\delta + r\beta_r}{\gamma_g + Q - S_p v_{00} + (S_p - 1)v_{0L} + A\nu_g^{-1} S_g k^{B-1}\varphi(k)} \\ &= \frac{\delta + A_r + A\nu_g^{-1} S_p k^{B-1}\varphi(k)}{A_k + A\nu_g^{-1} S_g k^{B-1}\varphi(k)} \end{aligned} \tag{4.58}$$

where $A_k \equiv \gamma_g + Q - S_p v_{00} + (S_p - 1)v_{0L}$ will be positive for large enough γ_g. If we again make assumptions about values as in Section 4.3 and (4.55) then we find the resulting tax rate $1 - \beta_k$ is as in Figure 4. The tax rate is in fact a subsidy for all but very low values of k, to induce the private sector to invest in capital in preference to consuming.

If we make the plausible assumption that $\beta_k = \beta_r$ then there is an explicit expression for the interest-rate process r:

$$r = \left\{ A_k + A\nu_g^{-1} S_g k^{B-1}\varphi(k) \right\} \frac{A_r + A\nu_g^{-1} S_p k^{B-1}\varphi(k)}{\delta + A_r + A\nu_g^{-1} S_p k^{B-1}\varphi(k)}. \tag{4.59}$$

This is illustrated in Figure 4. The interest rate r is thus expressed as a function of the diffusion process k which solves the SDE

$$dk = \sigma k dW + \left[(Q + \gamma_g - \gamma)k + Ak^B\varphi(k)/\nu_g\right]dt.$$

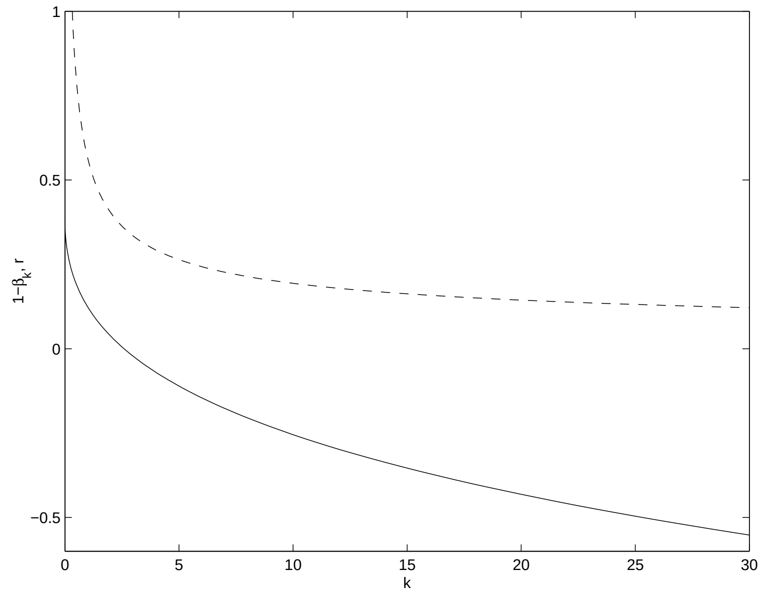

Figure 4: Plot of the capital income tax rate (solid line) and the interest rate r (dashed line) against total capital k for the explicit example with varying wage and consumption tax.

Specializing further by assuming that φ is constant, this can be reduced to *linear* form by considering instead the variable $\zeta \equiv k^{1-B}$, which solves

$$d\zeta = (1-B)\left[\,\sigma\zeta dW + (Q + \gamma_g - \gamma - \frac{1}{2}B\sigma^2)\zeta dt\,\right] + A(1-B)\varphi_0 dt/\nu_g.$$

Merton (1975) finds structurally similar interest rate processes in a study of a single-sector growth model, and Kloeden and Platen (1992) present this under the name of the stochastic Verhulst equation.

Approach 2 : Equation (4.52) tells us that $\beta_c\beta_w$ is constant so we choose both β_c and β_w to be constant for all k. Taking constants as in Section 4.3 and (4.55) again we find that $\beta_c\beta_w = 3/7$ so we can choose $\beta_w = 3/5$ and $\beta_c = 5/7$ giving rates of 40% for both wage and consumption taxes. Equation (4.53) gives

$$\psi = \beta_c \equiv A^{-(1+\nu_p)}\Theta_p k^{-Sp}\varphi^{-\alpha}$$

and then, as before, (PS1) and (PS4) can be used to determine β_k and r (assuming $\beta_r \equiv \beta_k$). Again β_k and r will be (complicated) functions of k but, as Figure 5 shows, the values obtained for this explicit example are very reasonable. The tax rate is a subsidy for small k and then becomes an increasing conventional tax rate for higher k. Figure 6 shows $1 - \beta_k$ and r again, for the numerical example as described in Section 4.3 and with the private sector's felicity function given as in equations (4.51) and (4.55). These tax rates are obtained using exactly the same sort of method and

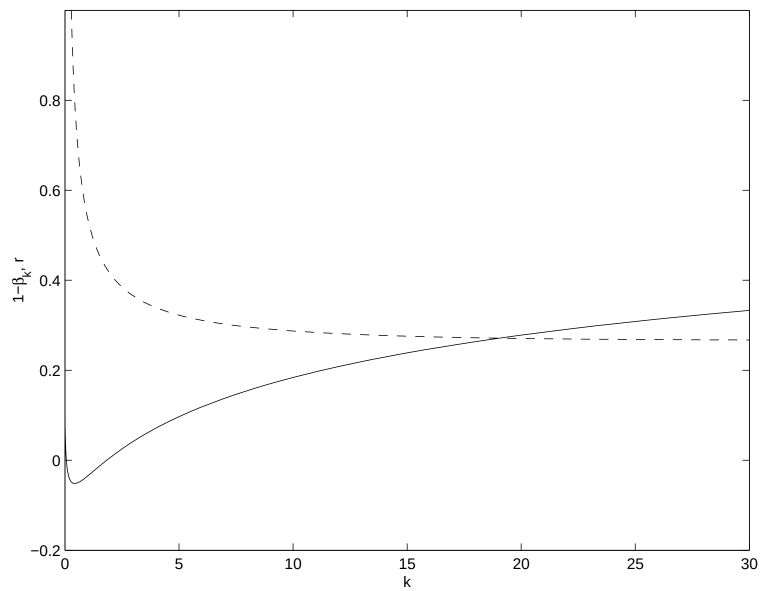

Figure 5: Plot of the capital income tax rate (solid line) and the interest rate r (dashed line) against total capital k for the explicit example with constant wage and consumption tax.

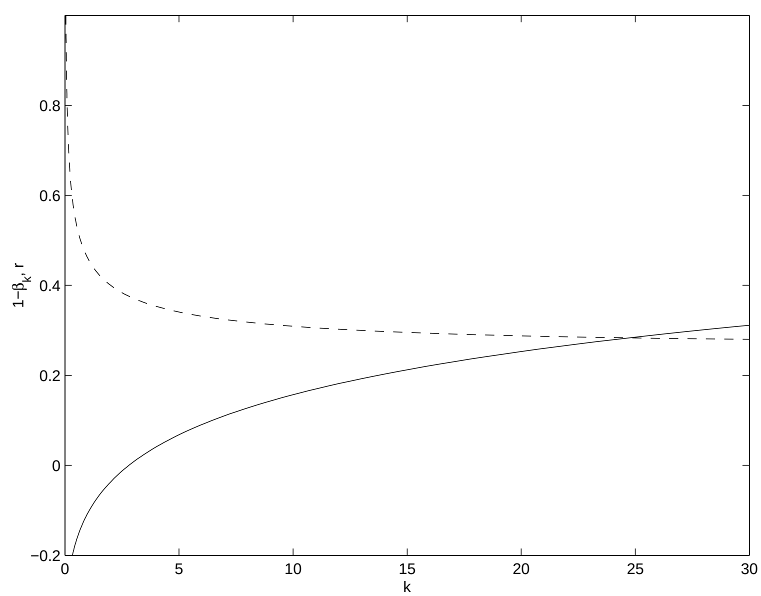

Figure 6: Plot of the capital income tax rate (solid line) and the interest rate r (dashed line) against total capital k for the numerical example with constant wage and consumption tax.

turn out to be very similar to our explicit example, again showing that this explicit example is genuinely useful.

One further calculation we can attempt now that we have the (short-term) interest rate as a function of capital is to compute bond prices. We shall write $B(t, x; T)$ for the time-t price of a zero-coupon bond paying one unit of capital at time T, where $x_t = \log k_t = x$ at time $t \leq T$. The price the private sector will be prepared to pay for such a bond is

$$B(t, x; T) \equiv \frac{E\left[\zeta_T e^{-\int_t^T r\beta_r\, ds} \mid x_t = x\right]}{\zeta_t}$$

where ζ is the private sector's state-price density process which will be given by

$$\zeta_t \equiv e^{-\lambda_p t} \eta_t^{-1} u_c(k_t) \beta_c(k_t).$$

Again following Hartley (2003) we can compute these bond prices numerically by solving a PDE and hence find the corresponding yields, given by

$$Y(t, x; T) = -\frac{1}{T-t} \log B(t, x; T)$$

for $0 \leq t < T$. Figure 7 shows the resulting yield surface for our numerical example. At high levels of capital the yield curve is a conventional increasing curve and at low levels of capital the yield curve is inverted. Figure 8 shows a selection of equally spaced (with respect to k) curves taken from the surface in Figure 7. We see that between the conventional and the inverted yield curves there are humped yield curves - another type of curve occasionally spotted in the real world as the yield curve makes a transition from increasing to inverted or vice-versa.

5 Conclusions

We have introduced stochastic terms into the model of Arrow and Kurz (1970) and also added a factor to account for the proportion of work devoted to labour, as in the original model of Ramsey (1928). With these modifications we have then solved the government's central-planning problem. Under the assumption that tax rates are chosen so that the private sector, optimising its own utility functional, follows the optimal path of the government we have found tax and interest rates as functions of per-capita capital, i.e. closed loop control. Furthermore we have been able to exhibit an explicit solution to these two problems. We have shown that the tax rates can be chosen to take sensible values and found a novel single-factor interest rate model.

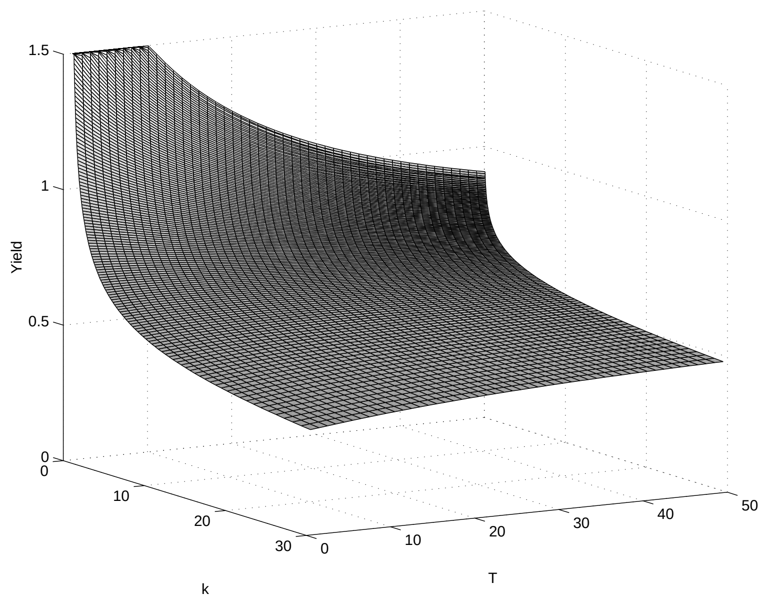

Figure 7: Yield at time-0 of a zero-coupon bond of maturity T, against T and time-0 capital level k for the numerical example

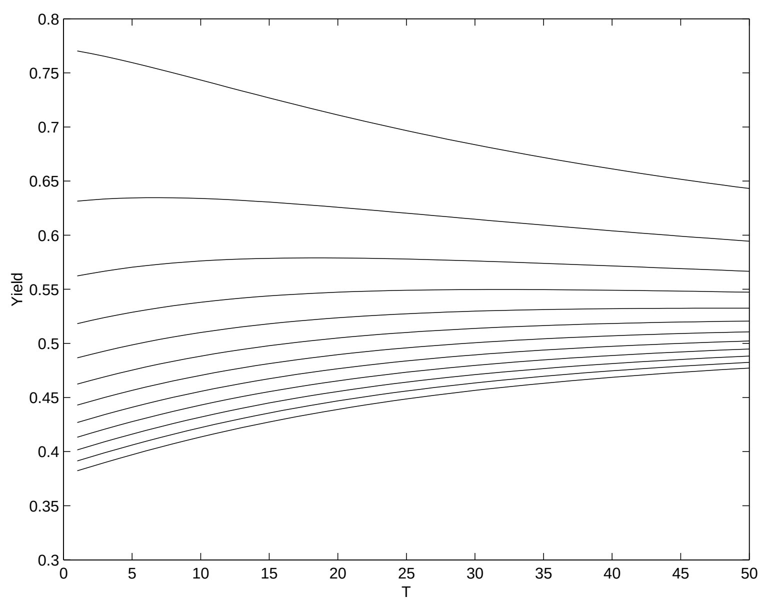

Figure 8: Yield at time-0 of a zero-coupon bond of maturity T, against T. Each line is for a different initial k with the line corresponding to the smallest k at the top of the picture.

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Appendix A Proofs

PROOF OF THEOREM 1. Suppose that the process k_t has dynamics given by (2.12) for some consumption process c_t and some choice $\theta_t = k_g(t)/k(t)$ of the proportion of capital held by the government. We define a P_g-Brownian motion w by $z^0 - z^L \equiv \sigma w$ and introduce a (Lagrangian) semimartingale $e^{-\lambda_g t}\Psi_t \equiv e^{-\lambda_g t}\Psi(k_t^*)$ where k^* is the conjectured optimal process, satisfying (2.21), and where

$$d\Psi_t \equiv \Psi_t(a_t dw + b_t dt).$$

We have for any stopping time τ that (omitting explicit appearance of t in most places)

$$\begin{aligned}\int_0^\tau e^{-\lambda_g t}U(c,k_g,\pi)dt &= \int_0^\tau e^{-\lambda_g t}\Big[U(c,k_g,\pi) + \Psi(F(k_p,k_g,\pi) - \gamma_g k - c) + \\ &\qquad k\Psi(b-\lambda_g) + \sigma a k\Psi\Big]\,dt + k_0\Psi_0 - e^{-\lambda_g \tau}k_\tau\Psi_\tau + M_\tau\end{aligned}$$

for some P_g-local martingale M; this is just obtained by integrating the process $e^{-\lambda_g t}\Psi_t k_t$ by parts. Taking a stopping time τ which reduces M strongly, we can now take expectations to obtain

$$\begin{aligned}E_g\int_0^\tau e^{-\lambda_g t}U(c,k_g,\pi)dt = E_g\int_0^\tau e^{-\lambda_g t}\Big[U(c,k_g,\pi) + \Psi(F(k_p,k_g,\pi) - \gamma_g k - c) + \\ k\Psi(b-\lambda_g) + \sigma a k\Psi\Big]\,dt + k_0\Psi_0 - E_g e^{-\lambda_g \tau}k_\tau\Psi_\tau.\end{aligned} \tag{A.1}$$

We now consider the maximisation over k, c, k_g and π of the integrand on the right-hand side of (A.1): the first-order conditions we obtain will be

$$\begin{aligned}\Psi(k^*)(F_p(k_p,k_g,\pi) - \gamma_g) &= (\lambda_g - b - a\sigma)\Psi(k^*)\\ U_c(c,k_g,\pi) &= \Psi(k^*)\\ U_g(c,k_g,\pi) &= \Psi(k^*)(F_p - F_g)\\ U_\pi(c,k_g,\pi) &= -\Psi(k^*)F_L.\end{aligned}$$

The last three of these are satisfied at $c = c^*(k^*)$, $k_g = k_g^*(k^*)$, $\pi = \pi^*(k^*)$ in view of (G2), (G3) and (G4). The first is satisfied due to (G1), since from the Itô expansion of $\Psi(k^*)$ we must have that

$$\begin{aligned} a &= \frac{\sigma k^* \Psi'(k^*)}{\Psi(k^*)} \\ b &= \frac{\Phi(k^*)\Psi'(k^*) + \frac{1}{2}\sigma^* k^{*2}\Psi''(k^*)}{\Psi(k^*)}. \end{aligned}$$

To summarise then: the integrand on the right-hand side of (A.1) is maximised at $c = c^*(k^*)$, $k_p = k_p^*(k^*)$, $k_g = k_g^*(k^*)$, $\pi = \pi^*(k^*)$. Reversing the integration-by-parts argument by which we arrived at (A.1), we conclude that[21]

$$\begin{aligned} E_g \int_0^\tau e^{-\lambda_g t} U(c, k_g, \pi)\, dt &\leq E_g \int_0^\tau e^{-\lambda_g t} U(c^*(k_t^*), k_g^*(k_t^*), \pi^*(k_t^*))\, dt \\ &\quad + E_g \left[e^{-\lambda_g \tau}(k_\tau^* - k_\tau)\Psi(k_\tau^*) \right] \\ &\leq E_g \int_0^\tau e^{-\lambda_g t} U(c^*(k_t^*), k_g^*(k_t^*), \pi^*(k_t^*))\, dt \\ &\quad + E_g \left[e^{-\lambda_g \tau} k_\tau^* \Psi(k_\tau^*) \right]. \end{aligned}$$

Now it only remains to let the reducing time τ tend to infinity, and appeal to the transversality condition (GT), together with the fact that U does not change sign to give us the required optimality result.

Finally, suppose that we take $V(k)$ given by

$$V(k) \equiv \int_1^k \Psi(y)\, dy + V_1$$

where

$$V_1 \equiv \frac{1}{\lambda_g} \left[\Psi(1)\Phi(1) + \frac{1}{2}\sigma^2 \Psi'(1) + U(c^*(1), k_g^*(1), \pi^*(1)) \right].$$

If we differentiate

$$-\lambda_g V(k) + V'(k)\Phi(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + U(c(k), k_g(k), \pi(k)) \qquad \text{(A.2)}$$

with respect to k, using the fact that $V'(k) = \Psi(k)$ we obtain

$$\begin{aligned} &\Psi(-\lambda_g + (1 - k_g')F_p + k_g' F_g + \pi' F_L - \gamma_g - c') \\ &\quad + \Psi'(\Phi + \sigma^2 k) + \frac{1}{2}\sigma^2 k^2 \Psi'' + c' U_c + k_g' U_g + \pi' U_\pi = 0 \end{aligned}$$

[21]There is a detail here: the stopping time τ which reduced M strongly may not reduce the corresponding local martingale for the optimal process. We can nevertheless replace τ by a stopping time which is no larger and which reduces both local martingales. Since we are interested in letting the reducing time tend to infinity, this little change affects nothing in the end.

by (G1)–(G4). Hence expression (A.2) is constant and this constant is 0 by the construction of V_1. □

PROOF OF THEOREM 2. The strategy is firstly to discover the dynamics faced by a single household optimising in an economy which is following the government's optimal path. Next we rework the private household's objective, expressing it in intensive variables. We then use the Lagrangian method to characterise the private household's optimal path,

So suppose we consider what happens if we add one more household to the (large) economy which is following the government's optimal path. The total labour available has increased by L_t/L_0, an $O(1)$ quantity, and the total amounts of both types of capital and of government debt will also have changed by an $O(1)$ quantity. If ΔC, ΔK_p, ΔD denote the changes in the corresponding aggregate quantities, and $\tilde{\pi}$ denote the proportion of effort which the new household devotes to production, then the perturbation of (3.7) to leading order is **not**

$$\begin{aligned} d\Delta K_p + d\Delta D &= \Delta K_p \big[\beta_k dZ^0 + (\beta_k F_p - \delta) dt \big] + r\beta_r \Delta D dt - \beta_c^{-1} \Delta C dt \\ &\quad + \beta_w \tilde{\pi}_t \frac{\eta_t}{L_0} F_L dt + \frac{\eta_t}{L_0} k_g (\beta_k \theta_p + \beta_w \theta_L)(dZ^0 + F_g dt). \quad \text{(A.3)} \end{aligned}$$

This is because if we consider the change in (3.7) when the new household joins, not only do the total amounts of capital, labour, consumption and debt change by the $O(1)$ amounts indicated in (A.3), *but the coefficients β. and the derivatives f. also get changed*, by amounts which are $O(1/L_0)$. Since these changes then get multiplied by quantities which are $O(L_0)$, the net impact on the budget equation of these changes is still $O(1)$. Nevertheless, *we argue that equation (A.3) is the correct equation for the evolution of the new household's wealth*, where the tax rates and all derivatives of f are evaluated along the *original* (government-optimal) path. This is because the quantities on the right-hand side of (A.3) are items directly visible to the new household: the return on *its* private capital, the wages for *its* labour, etc.. The other $O(1)$ changes in the budget equation, such as the changes in total wages due to the $O(1/L_0)$ shift in wage rates, get distributed among the population as a whole, and so have only an $O(1/L_0)$ effect on any one household.

This agreed, the problem facing the typical private sector household is to optimise the objective (3.1) with the dynamics given by (A.3), where the tax rates, the β., the f., r and f are all evaluated along the government's optimal path. As with the government's problem, we first reduce to technology-adjusted *per capita* variables, expressing the objective as

$$\begin{aligned} E \int_0^\infty e^{-\rho_p t} u\Big(\frac{L_0 \Delta C_t}{L_t}, \frac{K_g(t)}{L_t}, \tilde{\pi}_t \Big)\, dt, &= E \int_0^\infty e^{-\rho_p t} T_t^{1-R_p} u(c_t, k_g^*(t), \tilde{\pi}_t)\, dt \\ &= E \int_0^\infty e^{-\lambda_p t} u(c_t, k_g^*(t), \tilde{\pi}_t)\, dt, \quad \text{(A.4)} \end{aligned}$$

where we are reserving starred variables (k_p^*, k_g^*) for the government's optimal values, and are using the notations[22]

$$k_p \equiv \Delta K_p L_0/\eta, \quad \Delta_p \equiv \Delta D L_0/\eta, \quad c_t \equiv \Delta C_t L_0/\eta_t, \quad \lambda_p \equiv \rho_p - (1-R_p)\mu_T.$$

The dynamics (A.3) implies the following dynamics for the (technology-adjusted *per capita*) private-sector wealth process $x \equiv k_p + \Delta_p$:

$$\begin{aligned} dx = k_p \big[\beta_k dZ^0 - dZ^L + (\beta_k F_p - \gamma + v_{0L}(1-\beta_k))dt \big] + \beta_w \tilde{\pi} F_L dt \\ + \Delta_p \big[-dZ^L + (\mu_0 + r\beta_r)dt \big] - \beta_c^{-1} c dt + A dZ^0 + B dt, \end{aligned} \tag{A.5}$$

where we have used the abbreviations $\mu_0 = v_{LL} - \mu_L - \mu_T$, $A = (\beta_k \theta_p + \beta_w \theta_L) k_g^*$ and $B = k_g^* (\beta_k \theta_p + \beta_w \theta_L)(F_g - v_{0L})$.

Let us now combine the objective (A.4) with the dynamics (A.5) using a Lagrangian process $e^{-\lambda_p t} \psi_t^* \equiv e^{-\lambda_p t} \psi(k_t^*)$, where by Itô's formula

$$d\psi^* = \psi^* \big[a^* (dZ^0 - dZ^L) + b^* dt \big], \tag{A.6}$$

using the notation $a_t^* = a(k_t^*)$, $b_t^* = b(k_t^*)$, and where

$$a(k) \;=\; k\psi'(k)/\psi(k) \tag{A.7}$$

$$b(k) \;=\; \frac{\frac{1}{2}\sigma^2 k^2 \psi''(k) + \psi'(k)\tilde{\Phi}(k)}{\psi(k)}. \tag{A.8}$$

Again omitting superfluous appearances of the time variable, integrating $xe^{-\lambda_p t}\psi^*$

[22] This notation conflicts slightly with the earlier use of c, k_p for the technology-adjusted *per capita* consumption C/η and private capital K_p/η. For the remainder of this proof, we shall treat c and k_p as local variables, distinct from those discussed earlier, and to be freely chosen by the private sector household. It will turn out in the end that the private sector will choose $c_t = c^*(k_t^*)$, $k_p(t) = k_p^*(k_t^*)$, of course.

by parts gives us

$$\begin{aligned}
\int_0^\tau e^{-\lambda_p t} u(c, k_g^*, \tilde{\pi})\, dt &= \int_0^\tau e^{-\lambda_p t} \Big[u(c, k_g^*, \tilde{\pi}) + x\psi^*(b^* - \lambda_p) + \psi^* \big\{ \beta_w \tilde{\pi} F_L - \beta_c^{-1} c \\
&\quad + k_p(\beta_k F_p - \gamma + v_{0L}(1 - \beta_k)) + \Delta_p(r\beta_r + \mu_0) + B \big\} \\
&\qquad + a^* \psi^* \big\{ (\beta_k k_p + A)(v_{00} - v_{0L}) + x(v_{LL} - v_{0L}) \big\} \Big]\, dt \\
&\qquad\quad + x_0 \psi_0^* - x_\tau e^{-\lambda_p \tau} \psi_\tau^* + M_\tau \\
&= \int_0^\tau e^{-\lambda_p t} \Big[u(c, k_g^*, \tilde{\pi}) - \beta_c^{-1} \psi^* c + \beta_w F_L \psi^* \tilde{\pi} \\
&\quad + \psi^* k_p \big\{ \beta_k F_p - \gamma - \lambda_p + v_{0L}(1 - \beta_k) + b^* \\
&\quad + a^* \beta_k (v_{00} - v_{0L}) + a^*(v_{LL} - v_{0L}) \big\} \\
&\qquad + \psi^* \Delta_p \big\{ r\beta_r + \mu_0 - \lambda_p + b^* + a^*(v_{LL} - v_{0L}) \big\} \\
&\qquad\quad + \psi^*(B + Aa^*(v_{00} - v_{0L})) \Big]\, dt \\
&\qquad\qquad + x_0 \psi_0^* - x_\tau e^{-\lambda_p \tau} \psi_\tau^* + M_\tau \\
&= \int_0^\tau e^{-\lambda_p t} \Big[u(c, k_g^*, \tilde{\pi}) - \beta_c^{-1} \psi^* c + \beta_w F_L \psi^* \tilde{\pi} \\
&\quad + k_p \big\{ \psi^*(\beta_k F_p - \gamma - \lambda_p + v_{0L}(1 - \beta_k)) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*) \\
&\quad + \psi'(k^*)(\tilde{\Phi} + (\beta_k(v_{00} - v_{0L}) + v_{LL} - v_{0L})k) \big\} \\
&\qquad + \Delta_p \big\{ \psi^*(r\beta_r + \mu_0 - \lambda_p) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*) \\
&\qquad + \psi'(k^*)(\tilde{\Phi} + (v_{LL} - v_{0L})k) \big\} \\
&\qquad\quad + \psi^*(B + Aa^*(v_{00} - v_{0L})) \Big]\, dt \\
&\qquad\quad + x_0 \psi_0^* - x_\tau e^{-\lambda_p \tau} \psi_\tau^* + M_\tau
\end{aligned}$$

where M is some continuous local martingale. Now because we are assuming that the conditions

$$\begin{aligned}
0 &= \psi^*(\beta_k F_p - \gamma - \lambda_p + v_{0L}(1 - \beta_k)) \\
&\qquad + \psi'(k^*)(\tilde{\Phi} + \beta_k \sigma^2 k + (1 - \beta_k)(\gamma_g - \gamma)k) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*) \\
u_c(c^*, k_g^*, \pi^*) &= \beta_c^{-1} \psi^* \\
u_\pi(c^*, k_g^*, \pi^*) &= -\beta_w F_L \psi^* \\
0 &= \psi^*(r\beta_r + \mu_0 - \lambda_p) + \psi'(k^*)(\tilde{\Phi} + (\gamma_g - \gamma)k) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*)
\end{aligned}$$

of Theorem 2 hold, and using the identities $v_{LL} - v_{0L} = \gamma_g - \gamma$ and $v_{00} - v_{0L} =$

$\sigma^2 - (\gamma_g - \gamma)$ we deduce that

$$\begin{aligned}
\int_0^\tau e^{-\lambda_p t} u(c, k_g^* \tilde{\pi})\, dt \;\; &\leq \;\; \int_0^\tau e^{-\lambda_p t} \big[u(c^*, k_g^*, \pi^*) - \beta_c^{-1} \psi^* c^* + \beta_w F_L \psi^* \pi^* \\
&\quad + \psi^* (B + A a^* (v_{00} - v_{0L})) \big] dt + x_0 \psi_0^* - x_\tau e^{-\lambda_p \tau} \psi_\tau^* + M_\tau \\
&= \;\; \int_0^\tau e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt + (x_\tau^* - x_\tau) e^{-\lambda_p \tau} \psi_\tau^* + \tilde{M}_\tau \\
&\leq \;\; \int_0^\tau e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt + x_\tau^* e^{-\lambda_p \tau} \psi_\tau^* + \tilde{M}_\tau
\end{aligned}$$

for some other local martingale $\tilde{M}$. Here, we obtained the second line by reversing the integration-by-parts used on the Lagrangian form. Taking expectations gives us that

$$E \int_0^\tau e^{-\lambda_p t} u(c, k_g^*, \tilde{\pi})\, dt \leq E \int_0^\tau e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt + E x_\tau^* e^{-\lambda_p \tau} \psi_\tau^*,$$

and the transversality condition (PST) allows us to let $\tau \to \infty$ to conclude that

$$E \int_0^\infty e^{-\lambda_p t} u(c, k_g^*, \tilde{\pi})\, dt \leq E \int_0^\infty e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt$$

as required. □

Proof that (L3) holds for $\varphi(k) = \varphi_0 (1 + ak)^\varepsilon$

We need to show that the expression

$$\begin{aligned}
\Lambda_0(k) &\equiv \frac{1}{A} \left[(x + Sy) \frac{c}{k} - \omega y \, \xi + z \, c \right] \\
&= (x + Sy) k^{B-1} \varphi(k) - \omega y k^{B-1} \varphi(k)^{R_g/\omega} + z k^B \varphi(k)
\end{aligned}$$

attains its infimum over $k \geq 0$ uniquely for all non-negative x, y, z. If $x = y = 0$ it will attain its infimum at $k = 0$ as c is increasing, and similarly if $y = z = 0$ the infimum will be attained at $k = \infty$ as c/k is decreasing. We will assume from now on that either y is non-zero or both x and z are non-zero.

Differentiating Λ_0 with respect to k, and using the fact that $R_g/\omega = 1 + (1 + \nu)/\omega$, gives

$$\Lambda_0'(k) = k^{B-2}\varphi'\left[(x+Sy)\left(k+(B-1)\frac{\varphi}{\varphi'}\right)+z\left(k^2+Bk\frac{\varphi}{\varphi'}\right)\right.$$
$$\left. -\,\omega y\left((B-1)\frac{\varphi}{\varphi'}\varphi^{(1+\nu)/\omega}+k\frac{R_g}{\omega}\varphi^{(1+\nu)/\omega}\right)\right]$$
$$= k^{B-2}(1+ak)^{\varepsilon-1}\left[az(\varepsilon+B)k^2+\left(zB-a(x+Sy)(1-B-\varepsilon)\right)k\right.$$
$$\left. -(1-B)(x+Sy)+\omega y\varphi_0^{(1+\nu)/\omega}(1+ak)^{\varepsilon(1+\nu)/\omega}\left((1-B)+a(1-B-\varepsilon\frac{R_g}{\omega})k\right)\right]$$
$$\equiv k^{B-2}(1+ak)^{\varepsilon-1}\left[az(\varepsilon+B)k^2+\left(zB-a(x+Sy)(1-B-\varepsilon)\right)k\right.$$
$$\left. -(1-B)(x+Sy)+\omega y\varphi_0^{(1+\nu)/\omega}(1+ak)^{\varepsilon_0}(1-B+a_0k)\right]$$
$$\equiv k^{B-2}(1+ak)^{\varepsilon-1}f(k)$$

with the appropriate identifications. If we can show that the equation $f(k)=0$ holds for only one point k then the minimum of Λ_0 must be attained uniquely. Firstly observe that

$$f(0) = -(1-B)\left[x+y(S-\omega\varphi_0^{(1+\nu)/\omega})\right]$$

is negative due to equation (4.49). Secondly $a_0 = a(1-B-\varepsilon R_g/\omega)$ is positive because of (4.50) and so $f(k)\to\infty$ as $k\to\infty$. Finally

$$f''(k) = \omega y\varphi_0^{(1+\nu)/\omega}(1+ak)^{\varepsilon_0-2}\left[a_0a^2\varepsilon_0(\varepsilon_0+1)k+a\varepsilon_0(2a_0+(1-B)a(\varepsilon_0-1))\right]$$
$$+\,2az(\varepsilon+B)$$

and so a sufficient condition for f to be convex is

$$a\varepsilon_0(2a_0+(1-B)a(\varepsilon_0-1))\geq 0$$

which is easily seen to be equivalent to

$$\varepsilon\leq\frac{\omega(1-B)}{R_g+S}$$

and this is assumption (4.50). Putting these three facts together we can conclude that $f(k)$ has only one root and so $\Lambda_0(k)$ does attain its infimum uniquely. □

Appendix B The debt process Δ_p

We define a P-Brownian motion W by $\sigma W\equiv Z^0-Z^L$ so that $\sigma^2\equiv v_{00}-2v_{0L}+v_{LL}$ and the dynamics (2.23) of k are

$$dk = \sigma k dW + \tilde{\Phi}(k)dt. \tag{B.1}$$

From (A.5) and Itô applied to $k_p(k)$ the dynamics of Δ_p are given by

$$d\Delta_p = \Delta_p\big[-dZ^L + (\mu_0 + r\beta_r)dt\big] - k_p'(\sigma k dW + \tilde{\Phi}dt) - \frac{1}{2}\sigma^2 k^2 k_p'' dt + AdZ^0 + Bdt + k_p\big[\beta_k dZ^0 - dZ^L + (\beta_k F_p - \gamma + v_{0L}(1-\beta_k))dt\big] + \beta_w \tilde{\pi} F_L dt - \beta_c^{-1} c dt. \tag{B.2}$$

We wish to express Z^0 and Z^L in terms of W so we write

$$dZ^L \equiv adW + bdW'$$

where W' is a P-Brownian motion independent of W so that

$$a\sigma = v_{0L} - v_{LL} \qquad\qquad a^2 + b^2 = v_{LL}$$

and then dZ^0 is given by $dZ^0 = (a+\sigma)dW + bdW'$. Inserting these expressions into (B.2) and collecting Δ_p, dW, dW' and dt terms gives

$$\begin{aligned} d\Delta_p &= \Delta_p\big[-adW - bdW' + (\mu_0 + r\beta_r)dt\big] \\ &\quad + \big[(A + \beta_k k_p)(a+\sigma) - ak_p - \sigma k k_p'\big]dW + \big[Ab + b\beta_k k_p - bk_p\big]dW' \\ &\quad + \big[B - k_p'\tilde{\Phi} - \frac{1}{2}\sigma^2 k^2 k_p'' + k_p(\beta_k F_p - \gamma + v_{0L}(1-\beta_k)) + \beta_w\tilde{\pi}F_L - \beta_c^{-1}c\big]dt \\ &\equiv \Delta_p\big[-adW - bdW' + (\mu_0 + r\beta_r)dt\big] + A_0(k)dW + A_1(k)dW' + \Gamma_0(k)dt \end{aligned} \tag{B.3}$$

with the necessary identifications. To deal firstly with the Δ_p term we consider Z solving the homogeneous stochastic differential equation

$$dZ = Z\big[-adW - bdW' + (\mu_0 + r\beta_r)dt\big]. \tag{B.4}$$

The solution to this stochastic differential equation is given (up to a constant) by

$$Z_t = \exp\left(-aW_t - bW_t' - \frac{1}{2}(a^2+b^2)t + \int_0^t (\mu_0 + r(k_s)\beta_r(k_s))ds\right). \tag{B.5}$$

Observe that from (B.1)

$$\begin{aligned} \sigma dW &= \frac{dk}{k} - \frac{\tilde{\Phi}(k)}{k}dt \\ &= d(\log k) + \left(\frac{1}{2}\sigma^2 - \frac{\tilde{\Phi}(k)}{k}\right)dt \end{aligned}$$

and so we can write equation (B.5) as

$$Z_t = k_t^{-a/\sigma}\exp\left(-bW_t' - \frac{1}{2}b^2 t + \int_0^t G_0(k_s)ds\right), \tag{B.6}$$

where $G_0(k) \equiv \mu_0 + r(k)\beta_r(k) + \frac{1}{2}b^2 - \frac{1}{2}v_{0L} + a\tilde{\Phi}(k)/\sigma k$. Combining the dynamics for Δ_p (B.3) and Z (B.4) gives

$$\begin{aligned} d\left(\frac{\Delta_p}{Z}\right) &= \frac{d\Delta_p}{Z} + \Delta_p d\left(\frac{1}{Z}\right) + d\left\langle \Delta_p, \frac{1}{Z}\right\rangle \\ &= \frac{d\Delta_p}{Z} + \frac{\Delta_p}{Z}\left(-\frac{dZ}{Z} + v_{LL}dt\right) + \frac{1}{Z}(aA_0(k) + bA_1(k) - \Delta_p v_{LL})dt \\ &= \frac{1}{Z}\left(A_0(k)dW + A_1(k)dW' + (\Gamma_0(k) + aA_0(k) + bA_1(k))dt\right) \\ &= \frac{1}{Z}\left(A_0(k)dW + A_1(k)dW' + \Gamma_1(k)dt\right) \end{aligned}$$

where $\Gamma_1(k) \equiv \Gamma_0(k) + aA_0(k) + bA_1(k)$. Thus for $s < t$

$$\frac{\Delta_p(t)}{Z_t} = \frac{\Delta_p(s)}{Z_s} + \int_s^t Z_u^{-1}\left\{A_0(k_u)dW_u + A_1(k_u)dW_u' + \Gamma_1(k_u)du\right\}. \tag{B.7}$$

We can re-express the dW part of this integral. Define $G_1(k)$ so that

$$\sigma k G_1'(k) = k^{a/\sigma} A_0(k)$$

and then we have that

$$\begin{aligned} dG_1(k) &= G_1'(k)(\sigma k dW + \tilde{\Phi}(k)dt) + \frac{1}{2}\sigma^2 k^2 G_1''(k)dt \\ &= k^{a/\sigma} A_0(k)dW + \mathcal{L}G_1(k)dt \end{aligned}$$

where $\mathcal{L}$ is the generator of the process k_t. We can now rewrite the dW term in expression (B.7) as follows

$$\begin{aligned} \int_s^t Z_u^{-1} A_0(k_u)dW_u &= \int_s^t e^{bW_u' + \frac{1}{2}b^2 u - \int_0^u G_0(k_v)dv}\left\{dG_1(k_u) - \mathcal{L}G_1(k_u)du\right\} \\ &= \left[G_1(k_u)k_u^{-a/\sigma}Z_u^{-1}\right]_s^t - \int_s^t k_u^{-a/\sigma}Z_u^{-1}G_1(k_u)\left\{bdW_u' + (b^2 - G_0(k_u))du\right\} \\ &\quad - \int_s^t k_u^{-a/\sigma}Z_u^{-1}\mathcal{L}G_1(k_u)du. \end{aligned} \tag{B.8}$$

Hence expression (B.7) can be written as

$$\begin{aligned} \frac{\Delta_p(t)}{Z_t} &= \frac{\Delta_p(s)}{Z_s} + \frac{G_1(k_t)k_t^{-a/\sigma}}{Z_t} - \frac{G_1(k_s)k_s^{-a/\sigma}}{Z_s} \\ &\quad + \int_s^t Z_u^{-1}\left\{A_1(k_u) - bk_u^{-a/\sigma}G_1(k_u)\right\}dW_u' \\ &\quad + \int_s^t Z_u^{-1}\left\{\Gamma_0(k_u) - k_u^{-a/\sigma}\mathcal{L}G_1(k_u) + k_u^{-a/\sigma}(G_0(k_u) - b^2)\right\}du \\ &\equiv \frac{\Delta_p(s)}{Z_s} + \frac{G_1(k_t)k_t^{-a/\sigma}}{Z_t} - \frac{G_1(k_s)k_s^{-a/\sigma}}{Z_s} + \int_s^t Z_u^{-1}\left\{G_2(k_u)dW_u' + G_3(k_u)du\right\} \end{aligned} \tag{B.9}$$

and Δ_p is given by

$$\Delta_p(t) = G_1(k_t)k_t^{-a/\sigma} + (\Delta_p(s) - G_1(k_s)k_s^{-a/\sigma})\left(\frac{k_s}{k_t}\right)^{a/\sigma} e^{-b(W'_t - W'_s) - \frac{1}{2}b^2(t-s) + \int_s^t G_0(k_u)du}$$
$$+ \int_s^t \left(\frac{k_u}{k_t}\right)^{a/\sigma} e^{-b(W'_t - W'_u) - \frac{1}{2}b^2(t-u) + \int_u^t G_0(k_v)dv} \{G_2(k_u)dW'_u + G_3(k_u)du\}. \tag{B.10}$$

Suppose that we start with zero debt so that $\Delta_p(s) = 0$ at some time s in the past. Can we hold t fixed, let $s \to -\infty$ and get some meaningful limit? We would like something like

$$\lim_{s\to\infty} G_1(k_s)e^{bW'_s - \frac{1}{2}b^2 s + \int_0^s G_0(k_u)du} = 0,$$
$$E^\pi \int_0^\infty k_u^{2a/\sigma} G_2(k_u)^2 e^{b^2 u + 2\int_0^u G_0(k_s)ds} du < \infty,$$
$$E^\pi \int_0^\infty \left| k_u^{a/\sigma} G_3(k_u) \right| e^{\int_0^u G_0(k_s)ds} du < \infty$$

where π is the invariant law of k. An example of the sort of simpler (sufficient) conditions needed for these to hold would be

$$G_1(k), G_2(k), G_3(k) \qquad \text{all bounded,}$$
$$\frac{\tilde{\Phi}(k)}{k} \to -\varepsilon < 0 \quad \text{as} \quad k \to \infty,$$
$$\sup_k G_0(k) < -\frac{1}{2}b^2.$$

The condition on $\tilde{\Phi}$ makes the tail of the invariant law of k like a Gaussian, so all moments exist, and the condition on $\sup G_0$ makes the exponential term decreasing, so then we do get convergence as $s \to -\infty$, with

$$\Delta_p(t) = G_1(k_t)k_t^{-a/\sigma} \tag{B.11}$$
$$+ \int_{-\infty}^t \left(\frac{k_u}{k_t}\right)^{a/\sigma} e^{-b(W'_t - W'_u) - \frac{1}{2}b^2(t-u) + \int_u^t G_0(k_v)dv} \{G_2(k_u)dW'_u + G_3(k_u)du\}.$$

Appendix C The one-sector problem

In the one-sector problem there is no distinction between public and private capital, and we can follow a similar development; or we may alternatively deduce the one-sector results as special cases of the two-sector results above. Either way, we will assume that the private sector works all the hours available to them ($\pi = 1$ in the previous notation) so that the rate of production is given simply by $F(K, LT) \equiv LTf(k)$ and the objective of the government is to maximise

$$E\int_0^\infty e^{-\rho_g t} L_t U\left(\frac{C_t}{L_t}\right) dt = L_0 E_g \int_0^\infty e^{-\lambda_g t} U(c_t) dt$$

where we use exactly the same notation as in the two-sector problem, and again assume that U is homogeneous of order $1 - R_g$. The optimality equations corresponding to those of Theorem 1 are

$$
\begin{aligned}
0 &= U(c) - \lambda_g V(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + \Phi(k)V'(k) && \text{(C.1)} \\
\Phi(k) &= f(k) - \gamma_g k - c && \text{(C.2)} \\
U'(c) &= V'(k). && \text{(C.3)}
\end{aligned}
$$

We have assumed that U is homogeneous of order $1-R_g$ so it must have the Constant Relative Risk Aversion (CRRA) form

$$U(c) = \frac{c^{1-R_g}}{1-R_g},$$

with $R_g > 0$ and $R_g \neq 1$. Again it is possible to construct an explicit solution to the government's problem; choosing V, we find the optimal c from (C.3), then deduce Φ from (C.1), and then deduce f from (C.2). It remains only to check that the f so obtained is concave, increasing and non-negative.

As a simple example, if we pick a value function that is also CRRA

$$V(k) = \frac{A_V^{-R_g} k^{1-S}}{1-S},$$

with $A_V > 0$, $S > 0$ and $S \neq 1$ then (C.3) gives us

$$c(k) = A_V k^{S/R_g}$$

and then (C.1) yields

$$
\begin{aligned}
\Phi(k) &= \left(\frac{\lambda_g}{1-S} + \frac{1}{2}\sigma^2 S\right) k - \frac{A_V k^{S/R_g}}{1-R_g} \\
&\equiv Qk - \frac{A_V k^{S/R_g}}{1-R_g}.
\end{aligned}
$$

Finally (C.2) gives

$$
\begin{aligned}
f(k) &= (\gamma_g + Q)k + \left(1 - \frac{1}{1-R_g}\right) c \\
&= (\gamma_g + Q)k - \frac{R_g A_V k^{S/R_g}}{1-R_g}.
\end{aligned}
$$

For these last two equations to make economic sense we require that

$$Q + \gamma_g \geq 0, \quad R_g > S > 1.$$

Appendix D Summary of notation

A t argument/subscript denotes a quantity at time t. Other subscripts are used to denote partial differentiation in the case of functions of two or more variables (e.g $f_g \equiv \partial f/\partial k_g$). Notation unique to the section on explicit solutions (Section 4) is not covered in this appendix.

C	Consumption rate
D	Level of government debt
I_g	Amount invested in government capital
I_p	Amount invested in private capital
K	Total capital
K_g	Government capital
K_p	Private sector capital
L	Labour force / population size
T	Technology level
X	Total private sector wealth $K_p + D$
c, k, k_g, k_p, x	$\equiv C/LT, K/LT, K_g/LT, K_p/LT, X/LT$
$c^*(k), k_g^*(k), \pi^*(k)$	Optimal values of c, k_g, π for a given k
$1-\beta_c$	Tax rate on consumption
$1-\beta_k$	Tax rate on returns on private capital
$1-\beta_r$	Tax rate on returns on government debt
$1-\beta_w$	Tax rate on wages
Δ_p	$\equiv D/\eta$
$\Delta C, \Delta D, \Delta K_p$	Per household rate of consumption, holding in government debt and amount of private capital
η	$\equiv LT$
ξ	$\equiv C/K_g \equiv c/k_g$
π	Proportion of time devoted to production
$F(K_p, K_g, \pi LT)$	Production (rate) function
$U(c, k_g, \pi)$	Government felicity function
$u(c, k_g, \pi)$	Private sector felicity function
$V(k)$	Government value function
$\Phi(k)$	$\equiv F(k_p^*(k), k_g^*(k), \pi^*(k)) - \gamma_g k - c^*(k)$. The drift in k along the optimal path under P_g

$\tilde{\Phi}(k)$	$\equiv F(k_p^*(k), k_g^*(k), \pi^*(k)) - \gamma k - c^*(k)$. The drift in k along the optimal path under P
Ψ	$\equiv V'(k)$. The Lagrange multiplier process corresponding to the government's optimization problem.
ψ	The Lagrange multiplier process corresponding to the private sector's optimization problem.
E, E_g	Expectation taken under P, P_g respectively
P	Real world probability measure
P_g	Government's valuation measure
v_{ij}	Covariation (per unit time) of Z^i and Z^j, $i, j \in 0, L$
W	A P-Brownian motion defined by $\sigma W \equiv Z^0 - Z^L$
w	A P_g-Brownian motion defined by $\sigma w \equiv z^0 - z^L$
Z^0, Z^L	Multiples of standard Brownian motions
(z^0, z^L)	Two P_g-Brownian motions with same covariance structure as (Z^0, Z^L)
R_g	U is homogeneous of order $1 - R_g$ in c,k_g
R_p	u is homogeneous of order $1 - R_p$ in c,k_g
δ	Rate of depreciation of capital
γ	$\equiv \delta + \mu_L + \mu_T + v_{0L} - v_{LL}$
γ_g	$\equiv \gamma - v_{0L} + v_{LL}$
θ_p, θ_L	Proportion of return on government's capital included in returns to private sector capital and labour respectively
λ_g	$\equiv \rho_g - (1 - R_g)\mu_T - \mu_L$
λ_p	$\equiv \rho_p - (1 - R_p)\mu_T$
μ_0	$\equiv v_{LL} - \mu_L - \mu_T$. Exponential drift of η^{-1}
μ_L	Exponential drift term of labour
μ_T	Exponential growth rate of technology level
ρ_g, ρ_p	Government and private sector utility time-discount factors
σ^2	$\equiv v_{00} - 2v_{0L} + v_{LL}$