Modelling liquidity effects in discrete time

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Abstract

We study optimal portfolio choices for an agent with the aim of maximising utility from terminal wealth within a market with liquidity costs. Under some mild conditions, we show the existence of optimal portfolios and that the marginal utility of the optimal terminal wealth serves as a change of measure to turn the marginal price process of the optimal strategy into a martingale. Finally, we illustrate our results numerically in a Cox-Ross-Rubinstein binomial model with liquidity costs and find the reservation ask prices for simple European put options.

KEYWORDS: liquidity risk, utility maximisation from terminal wealth, Bellman equation, equivalent martingale measure, Cox-Ross-Rubinstein model.

1 Introduction

After market risk and credit risk, liquidity risk is arguably the most important risk faced by the finance industry. There have been numerous approaches to modelling liquidity risk over the years, and the literature on illiquid financial markets can roughly be divided into two categories: (i) studies on the effect of a large trader, and (ii) studies on price impact due to immediacy provision by market makers. The research falling into the first category studies the implications of a large trader who can move the asset prices by his actions on pricing and hedging.

In the discrete-time model of Jarrow [9] the asset price depends on the holdings of the large trader via a certain reaction function. This paper studies the sufficient conditions to rule out the arbitrage opportunities for the large trader and analyze the optimal hedging strategies as well. Frey [5], Frey & Stremme [6], and Platen & Schweizer [11] study hedging strategies for the large trader in similar reaction settings in continuous time. Cuoco & Cvitanič [3] and Cvitanič & Ma [4] study a diffusion model for the price dynamics where the indirect feedback effect is modelled by making the drift and volatility coefficients depend on the large trader's trading strategy. Recently, Bank and Baum [1] extended Jarrow's result to continuous time using tools from Kunita's non-linear integral. Liquidity cost as the market maker's cost of providing immediacy is introduced into the literature by Grossmann and Miller [7] in a model for determining the equilibrium level of liquidity in a market.

Our approach in this paper is related to a couple of more recent contributions, Çetin, Jarrow & Protter [2] and Rogers & Singh [14], that could fit into the second category of research as outlined above. Although the formulations of the liquidity costs are different, due to different limiting arguments for the liquidity costs incurred by the continuous strategies, the common approach adopted in both is based on equalisation of supply and demand in the short-term market which is relevant if an agent is attempting to trade large volumes in a short time. As this market is localised in time, prices paid do not impact prices at other times when there is no abnormal buying or selling pressure. This has the strong advantage that the actions of agents do *not* influence prices except at the times when they are trading, with the result that the price process of the share has a dynamic that is not influenced by the actions of the agent. This is important practically and conceptually because if the actions of one agent affect the share price, then the actions of *all* agents must be allowed to affect the share price, and the analysis of such a complex system becomes impractical.

The plan of the paper is as follows. In Section 2 we set out the modelling assumptions. An agent trades in an illiquid market with the aim of maximising his expected utility of terminal wealth; when he changes his portfolio, the price he pays for the stock is the notional price plus a liquidity cost, which enters like a non-linear transaction cost[1].

Maximising expected utility of terminal wealth in a liquid market is of interest for a number of reasons. Firstly, there is the result (well known to economists) that if an optimum can be achieved then the marginal utility of optimal wealth is a state-price density; see, for example, the expository article [13] which gives a sketch of the ideas, and [12] for a proof of the discrete-time Fundamental Theorem of Asset Pricing using exactly this approach. More generally, Kramkov

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[1]See Rogers & Singh [14] for a derivation of this modelling idea.

& Schachermayer [10] and others examine the maximisation of expected utility of terminal wealth in the context of a general semimartingale model; the simple results of the economists need to be modified in subtle ways. Secondly, the maximisation of expected utility of terminal wealth can be used as a way of determining utility-indifference prices in an incomplete market, following the seminal paper of Hodges & Neuberger [8].

How are these results affected by including liquidity effects? We shall show that under certain mild conditions the optimisation of expected utility of terminal wealth does have a solution; and that the marginal utility of optimal terminal wealth is an equivalent martingale measure. However, in the transformed measure, the process which becomes the martingale is *not* the (notional) stock price process, but rather the *marginal* price process, that is, the price to be paid per unit for an infinitesimal extra amount of the stock. Looking at the simple argument preceding Theorem 4.1, this is hardly surprising. Moreover, no hypothesis of absence of arbitrage is needed, unlike the liquid case; again, this is not surprising when one realises that under strictly convex liquidity costs there is limited scope to exploit an arbitrage because of the increasing costs of taking ever larger positions in the advantageous portfolio.

The final part of our paper takes a simple example of the Cox-Ross-Rubinstein binomial model with liquidity effects, and an agent with CARA utility (as usual, this is done for numerical tractability in that the dimension of the problem reduces by 1; more general utilities could in principle be dealt with, but this example serves already to illustrate various properties). We find reservation ask prices for simple European put options, and see what hedging strategy is carried out by a liquidity-constrained agent. The appendix contains the proofs of the theorems that are not given in the text.

2   The modelling framework

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with the filtration $(\mathcal{F}_n)_{0 \leq n \leq N+1}$. All the random variables and stochastic processes in this and subsequent sections will be defined on this base. In a discrete time setting let $(S_n)_{0 \leq n \leq N}$ denote the strictly positive asset price process, which we shall suppose has the property

$$S_n \in L^1 \qquad \forall n \leq N, \tag{2.1}$$

and is adapted to $(\mathcal{F}_n)_{0 \leq n \leq N+1}$. Suppose that portfolio rebalancing occurs between two time points; between time $n-1$ and time n, we change the number of shares held from X_{n-1} to X_n, and the cash held changes from Y_{n-1} to

$$Y_n = Y_{n-1} - \varphi(\Delta X_n) S_{n-1}, \tag{2.2}$$

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where $\Delta X_n \equiv X_n - X_{n-1}$. For now, assume that interest rates are zero. We further assume that X_n is $\mathcal{F}_{n-1}$-measurable, which in turn implies the processes X and Y are previsible. We shall assume that $\varphi : \mathbb{R} \to (-\infty, \infty]$ is strictly convex and strictly increasing where finite, and has the properties

$$\inf_x \varphi'(x) = 0, \quad \sup_x \varphi'(x) = \infty, \quad \varphi(0) = 0. \tag{2.3}$$

We define the concave dual function of φ by

$$\tilde{\varphi}(w) \equiv \inf\{\varphi(x) + wx\}. \tag{2.4}$$

We suppose that an agent has the task of maximising $EU(Y_{N+1})$, where we understand that after S_N is revealed, the agent liquidates his holding of the share, so that at time $N+1$ he has only cash. The utility function $U : \mathbb{R} \to [-\infty, \infty)$ is assumed to be strictly increasing and strictly concave in its domain of finiteness $\mathcal{D} = \{x : U(x) > -\infty\}$. We also suppose the Inada conditions,

$$\sup_{x \in \mathcal{D}^o} U'(x) = +\infty, \qquad \inf_{x \in \mathcal{D}^o} U'(x) = 0, \tag{2.5}$$

where $\mathcal{D}^o$ denotes the interior of $\mathcal{D}$ as a subset of the real line. Without loss of generality, we shall further suppose $\mathcal{D} = \mathbb{R}$.

At time n, the agent's optimisation problem can be thought of as choosing the process $(\Delta X_j)_{j=n+1}^N$ of changes of portfolio to be applied from the present time up to N. Once these are chosen, the cash value at time $N+1$ is just

$$y - \varphi(-x - \sum_{j=n+1}^N \Delta X_j)S_N - \sum_{j=n+1}^N \varphi(\Delta X_j)S_{j-1},$$

leading us to define

$$\Phi_n(x, y, (\Delta X)) = U\Big(y - \varphi(-x - \sum_{j=n+1}^N \Delta X_j)S_N - \sum_{j=n+1}^N \varphi(\Delta X_j)S_{j-1}\Big). \tag{2.6}$$

Now we define for each integer $n \in [0, N]$

$$\begin{aligned}
v_n(x, y) &\equiv \text{ess sup } E_n[U(Y_{N+1})|X_n = x, Y_n = y] \tag{2.7} \\
&\equiv \text{ess sup } E_n[\Phi_n(x, y, (\Delta X))],
\end{aligned}$$

where E_n denotes the conditional expectation with respect to the σ-field $\mathcal{F}_n$, and the essential supremum is taken over all previsible processes (ΔX). Note that $v_n(x, y)$ is not a deterministic function but a random process for each (x, y). Thus for example

$$v_N(x, y) = U(y - \varphi(-x)S_N).$$

There may be an issue of the sense in which v_n is defined, in view of the uncountably many values of (x, y) for each of which a conditional expectation is required. However, we shall suppose that v_n is only defined in the first place for dyadic rational x and y; Proposition 2.1 will show that v_n is a concave increasing function of its two arguments, and so will extend uniquely off the rationals to all real (x, y).

So that we are not considering a vacuous question, we shall make the

ASSUMPTION A: *For all x, y, and for all n, $v_n(x, y) < \infty$, a.s.*

Under this assumption, we have our first result.

Proposition 2.1 *For each n, $v_n(x, y)$ is concave and increasing in x and y, almost surely.*

PROOF.   See Appendix.

3   Existence of optimal hedging strategies.

We aim to prove in this section that the supremum in (2.7) is actually a maximum. For this, it is helpful (though quite possibly unnecessary) to make the further

ASSUMPTION B: *For all n, for all $t > 0$, $S_n \tilde{\varphi}(-t/S_n) \in L^1$.*

Next define the convex dual functions

$$\tilde{v}_n(\eta) \equiv \sup_{x,y} \{ v_n(x, y) - \eta_1 x - \eta_2 y \} \tag{3.8}$$

for $\eta = (\eta_1, \eta_2) >> 0$. By considering the case where $x = 0$, $y = 1$, and where we use the suboptimal policy of never investing in the risky asset, we see that $\tilde{v}_n(\eta) \geq U(1) - \eta_2$, which is a constant lower bound. Before we bound $\tilde{v}_n(\eta)$ from above, note that for any (ΔX), any x, y, we have (writing $y' = y - \varphi(\Delta X_n) S_{n-1}$, $x' = x + \Delta X_n$)

$$
\begin{aligned}
E_n[\Phi_n(x, y, (\Delta X))] &= E_n \Big[E_{n+1}[U(y' - \varphi(-x' - \sum_{n+2}^{N} \Delta X_j) S_N - \sum_{n+2}^{N} \varphi(\Delta X_j) S_{j-1})] \Big] \\
&\leq E_n[v_{n+1}(x', y')].
\end{aligned}
$$

Keeping this in mind, one can show

Lemma 3.1 *For all n and for all $\eta >> 0$,*

$$\tilde{v}_n(\eta) \leq -\eta_2 S_n \tilde{\varphi}(-\eta_1/\eta_2 S_n) + E_n \tilde{v}_{n+1}(\eta).$$

5

PROOF. See Appendix.

It is simple to confirm that $\tilde{v}_N(\eta) \in L^1$ for any $\eta >> 0$, and in view of Assumption B we deduce that $\tilde{v}_n(\eta) \in L^1$ for all n, for all $\eta >> 0$. It follows immediately that $v_n(x, y) \in L^1$ for all x, y, for all n.

Now if we fix $\eta >> 0$

$$
\begin{aligned}
v_n(x, y) &\leq \sup_{\Delta x} E_n v_{n+1}(x + \Delta x, y - \varphi(\Delta x)S_n) \\
&\leq \sup_{\Delta x} E_n[\tilde{v}_{n+1}(\eta) + \eta_1(x + \Delta x) + \eta_2(y - \varphi(\Delta x)S_n) \\
&= E_n \tilde{v}_{n+1}(\eta) + \eta_1 x + \eta_2 y + \sup_{\Delta x}[\eta_1 \Delta x - \eta_2 \varphi(\Delta x)S_n]; \quad (3.9)
\end{aligned}
$$

the point of this is that the expression involving Δx in (3.9) tends to $-\infty$ as $\Delta x \to \infty$ and as $\Delta x \to -\infty$, so the supremum is actually a maximum. Thus, we have the following theorem:

Theorem 3.1 *Suppose Assumptions A and B hold. Then, $v_n(x, y) \in L^1$ for all n and x, y. Moreover, the solution to (2.7) is attained in the set of previsible processes, (X, Y), such that $X_n = x$ and $Y_n = y$.*

Our next result is a Bellman equation in this setting which is made precise in the next proposition.

Proposition 3.1 *We have for $0 \leq n < N$*

$$
v_n(x, y) = \sup_{\Delta x} E_n[\, v_{n+1}(x + \Delta x, y - \varphi(\Delta x)S_n)\,]. \quad (3.10)
$$

PROOF. See Appendix.

4 Risk-neutral marginal pricing

Suppose that we have $X_n = x$, $Y_n = y$, and that the process $(\Delta X_m)_{m=n+1}^N$ is optimal for these particular initial values. We are going to consider a perturbation $(\Delta X_m^{(\varepsilon)})_{m=n+1}^N$ of this optimal policy defined by

$$
\begin{aligned}
\Delta X_m^{(\varepsilon)} &= \varepsilon + \Delta X_{n+1} \quad (m = n + 1) \\
&= \Delta X_m \quad (m > n + 1).
\end{aligned}
$$

At time $N + 1$, the cash $Y_{N+1}^{(\varepsilon)}$ under this new policy is

$$
\begin{aligned}
Y_{N+1}^{(\varepsilon)} &= Y_{N+1} - [\varphi(\Delta X_{n+1}^{(\varepsilon)}) - \varphi(\Delta X_{n+1})]S_n - [\varphi(-X_N - \varepsilon) - \varphi(-X_N)]S_N \\
&= Y_{N+1} - [\varphi(\Delta X_{n+1} + \varepsilon) - \varphi(\Delta X_{n+1})]S_n - [\varphi(-X_N - \varepsilon) - \varphi(-X_N)]S_N,
\end{aligned}
$$

which is clearly concave in ε. Because of optimality, we know that for any ε,

$$E_n[U(Y_{N+1})] \geq E_n[U(Y_{N+1}^{(\varepsilon)})],$$

so[2] we learn that

$$E_n[U'(Y_{N+1})(\varphi'(\Delta X_{n+1})S_n - \varphi'(-X_N)S_N)] = 0. \qquad (4.11)$$

In summary, this says that the process[3]

$$M_n \equiv \varphi'(\Delta X_{n+1})S_n \qquad (4.12)$$

is a martingale under the measure Q defined by

$$\frac{dQ}{dP} \propto U'(Y_{N+1})$$

This heuristic argument leads to the following theorem whose proof is provided in the appendix.

Theorem 4.1 *Suppose Assumptions A and B hold and let the process* $(\Delta X_m)_{m=n+1}^{N}$ *be the optimal solution of* $v_n(x, y)$ *with the optimal terminal wealth* Y_{N+1}. *Then*

i) the value function v_n *is a.s. differentiable with respect to both arguments and, moreover,*

$$
\begin{aligned}
D_x v_n(x, y) &= E_n\left[S_N \varphi'(-X_N)U'(Y_{N+1}) \right] = S_n \varphi'(\Delta X_{n+1})E_n[U'(Y_{N+1})], \\
D_y v_n(x, y) &= E_n\left[U'(Y_{N+1}) \right].
\end{aligned}
$$

ii) The process

$$M_n \equiv \varphi'(\Delta X_{n+1})S_n$$

is a martingale under the measure Q *defined by*

$$\frac{dQ}{dP} \propto U'(Y_{N+1}).$$

5 Arbitrage opportunities and equivalent martingale measures

As seen, we have not assumed absence of arbitrage in the previous sections to show the existence of optimal strategies. Indeed, as the following example shows, arbitrage opportunities and optimal portfolios could co-exist in an illiquid market. We first make precise what we mean by an arbitrage opportunity.

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[2]... *assuming* we can differentiate inside the expectation! This needs justification, and is dealt with in Section A.1.

[3]Is the process M defined by (4.12) integrable? We shall show in Section A.1 that it is.

Definition 5.1 (X, Y) *is said to be an* arbitrage opportunity *if X and Y are previsible processes with $X_0 = Y_0 = X_{N+1} = 0$, satisfying (2.2), and*

$$Y_{N+1} \geq 0 \qquad and \qquad P(Y_{N+1} > 0) > 0.$$

Example. Consider the one-period market where $S_0 = 1$, $\varphi(x) = e^x - 1$, and $P(S_1 = 1/4) = P(S_1 = 1/2) = 1/2$. Choosing the initial trade $x \in (\log \frac{1}{2}, 0)$ will generate cash

$$Y_2 = (e^x - 1)(e^{-x} S_1 - 1)$$

at time 2, which is strictly positive, therefore an arbitrage opportunity. Clearly, this model satisfies Assumptions A and B; thus, given a utility function, the optimal portfolio exists. Note that a trading strategy in this model is buying x units at time 0 and selling all at time 1. Thus, the associated marginal prices with this strategy are e^x at time 0 and $e^{-x} S_1$ at time 1. The only way that the marginal price process of this strategy is a martingale under some equivalent measure is when

$$e^{-x} \frac{1}{4} < e^x \qquad and \qquad e^{-x} \frac{1}{2} > e^x,$$

which can hold only if $x \in (\log \frac{1}{2}, \log \frac{1}{\sqrt{2}})$. Therefore, the optimal strategy with respect to a given utility function always satisfies these bounds.

Co-existence of arbitrage and the optimal strategy is due to the fact that infinite arbitrage opportunities are not possible in an illiquid market in our sense.

6 Numerical study

In this Section we consider an agent with utility

$$U(x) = -\exp(-\gamma x),$$

and the simple binomial model. We shall let S_n denote the stock price at time n, and suppose that $S_0 = 1$. Given the value S_{n-1} of the stock at time $n-1$, the value of the stock at time n is either uS_{n-1} with probability $p \in (0, 1)$ or dS_{n-1} with probability $q = 1-p$, where $d < u$. There is also a riskless money-market account, which grows by a factor r each period; in the conventional binomial model, we require that $d < r < u$ to avoid arbitrage, but this is no longer necessary, since the presence of liquidity costs will limit the possibilities for unbounded riskless gain.

We shall be concerned only with the pricing of European-style contingent claims, so the state-space of this system is a recombining lattice. If ξ denotes one of the nodes of this lattice, then ξu will denote the node one unit of time later

resulting from a favourable stock move (upward to uS), and ξd will denote the node one unit of time later resulting from an unfavourable stock move. We shall write $V_\xi(x)$ for the value to the agent of being at node ξ and holding x units of stock, and zero units of the money-market account; the CARA form of the utility permits us to factor out the dependence on the money-market component, and leads to the Bellman equation

$$V_\xi(x) = \sup_{\Delta x} e^{\gamma \varphi(\Delta x) S_\xi r^m} [p V_{\xi u}(x + \Delta x) + q V_{\xi d}(x + \Delta x)], \qquad (6.13)$$

where m denotes the number of time steps from ξu to the end. We suppose

$$\varphi(x) = \frac{e^{\alpha x} - 1}{\alpha}.$$

We investigate the optimal hedging strategy in a 3-period economy for a trader short one European put option and with zero initial position in the stock and zero cash at time 0. We run the above model with parameters $p = 0.7, u = 1/d = \exp(0.1)$ and $r = 1.05$. Tables I-III give a comparison of optimal hedging strategies with respect to different liquidity parameters, α, with differing strikes and risk aversion parameters. Notice that $\alpha = 0$ corresponds to a perfectly liquid market, i.e. $\varphi(x) = x$. As seen, the presence of liquidity costs forces the trader to trade more cautiously and much less, in absolute quantities, compared to one in a liquid market . This behaviour does not change even if the risk aversion parameter, γ, changes. The less liquid the market the less the agent trades. For instance, if one increases the liquidity parameter to 0.5, the initial optimal hedge becomes selling 0.04 units of the stock short.

On the other hand, one can see that as the liquidity parameter gets smaller, i.e. $\varphi(x) \to x$, the optimal hedge ratios tends to be closer to the ones in a perfectly liquid market. The speed of convergence is not uniform in trading dates and the closer to the maturity the smaller α may be needed to converge to the standard CRR limit.

Table IV reports the marginal prices associated with the optimal hedges. Notice that the asset price S admits an equivalent martingale measure. Although the table shows the values with 2 decimal points, the marginal prices corresponding to the optimal strategy when $\alpha = 5e-5$, coincide with S up to 3 decimal points at all nodes.

Let C denote the random variable representing a European contingent claim in the binomial model specified above. Define

$$v^C(x, y) = \text{ess sup } E[U(Y_{N+1} - C)].$$

The reservation ask price, $p(x, y)$, of the claim C is the real number satisfying

$$v^C(x, y + p) = v(x, y).$$

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	α	t=0	u	d	uu	ud	du	dd
Δ	0	-0.75	0.15	-0.49	-8.26	-9.26	-8.66	-9.86
Δ	5e-5	-0.75	0.11	-0.54	-8.14	-9.14	-8.46	-9.63
Δ	5e-2	-0.32	-0.22	-0.30	-0.2	-0.24	-0.20	-0.24

Table I: $K = 1$ $\gamma = 1$

	α	t=0	u	d	uu	ud	du	dd
Δ	0	-0.23	0.12	-0.40	-2.85	-4.01	-3.49	-4.77
Δ	5e-2	-0.24	-0.09	-0.33	-0.17	-0.31	-0.16	-0.30

Table II: $K = 1$ $\gamma = 5$

	α	t=0	u	d	uu	ud	du	dd
Δ	0	-0.89	0.25	-0.83	-8.26	-9.3	-8.22	-9.37
Δ	5e-2	-0.36	-0.23	-0.34	-0.17	-0.25	-0.19	-0.19

Table III: $K = 1.1.$ $\gamma = 1$

	α	t=0	u	d	uu	ud	du	dd
$\varphi'(\Delta)S$	0	1	1.10	0.90	1.22	1.00	1.00	0.82
$\varphi'(\Delta)S$	5e-5	1	1.10	0.90	1.22	1.00	1.00	0.82
$\varphi'(\Delta)S$	5e-2	0.98	1.09	0.89	1.21	0.99	0.99	0.81

Table IV: Marginal prices associated with the optimal hedging strategies with different liquidity parameters.

Given the exponential form of the utility function, one finds

$$p(x,y) = \frac{1}{\gamma} \log \frac{v^C(x,y)}{v(x,y)}. \tag{6.14}$$

Figures 1 and 2 present the reservation ask prices, as a function of initial holdings of the stock (assuming no initial cash position), of European put options with different strikes, liquidity and risk aversion parameters. Again, as the liquidity parameter gets smaller, the price converges to the standard CRR price. Moreover, using Theorem 4.1, one obtains

$$D_x p(x,y) = \varphi'(\Delta X_1)S_0 - \varphi'(\Delta X_1^C)S_0 \tag{6.15}$$

where ΔX and ΔX^C denote the optimal trading strategies with no position and short position in the option, respectively. The above expression indicates that the slope of the price as a function of initial stock holding equals the difference between the marginal prices defined by the optimal strategies for two identical
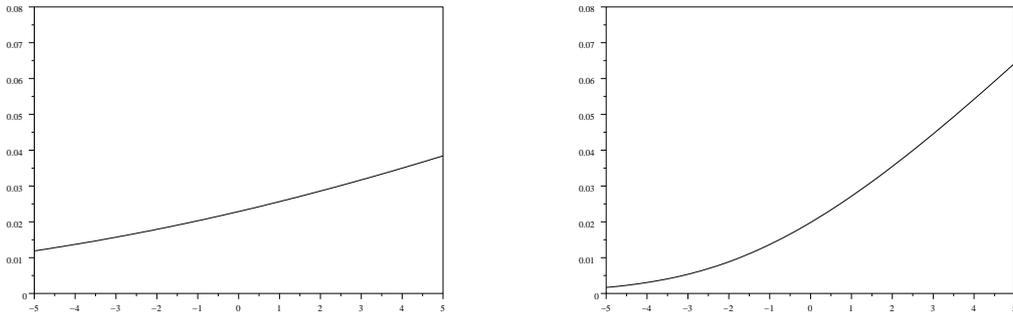
Figure 1: $K = 1, \alpha =$5e-2. Price of the option in the corresponding liquid market is 0.01. Left plot corresponds to $\gamma = 1$ while the right corresponds to $\gamma = 5$
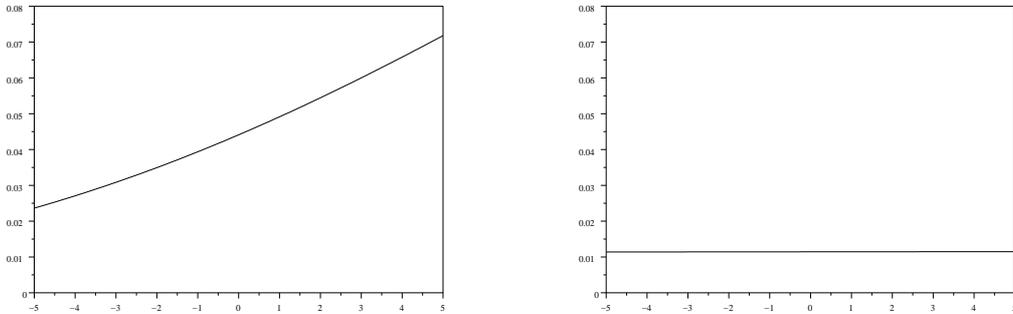

Figure 2: Left plot corresponds to $K = 1.1, \alpha =$5e-2, $\gamma = 1$ and the price of the option in the corresponding liquid market is 0.02. Right plot corresponds to $K = 1, \alpha =$5e-5, $\gamma = 1$ and price of the option in the corresponding liquid market is 0.01.

agents, one having no position in the option and the other short one option. Since φ is strictly convex and increasing one may deduce from (6.15) that for fixed y, $\Delta X_1(x, y) > $ (resp. $<$) $\Delta X_1^C(x, y)$ at x where $p(x, y)$ is increasing (resp. decreasing).

Although the above figures may seem to indicate a convex price curve, this is not always the case as the next plot shows.
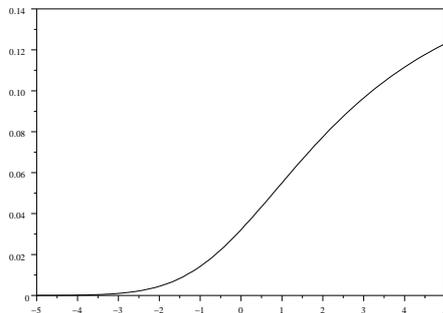
Figure 3: $K = 1, \alpha = 0.9, \gamma = 1$

7 Conclusions

In this paper, we have studied the maximisation of expected utility of terminal wealth in a simple model for liquidity effects, based on Rogers & Singh [14]. What in a liquid market would be arbitrage opportunities can perfectly well exist in such a model, because the cost of liquidity prevents unbounded exploitation of the apparent arbitrage; an agent will exploit the opportunity until the increasing cost of liquidity makes it unprofitable to proceed further. Along the optimal path, the marginal price of the stock becomes a martingale in the measure given by the marginal utility of optimal terminal wealth.

We also investigate the effects of liquidity on the optimal hedging strategy for a European put option in a binomial model with CARA utility. Even small liquidity costs can make a big difference to the extent to which one should hedge, even in this simple discrete-time model. A similar analysis in continuous time is performed by Rogers & Singh [14], who find comparable results, as well as an asymptotic expansion for the cost of liquidity and its effect on hedging.

A Appendix

A.1 Proofs of theorems

Define for each $n, x,$ and $y,$

$$C_n(x, y) = \{E_n[\Phi_n(x, y, (\Delta X))] : (\Delta X) \text{ previsible}, X_n = x, Y_n = y\}.$$

Thus, we can rewrite the optimisation problem of the trader as follows:

$$v_n(x, y) = \text{ess sup}_{H \in C_n(x,y)} H. \qquad (A.16)$$

Lemma A.1 $C_n(x, y)$ *is a lattice for all* n, x, *and* y.

PROOF. If $Z_i = E_n[\Phi_n(x, y, (\Delta X^i))]$, $i = 1, 2$, then taking $(\Delta X) = I_{\{Z^1 > Z^2\}}(\Delta X^1) + I_{\{Z^1 \leq Z^2\}}(\Delta X^2)$, we obtain $E_n[\Phi_n(x, y, (\Delta X))] = Z_1 \vee Z_2$. □

PROOF OF PROPOSITION 2.1. Monotonicity is obvious. For the concavity, suppose we consider (x_1, y_1) and (x_2, y_2) as two possible starting values at time n. Pick $p \in (0, 1)$ and set $q = 1 - p$. If for some $\varepsilon > 0$ we had

$$v_n(px_1 + qx_2, py_1 + qy_2) < pv_n(x_1, y_1) + qv_n(x_2, y_2) - 2\varepsilon,$$

on some $\Lambda \in \mathcal{F}$ with $P(\Lambda) > 0$, take ε-optimal policies $(\Delta X_m^j)_{n < m \leq N}$, $j = 1, 2$ with corresponding cash processes $(Y_m^j)_{n < m \leq N+1}$ and

$$E[U(Y_{N+1}^j)1_\Lambda] \geq E[v_n(x_j, y_j)1_\Lambda] - \varepsilon, \qquad (A.17)$$

$j = 1, 2$ thanks to Lemma A.1. Now if we use the strategy

$$\Delta X_m = p\Delta X_m^1 + q\Delta X_m^2,$$

the corresponding cash process Y satisfies

$$\Delta Y_m \geq p\Delta Y_m^1 + q\Delta Y_m^2$$

because of convexity of φ and positivity of S. Therefore $Y_{N+1} \geq pY_{N+1}^1 + qY_{N+1}^2$, and by concavity of U we have

$$
\begin{aligned}
E[v_n(px_1 + qx_2, py_1 + qy_2)1_\Lambda] &\geq E[U(Y_{N+1})1_\Lambda] \\
&\geq pE[U(Y_{N+1}^1)1_\Lambda] + qE[U(Y_{N+1}^2)1_\Lambda] \\
&\geq E[pv_n(x_j, y_j)1_\Lambda + qv_n(x_j, y_j)1_\Lambda] - \varepsilon \\
&\geq E[v_n(px_1 + qx_2, py_1 + qy_2)1_\Lambda] + \varepsilon,
\end{aligned}
$$

a contradiction.

□

PROOF OF LEMMA 3.1.

$$\tilde{v}_n(\eta) = \sup_{x,y}\{v_n(x, y) - \eta_1 x - \eta_2 y\}$$

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$$\leq \sup_{x,y,\Delta x} \{ E_n v_{n+1}(x + \Delta x, y - \varphi(\Delta x)S_n) - \eta_1 x - \eta_2 y \}$$

$$= \sup_{x',y',\Delta x} \{ E_n v_{n+1}(x', y') - \eta_1(x' - \Delta x) - \eta_2(y' + \varphi(\Delta x)S_n) \}$$

$$= \sup_{x',y',\Delta x} \{ \eta_1 \Delta x - \eta_2 \varphi(\Delta x)S_n + E_n(v_{n+1}(x', y') - \eta_1 x' - \eta_2 y') \}$$

$$\leq \sup_{\Delta x} \{ \eta_1 \Delta x - \eta_2 \varphi(\Delta x)S_n + E_n \tilde{v}_{n+1}(\eta) \}$$

$$= -\eta_2 S_n \tilde{\varphi}(-\eta_1/\eta_2 S_n) + E_n \tilde{v}_{n+1}(\eta).$$

$\square$

PROOF OF PROPOSITION 3.1. By Lemma A.1, $C_n(x, y)$ is a lattice. Therefore for each k we can find some $(\Delta X^{n,k,x,y})$ such that

$$E[Z^{n,k,x,y})] \geq \sup\{EZ : Z \in C_n(x, y)\} - 2^{-k}.$$

where $Z^{n,k,x,y} \equiv \Phi(x, y, (\Delta X^{n,k,x,y}))$, and (for fixed n, x, y) the random variables $Z^{n,k,x,y}$ increase with k. Such choices can be made simultaneously for all (x, y) in[4] $\mathbb{D}_k \times \mathbb{D}_k$. These strategies are in some sense good if we start with portfolio (x, y) at time n. We extend these good portfolio choices to all (x', y') by setting

$$(\Delta X^{n,k,x',y'}) = (\Delta X^{n,k,x,y}) \quad \text{if } x \leq x' < x + 2^{-k}, \ y \leq y' < y + 2^{-k},$$

where $(x, y) \in \mathbb{D}_k \times \mathbb{D}_k$. Thus if we set

$$v_{n,k}(x', y') \equiv E_n[\Phi_n(x', y', (\Delta X^{n,k,x',y'}))],$$

then it is clear that always (when $x, y \in \mathbb{D}_k$, $x' - x, y' - y \in [0, 2^{-k})$)

$$v_{n,k}(x, y) \leq v_{n,k}(x', y') \leq v_n(x', y') \leq v_n(x + 2^{-k}, y + 2^{-k}), \qquad \text{(A.18)}$$

the first because (from (2.6)) Φ is increasing in its first two arguments, and the second because $v_{n,k}(x', y')$ is the value of *some* strategy starting from (x', y') at time n, which is therefore no more than the supremum $v_n(x', y')$. Hence almost surely for all (x, y) we have

$$v_{n,k}(x, y) \to v_n(x, y) \qquad (k \to \infty).$$

Fixing some $\mathcal{F}_n$-measurable ξ, and considering the policy defined by $\Delta X_{n+1} = \xi$ followed by $(\Delta X^{n+1,k,x+\xi,y-\varphi(\xi)S_n})$, we shall have

$$\begin{aligned} v_n(x, y) &\equiv \operatorname{ess\,sup} E_n[\Phi_n(x, y, (\Delta X))] \\ &\geq E_n[\Phi_{n+1}(x + \xi, y - \varphi(\xi)S_n, (\Delta X^{n+1,k,x+\xi,y-\varphi(\xi)S_n}))] \\ &= E_n[v_{n+1,k}(x + \xi, y - \varphi(\xi)S_n)] \end{aligned}$$

---

[4]We use the notation $\mathbb{D}_k = 2^{-k}\mathbb{Z}$.

and letting $k \uparrow \infty$ gives

$$v_n(x, y) \geq E_n[v_{n+1}(x + \xi, y - \varphi(\xi)S_n)]$$

by Fatou's Lemma. Now take the essential supremum over ξ to deduce that

$$v_n(x, y) \geq \text{ess sup }_\xi E_n[v_{n+1}(x + \xi, y - \varphi(\xi)S_n)].$$

For the other inequality, for any (ΔX), any x, y, we have (writing $\tilde{y} = y - \varphi(\Delta X_n)S_{n-1}$, $\tilde{x} = x + \Delta X_n$)

$$E_n[\Phi_n(x, y, (\Delta X))] = E_n\left[E_{n+1}[U(\tilde{y} - \varphi(-\tilde{x} - \sum_{n+2}^{N} \Delta X_j)S_N - \sum_{n+2}^{N} \varphi(\Delta X_j)S_{j-1})] \right]$$

$$\leq E_n[v_{n+1}(\tilde{x}, \tilde{y})]$$

so taking the essential supremum over ΔX_n on the right-hand side, then the essential supremum over (ΔX) on the left-hand side gives the reverse inequality.

$\square$

PROOF OF THEOREM 4.1. If we consider a particular portfolio (x, y) at time n, and let $(X_m, Y_m)_{n \leq m \leq N}$ denote the optimal portfolio process from that time on, we have for any $\varepsilon > 0$ that

$$v_n(x+\varepsilon, y) \geq v_n(x, y) + E_n\left[U(Y_{N+1} + \{-\varphi(-X_N - \varepsilon) + \varphi(-X_N)\}S_N) - U(Y_{N+1}) \right],$$
(A.19)

by comparing the optimal outcome from $(x + \varepsilon, y)$ with what would happen if we simply held the ε units of stock until the end, while following the optimal policy from (x, y) with the rest of the portfolio.

Rearranging (A.19) gives

$$\frac{v_n(x + \varepsilon, y) - v_n(x, y)}{\varepsilon} \geq E_n\left[R_1(\varepsilon)R_2(\varepsilon) \right]$$
(A.20)

where the two positive random variables $R_1(\varepsilon)$ and $R_2(\varepsilon)$ are defined by

$$R_1(\varepsilon) \equiv S_N \frac{\varphi(-X_N) - \varphi(-X_N - \varepsilon)}{\varepsilon}$$

$$R_2(\varepsilon) \equiv \frac{U(Y_{N+1} + S_N(\varphi(-X_N) - \varphi(-X_N - \varepsilon))) - U(Y_{N+1})}{S_N(\varphi(-X_N) - \varphi(-X_N - \varepsilon))}$$

Notice that as $\varepsilon \downarrow 0$ we have $R_1(\varepsilon)$ increases (because of convexity of φ) and $R_2(\varepsilon)$ increases (because of concavity of U), leading to the conclusion

$$D_{x+}v_n(x, y) \geq E_n\left[S_N\varphi'(-X_N)U'(Y_{N+1}) \right]$$
(A.21)

where D_{x+} denotes the right derivative with respect to x. If we now consider what happens if we perturb (x,y) to $(x-\varepsilon,y)$ we obtain similarly

$$\frac{v_n(x-\varepsilon,y)-v_n(x,y)}{\varepsilon} \geq E_n\Big[\, R_3(\varepsilon)R_4(\varepsilon)\,\Big] \qquad\text{(A.22)}$$

where the two positive random variables $R_3(\varepsilon)$ and $R_4(\varepsilon)$ are defined by

$$\begin{aligned}
R_3(\varepsilon) &\equiv\; S_N\frac{\varphi(-X_N+\varepsilon)-\varphi(-X_N)}{\varepsilon}\\
R_4(\varepsilon) &\equiv\; \frac{U(Y_{N+1})-U(Y_{N+1}-S_N(\varphi(-X_N+\varepsilon)-\varphi(-X_N)))}{S_N(\varphi(-X_N+\varepsilon)-\varphi(-X_N))}
\end{aligned}$$

As before, these converge monotonically downwards as $\varepsilon\downarrow 0$, by the convexity of φ and concavity of U, leading to the conclusion that

$$D_{x-}v_n(x,y) \leq E_n\Big[\, S_N\varphi'(-X_N)U'(Y_{N+1})\,\Big] \qquad\text{(A.23)}$$

However, the concavity of v_n guarantees that $D_{x-}v_n(x,y) \geq D_{x+}v_n(x,y)$, and together with (A.21) and (A.23) all that can happen is that

$$D_{x+}v_n(x,y) = D_{x-}v_n(x,y) = D_xv_n(x,y) = E_n\Big[\, S_N\varphi'(-X_N)U'(Y_{N+1})\,\Big] \quad\text{(A.24)}$$

A similar but simpler argument gives us

$$D_{y+}v_n(x,y) = D_{y-}v_n(x,y) = D_yv_n(x,y) = E_n\Big[\, U'(Y_{N+1})\,\Big] \qquad\text{(A.25)}$$

Observe that in view of Assumption A we have for any $(x,y)\in\mathcal{D}^o$ that $U'(Y_{N+1})\in L^1$, $S_N\varphi'(-X_N)U'(Y_{N+1})\in L^1$.

However, we have not finished with these perturbations ideas yet. We analysed $D_xv_n(x,y)$ by considering perturbing x to $x+\varepsilon$, and then using the suboptimal policy of holding the additional ε units of stock until the end. Alternatively, we could consider the suboptimal policy of immediately converting the extra ε units of stock into cash, and we would obtain (as in (A.19))

$$v_n(x+\varepsilon,y) \geq v_n(x,y)+E_n\Big[\, U(Y_{N+1}-\{\varphi(\Delta X_{n+1}-\varepsilon)-\varphi(\Delta X_{n+1})\}S_n)-U(Y_{N+1})\,\Big],$$
$$\text{(A.26)}$$

Carrying out the analogues of steps (A.20) to (A.24) leads us to the similar but different conclusion that

$$D_xv_n(x,y) = S_n\varphi'(\Delta X_{n+1})E_n[\,U'(Y_{N+1})\,], \qquad\text{(A.27)}$$

which combines with (A.24) to give

$$E_n\Big[\, U'(Y_{N+1})\{S_n\varphi'(\Delta X_{n+1})-S_N\varphi'(-X_N)\}\,\Big] = 0,$$

in other words,

the process $M_n \equiv \varphi'(\Delta X_{n+1}) S_n$ is a martingale with respect to Q,   (A.28)

where Q is the probability measure equivalent to P, with density

$$\frac{dQ}{dP} \propto U'(Y_{N+1})$$

$\square$

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