Finance Stoch manuscript No.
(will be inserted by the editor)

Trading to stops

Nora Imkeller · **L. C. G. Rogers**

Received: date / Accepted: date

Abstract The use of trading stops is a common practice in financial markets for a variety of reasons: it reduces the frequency of trading and thereby transaction costs; it provides a simple way to control losses on a given trade, while also ensuring that profit-taking is not deferred indefinitely; and it allows opportunities to consider re-allocating resources to other investments. In this study, we try to explain why the use of stops may be desirable, by proposing a simple objective to be optimized. We investigate a number of possible rules for the placing and use of stops, either fixed or moving, with fixed costs, showing how to identify optimal levels at which to set stops, and compare the performance of different rules and strategies.

Keywords Barriers · Trailing stop · Transaction costs · Stopping time · Laplace transform

Mathematics Subject Classification (2010) 60G40 · 91B60 · 91G10

JEL Classification G11

1 Introduction.

When an investor acquires fund shares, it is common to set stops at which he will come out of the position; for example, he may decide to come out of the position when the value has either risen by 0.1 or fallen by 0.03. Such a fixed-stop trading rule is the simplest to describe, but there are other possibilities, where perhaps the lower stop rises as the value of the position rises, thereby locking in any gain, while

Nora Imkeller (✉)
Fraunhofer ITWM, Fraunhofer-Platz 1, D-67663 Kaiserslautern, Germany
E-mail: nora.imkeller@itwm.fraunhofer.de

L. C. G. Rogers
University of Cambridge, Statistical Laboratory, Wilberforce Road, Cambridge, CB3 0WB, UK
E-mail: L.C.G.Rogers@statslab.cam.ac.uk

allowing the position to continue to rise in value. The latter kind of trading rule is called a trailing stop and it is commonly used in practice.

In this paper, we shall study some simple explicit instances of trading to stops, and try to answer four questions: Is it a good idea to trade to stops in some way? Is it sufficient to consider simple stopping rules? Given that we intend to trade to stops in some way, how would we go about placing them? And when we have reached one stop, how should we act then? To answer the third of these questions, we shall propose a simple objective which must be maximized over the parameters defining the stopping rule. The answer to the first question is more subtle. If we (just for now) restrict the discussion to rules which trade to fixed stops, what we find is that in most instances the best thing to do is to put the lower stop at $-\infty$, which is counter-intuitive. It is counter-intuitive, because one of the reasons to use stops is to prevent the trade running up huge losses, and yet it seems from the theory that this is exactly what we should be doing. However, the theoretical predictions are based on very precise assumptions about the dynamics of the fund; if we relax these strong assumptions, we find a different picture emerging. Specifically, we shall assume that the value of the position evolves as a Brownian motion with constant drift and constant volatility; the volatility will always be assumed to be known, but we will relax the assumption that the drift is known with certainty to the more realistic assumption that we have some (finite atomic) prior over the possible values of the drift. Given this, we find that there is good reason to place stops, either fixed or moving, as a means to protect against model uncertainty, and we compare various different ways of placing the stops. As a stop gives the opportunity of reallocating the investment capital to a different fund or to stick with the original one, the outcome of the trade may help to decide which action should be performed which gives an answer to the fourth question. To find out whether simple rules are sufficient, the results will be compared to an optimal stopping problem. There we see that we can get quite close to the optimal value by using a very simple stopping rule with fixed stops and a time dependent slope.

2 Model set-up.

We shall suppose that a wealthy individual invests into a fund at time 0, requiring the commitment of unit capital. The trading gains of this position at time t is $X_t = \sigma W_t + \mu t$, where W is a standard Brownian motion[1]. Now this gain is not realized until the position is closed out, at some stopping time $T = T_1$, when the investor will be able to book a gain equal to X_T, which may be negative. We shall suppose that when the position is closed, a constant cost equal to c will be paid. Having closed out the position, we will suppose that the investor repeats the process, once again investing unit capital in the position, and using the same stopping rule applied to the rebased process $(X(T_1+t) - X(T_1))_{t\geq 0}$. Thus the stopping times T_n (which are the times at which the position gets closed and immediately re-opened) form a renewal

[1] It might be considered more natural to use geometric Brownian motion to model the gain process, taking a CRRA utility to express the investor's preferences. However, it turns out that in the case of fixed stops the optimization problem results in an uninteresting solution; the optimal placing of the upper stop is either at infinity, or at the starting value.

process. The time-0 value of this repeated trading activity will be

$$\varphi \equiv E\big[\sum_{n\geq 0} e^{-\rho T_{n+1}}\, U(X(T_{n+1})-X(T_n)-c)\big] \tag{2.1}$$

where ρ is the (constant) rate of discounting , and the utility U is some smooth concave strictly increasing function. If we cared only about the net present value of all the gains from trade over time, we would take $\rho = r$, the riskless rate of interest, and $U(x) = x$, and take expectations with respect to the pricing measure. However, this is not the only possible case of interest. Indeed, we shall see that we must allow strict concavity of U to explain why an investor would wish to place stops; when it comes to studying this, we shall always take the exponential utility function

$$U(x) = 1-\exp(-\gamma x) \tag{2.2}$$

for some $\gamma > 0$, the coefficient of absolute risk aversion. The (risk-neutral) case of linear U is regarded as a limiting case, using the limit as $\gamma \downarrow 0$ of $\gamma^{-1}(1-e^{-\gamma x})$

The following simple result reduces the calculation of φ to two simpler calculations.

Proposition 2.1 *The value φ of the trading strategy is*

$$\varphi = \frac{E[e^{-\rho T}\, U(X_T-c)]}{1-E[e^{-\rho T}]} \tag{2.3}$$

where $T \equiv T_1$.

PROOF. By the strong Markov property, by decomposing the objective (2.1) at the first time $T = T_1$ that the position gets closed out we see that

$$\begin{aligned}
\varphi &= E[e^{-\rho T} U(X_T-c)] + E\big[\sum_{n\geq 1} e^{-\rho T_{n+1}} U(X(T_{n+1})-X(T_n)-c)\big] \\
&= E[e^{-\rho T} U(X_T-c)] + E\big[e^{-\rho T} E[\sum_{n\geq 1} e^{-\rho (T_{n+1}-T)} U(X(T_{n+1})-X(T_n)-c)|\mathscr{F}_T]\big] \\
&= E[e^{-\rho T} U(X_T-c)] + E[e^{-\rho T}]\varphi.
\end{aligned}$$

Rearrangement gives the result (2.3).

□

To set the stage, we now offer a few natural examples which we will study in more detail later.

Example 1: fixed stops. This is the easiest example of all. We take $a > 0$, $b > 0$ and set

$$T \equiv \inf\{t : X_t = -a \text{ or } X_t = b\}. \tag{2.4}$$

Example 2: trailing stop. Fix $a > 0$ and let $\bar{X}_t \equiv \sup_{0\leq s\leq t} X_s$. Then the trailing stop is defined by the stopping time

$$T \equiv \inf\{t : \bar{X}_t - X_t = a\}. \tag{2.5}$$

Example 3: trailing stop and fixed stop. This time we fix $a > 0$ and $b > 0$, and set

$$T \equiv \inf\{t : \bar{X}_t - X_t = a \text{ or } X_t = b\}, \tag{2.6}$$

which gives the trailing stop of Example 2 but with a take-profit stop at $b > 0$.

Example 4: converging stops. Fix $a > 0$ and $\varepsilon > 0$. Then we use the stopping time

$$T \equiv \inf\{t : (1+\varepsilon)\bar{X}_t - X_t = a\}. \tag{2.7}$$

In this situation, it is easy to see that the trade stops out before X first hits a/ε; it has similarities to Example 3, and in the special case $\varepsilon = 0$ we recover Example 2.

Since our main interest is in the case of CARA utility U (2.2), we see that the value of the problem can be expressed as

$$\begin{aligned} \varphi &= \frac{E[e^{-\rho T}] - e^{\gamma c} E[e^{-\rho T - \gamma X_T}]}{1 - E[e^{-\rho T}]} \\ &= \frac{L(\rho,0) - e^{\gamma c} L(\rho,\gamma)}{1 - L(\rho,0)} \end{aligned} \tag{2.8}$$

where for arbitrary $\rho,\ \gamma \geq 0$

$$L(\rho,\gamma) \equiv E[e^{-\rho T - \gamma X_T}] \tag{2.9}$$

is the joint Laplace transform of the time and place of stopping. Thus the first objective is to identify the joint Laplace transform L as explicitly as possible in each of the examples under investigation. As we shall see, this is not the end of the story, merely the start.

3 Analysis of the examples.

In this Section, we shall analyze the examples presented in Section 2 and derive explicit solutions for the joint Laplace transform L in each case. The first example is solved using differential equations techniques, which we can think of as an application of Itô calculus. Similar techniques may also be used to solve the other examples, but as the state variable is no longer one-dimensional, the construction of the correct functions is not as simple or transparent. For this reason, we prefer to derive the answers using Itô excursion theory, introduced by Itô in [4]; see [6] or [7] for accessible accounts.

3.1 Example 1: fixed stops.

We write

$$\mathscr{L} \equiv \tfrac{1}{2}\sigma^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu \frac{\mathrm{d}}{\mathrm{d}x} - \rho \tag{3.1}$$

for the generator of the diffusion X with killing rate ρ. If $f : \mathbb{R} \mapsto \mathbb{R}$ is C^2 and satisfies $\mathscr{L} f = 0$, then by an application of Itô's formula we have that

$$M_t \equiv e^{-\rho t} f(X_t) \text{ is a local martingale}$$

which is bounded on the interval $[0,T]$, and therefore[2] $(M(t \wedge T))_{t \geq 0}$ is a martingale. By the Optional Sampling Theorem, it follows[3] that

$$f(0) = E^0[e^{-\rho T} f(X_T)] \tag{3.2}$$

so in order to compute the numerator and denominator in (2.3) it is enough to solve the ODE $\mathscr{L} f = 0$ in $[-a,b]$ with the appropriate boundary conditions.

If we let $-\alpha < 0 < \beta$ be the roots of the quadratic

$$\tfrac{1}{2}\sigma^2 z^2 + \mu z - \rho = 0, \tag{3.3}$$

then the solution to the ODE

$$\mathscr{L} f = 0, \quad f(-a) = A, \quad f(b) = B$$

is

$$f(x) = \frac{(Ae^{\beta b} - Be^{-\beta a})e^{-\alpha x} + (Be^{\alpha a} - Ae^{-\alpha b})e^{\beta x}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}.$$

Evaluating at $x = 0$ simplifies to

$$f(0) = \frac{A(e^{\beta b} - e^{-\alpha b}) + B(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}. \tag{3.4}$$

If we now take $A = \exp(\gamma a)$ and $B = \exp(-\gamma b)$ we read off the joint Laplace transform L_1 for this first example:

$$L_1(\rho, \gamma) = \frac{e^{\gamma a}(e^{\beta b} - e^{-\alpha b}) + e^{-\gamma b}(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}. \tag{3.5}$$

Substituting the form of L_1 into the expression (2.8) gives the value φ for this stopping rule. The dependence of the right-hand side on ρ is of course through the dependence of α, β on ρ as solutions to (3.3). The mean of the hitting time can be derived from the Laplace transform as

$$\begin{aligned} E[T] &= -\frac{\partial L_1}{\partial \rho}(0,0) \\ &= \frac{b(e^{ka} - 1) - a(1 - e^{-kb})}{\mu(e^{ka} - e^{-kb})} \end{aligned} \tag{3.6}$$

after some calculations, where $k \equiv 2\mu/\sigma^2$.

2 Here, of course, T is given by (2.4).

3 The notation E^x denotes expectation under the initial condition $X_0 = x$.

3.2 Example 3: trailing stop and fixed stop.

We deal with this example first, and read off the solution to Example 2 as the special case $b = \infty$. Recall that we take the stopping time

$$T \equiv \inf\{t : \bar{X}_t - X_t = a \text{ or } X_t = b\}, \tag{3.7}$$

where $\bar{X}_t \equiv \sup_{0 \le s \le t} X_s$. The process $Y \equiv X - \bar{X}$ is a continuous strong Markov process with values in $\mathscr{X} \equiv (-\infty, 0]$, and 0 is a recurrent point for this process. The Itô theory of excursions [4] applies to this process, and we will make use of it. Let U denote the space of all excursions of Y away from 0, that is, continuous functions $f : \mathbb{R}^+ \to \mathscr{X}$ with the property that for some $\zeta = \zeta(f) \in (0, \infty]$, the *lifetime* of the excursion, the set $f^{-1}((-\infty, 0))$ is of the form $(0, \zeta)$. Regarding U as a subset of $C(\mathbb{R}^+, \mathbb{R})$ induces the subset topology on U, and in fact U is a Polish space; see, for example, [7] for definitions and basic properties. The process $\bar{X}$ is a continuous homogeneous additive functional of Y, growing only when $Y = 0$, and acts as the local time at zero for Y. The open set $Y^{-1}((-\infty, 0))$ is the disjoint union of countably many excursion intervals I_j, and the point process $\Pi \equiv \{(L_j, \xi^j) : j \in \mathbb{Z}\}$ is a Poisson point process in $(0, \infty) \times U$, where

$$L_j = \bar{X}(I_j),$$
$$\xi^j = Y|_{I_j}\,.$$

The mean measure of Π is Leb $\times\, n$, where n is the σ-finite *excursion measure*: see Itô [4]. The key to effective use of Itô excursion theory is an explicit characterization of the excursion measure n. Once the excursion has escaped from 0, it evolves like the diffusion $X - \bar{X}$ until it first hits zero, and it leaves 0 according to an entrance law.

We shall use excursion theory to calculate for any $\gamma \ge 0$ the expectation

$$L(\rho, \gamma) \equiv E[\,\exp(-\rho T - \gamma X_T)\,]; \tag{3.8}$$

evidently, once we have this, we can obtain the numerator and denominator in (2.3) by suitable substitutions and combinations. As explained in [6], we deal with expectations such as (3.8) by introducing an independent $\exp(\rho)$ time τ, and writing

$$E[\,\exp(-\rho T - \gamma X_T)\,] = E[\,e^{-\gamma X_T} : T < \tau]. \tag{3.9}$$

The way this is handled by excursion theory is to think of τ as being the first event time τ_1 in a Poisson process on $\mathbb{R}^+$ of intensity ρ, with event times $\tau_1 < \tau_2 < \ldots$. This Poisson process of times can be dealt with by *marking* the excursions of Y, each independently of all others, according to a Poisson process of intensity ρ. The excursion point process Π gets modified to the marked excursion point process $\tilde{\Pi}$, where each excursion ξ^j gets augmented to $\tilde{\xi}^j \equiv (\xi^j, N^j)$, where N^j is an increasing $\mathbb{Z}^+$-valued path, representing the path of the marking process restricted to the excursion ξ^j. We observe the marked excursion process $\tilde{\Pi}$ until *either* local time $\bar{X}$ reaches b; *or* we see an excursion which gets to $-a$ before any mark; *or* we see an excursion which gets marked before it reaches $\{0, -a\}$. To set some notation, let

$$A \equiv \{\text{excursions which are marked before reaching 0 or } -a\}; \tag{3.10}$$

$$B \equiv \{\text{excursions which get to } -a \text{ with no mark before reaching } -a\,\}. \tag{3.11}$$

We shall calculate $n(A)$ and $n(B)$ quite simply, but for this we need to characterize the excursion measure effectively. Let $-\alpha < 0 < \beta$ be the roots of the quadratic $\frac{1}{2}\sigma^2 t^2 + \mu t - \rho$; then routine calculations lead to the conclusion that for any $-a < x < 0$

$$E^x[1 - e^{-\rho H_0 \wedge H_{-a}}] = \frac{1 - e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}}(1 - e^{-\alpha x}) + \frac{e^{\alpha a} - 1}{e^{\alpha a} - e^{-\beta a}}(1 - e^{\beta x}) \tag{3.12}$$

$$E^x[e^{-\rho H_{-a}} : H_{-a} < H_0] = \frac{e^{-\alpha x} - e^{\beta x}}{e^{\alpha a} - e^{-\beta a}} \tag{3.13}$$

where $H_z \equiv \inf\{t : X_t = z\}$ is the hitting time of z. Since the measure of excursions which reach $-\varepsilon$ is asymptotic to ε^{-1} as $\varepsilon \downarrow 0$ (see Williams' decomposition of the Brownian excursion law [9], II.67), we conclude that

$$n(A) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^{-\varepsilon}[1 - e^{-\rho H_0 \wedge H_{-a}}] = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a} - (\alpha + \beta)}{e^{\alpha a} - e^{-\beta a}}, \tag{3.14}$$

$$n(B) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^{-\varepsilon}[e^{-\rho H_{-a}} : H_{-a} < H_0] = \frac{\alpha + \beta}{e^{\alpha a} - e^{-\beta a}}. \tag{3.15}$$

The first excursion in $A \cup B$ comes at local time rate

$$\nu \equiv n(A \cup B) = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}}. \tag{3.16}$$

We shall stop the point process either at the first time we see an excursion in $A \cup B$, or when local time reaches b, whichever comes sooner.

Now we come back to the expectation (3.9), and consider how the event $T < \tau$ could happen: this could *either* be because $\bar{X}$ reaches b before the first excursion in $A \cup B$; *or* because the first excursion in $A \cup B$ happens before $\bar{X}$ reaches b, and is in fact an excursion in B. By simple properties of Poisson processes, we discover after a little thought that

$$\begin{aligned} L_3(\rho, \gamma) &\equiv E[\exp(-\rho T - \gamma X_T)] \\ &= E[e^{-\gamma X_T} : T < \tau] \\ &= e^{-\nu b - \gamma b} + \int_0^b \nu e^{-\nu y} \frac{n(B)}{\nu} e^{-\gamma(y-a)}\, \mathrm{d}y \\ &= e^{-(\nu+\gamma)b} + \frac{n(B) e^{\gamma a}}{\nu + \gamma}(1 - e^{-(\nu+\gamma)b}). \end{aligned} \tag{3.17}$$

As before, the mean of T can be computed by differentiating the Laplace transform with respect to ρ at zero. We find[4] that

$$E[T] = \frac{\sigma^2}{2\mu^2}(e^{ka} - 1 - ka)(1 - e^{-mb}) \tag{3.18}$$

where $k = 2\mu/\sigma^2$ as before, and $m = k/(e^{ka} - 1)$.

[4] The calculations were carried out by a symbolic mathematics package, and by traditional methods.

3.3 Example 2: trailing stop.

When $b=\infty$, the results of subsection 3.2 reduce to simpler expressions

$$L_2(\rho,\gamma)=\frac{n(B)e^{\gamma a}}{\nu+\gamma}, \qquad E[T]=\frac{\sigma^2}{2\mu^2}(e^{ka}-1-ka). \tag{3.19}$$

The first of these agrees with the result of Taylor [8], equation (1.1) and can easily be obtained from a result of Glynn & Iglehart [3]. Lehoczky determined this quantity in [5], equation 4, for the larger class of time homogeneous processes.

3.4 Example 4: converging stops.

In this example, the stopping time is given by (2.7):

$$T\equiv \inf\{t:(1+\varepsilon)\bar{X}_t-X_t=a\}.$$

The analysis of this example is quite similar to Example 3, except that the excursion measure of the excursions which stop the process now depends on how much local time has elapsed. When local time $\bar{X}$ has reached ℓ, then any excursion which either contains a mark, or reaches $-a+\varepsilon\ell$ will stop the Poisson point process. Exactly as at (3.14), (3.15), the intensity of excursions which are marked before reaching $-a+\varepsilon\ell$ or zero is

$$n_A(\ell)\equiv\frac{\beta e^{\alpha(a-\varepsilon\ell)}+\alpha e^{-\beta(a-\varepsilon\ell)}-(\alpha+\beta)}{e^{\alpha(a-\varepsilon\ell)}-e^{-\beta(a-\varepsilon\ell)}}\,, \tag{3.20}$$

and the intensity of excursions which get to $-a+\varepsilon\ell$ before getting marked is

$$n_B(\ell)\equiv\frac{\alpha+\beta}{e^{\alpha(a-\varepsilon\ell)}-e^{-\beta(a-\varepsilon\ell)}}\,. \tag{3.21}$$

So in total, the intensity of excursions which stop the Poisson point process is

$$n_{A\cup B}(\ell)=\frac{\beta e^{\alpha(a-\varepsilon\ell)}+\alpha e^{-\beta(a-\varepsilon\ell)}}{e^{\alpha(a-\varepsilon\ell)}-e^{-\beta(a-\varepsilon\ell)}}\,. \tag{3.22}$$

We can now calculate

$$\begin{aligned}\bar{F}(t)&\equiv P(\bar{X}\text{ reaches }t\text{ before the stopping excursion})\\ &=\exp\left[-\int_0^t n_{A\cup B}(s)\,\mathrm{d}s\right]\\ &=\exp\left\{-\beta t-\varepsilon^{-1}\log\left(\frac{1-e^{-(\alpha+\beta)a}}{1-e^{-(\alpha+\beta)(a-\varepsilon t)}}\right)\right\}\\ &=e^{-\beta t}\left(\frac{1-e^{-(\alpha+\beta)(a-\varepsilon t)}}{1-e^{-(\alpha+\beta)a}}\right)^{1/\varepsilon},\end{aligned}$$

which we notice is decreasing with t, and vanishes when $t = a/\varepsilon$ as it must. Using this, we deduce after some calculations that

$$
\begin{aligned}
E[e^{-\gamma X_T - \rho T}] &= \int_0^{a/\varepsilon} e^{-\gamma((1+\varepsilon)x-a)} \, n_B(x) \bar{F}(x) \, \mathrm{d}x \\
&= \int_0^{a/\varepsilon} e^{-\gamma((1+\varepsilon)x-a)} \, \frac{\alpha+\beta}{e^{\alpha(a-\varepsilon x)} - e^{-\beta(a-\varepsilon x)}} \, \bar{F}(x) \, \mathrm{d}x \\
&= \frac{1}{\varepsilon} \left(\frac{e^{-(\gamma+\beta)a}}{1-e^{-(\alpha+\beta)a}} \right)^{1/\varepsilon} \int_{\exp\{-(\alpha+\beta)a\}}^{1} (1-t)^{(1-\varepsilon)/\varepsilon} t^{-c} \, \mathrm{d}t \qquad (3.23)
\end{aligned}
$$

where $c = (\gamma+\beta)(1+\varepsilon)/\varepsilon(\alpha+\beta)$. The answer is therefore available in terms of incomplete beta functions.

4 Placing of the stops.

The identification of the joint Laplace transform of T and X_T in each of the previous examples now allows us to evaluate the objective φ (2.8), and by varying the parameters a and b we are able to optimize φ. However, numerical investigation shows that in many cases it is optimal to let $a \to \infty$. If this happens, then there would be no reason to place a lower stop, which is somewhat unexpected. We can analyze this phenomenon quite completely for the case of fixed stops, which we shall now do. The other examples are more complicated, and we have not pursued the analysis of this phenomenon in those instances; numerical investigations show similar behavior. In any case, since we observe that often for fixed stops the best thing is to use *no* lower stop, we are forced to re-assess the modeling assumptions.

Accordingly, we will until further notice restrict attention to the fixed-stops example, Example 1. The joint Laplace transform L_1 of T and X_T has been found (3.5), and so we are able to obtain an explicit expression for the value φ using (2.8). Since we are concerned with the behavior of this as $a \to \infty$ with all other parameters fixed, we shall use the (local) notation $\varphi(a)$, where we have explicitly

$$
\begin{aligned}
\varphi(a) &= \frac{L(\rho,0) - e^{\gamma c} L(\rho,\gamma)}{1-L(\rho,0)} \\
&= -1 + \frac{1 - e^{\gamma c} L(\rho,\gamma)}{1-L(\rho,0)} \\
&= -1 + \frac{e^{\alpha a} - B_1 e^{-\beta a} - (1-B_1) e^{\gamma(a+c)} - B_2 e^{\gamma c}(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a} - B_1 e^{-\beta a} - (1-B_1) - B_3(e^{\alpha a} - e^{-\beta a})} \qquad (4.1)
\end{aligned}
$$

where $B_1 = e^{-(\alpha+\beta)b}$, $B_2 = e^{-(\gamma+\beta)b}$, $B_3 = e^{-\beta b}$, all positive constants less than 1. The large-a behavior of this expression is determined in the following little result.

Proposition 4.1 *Consider the behavior of the objective* (2.8) *in the case of fixed stops* (2.4) *as $a \to \infty$, with b fixed.*

(i) If $\gamma > \alpha$ then

$$\lim_{a\to\infty} \varphi(a) = -\infty \qquad (4.2)$$

(ii) If $\alpha > \gamma$ and $b > c$ then

$$\varphi(a) < \varphi(\infty) \tag{4.3}$$

for all $a > 0$.

PROOF. The proof is given in the Appendix A.

REMARKS. It is easy to understand the content of Proposition 4.1. In the case where $\gamma > \alpha$, it is *not* advantageous to let $a \to \infty$ because although the expectation

$$E[e^{-\rho T} : X_T = -a] \sim e^{-\alpha a} \tag{4.4}$$

is getting exponentially small, the utility when this event happens is getting large negative exponentially, and *at a greater rate*. In contrast, if $\gamma < \alpha$, the exponential decay of the expectation (4.4) beats the growth of the penalty, and the investor can ignore the penalty for stopping at a low negative level. The condition $b > c$ is needed for the proof, but has a natural interpretation; if $b < c$, we are certain to be losing money every time we review our portfolio, so we would never consider entering this trade.

For a reasonable solution, then, it seems that we require $\gamma > \alpha$. However, in typical examples, this can lead to coefficients γ of absolute risk aversion so high that the value φ is always negative, so we would never engage in this trade! The point is that $-\alpha$ solves the quadratic (3.3), and if $\mu > 0$, we will always have $\alpha > 2\mu/\sigma^2$, a lower bound which need not be small. So for a solution with realistic values of γ, and with a rationale for a lower stop at a finite position, it seems that we are forced to consider situations where μ is negative. But if the growth rate of the trade was negative, and we are paying transaction costs, we would certainly never want to enter into it!

The resolution of these seemingly inconsistent requirements is to suppose that *we are not certain of the true value of μ*. If we have some prior distribution over possible μ values which allows positive probability that μ is negative, we will find that even for small values of γ the punishment for stopping at very low levels really hurts, and we will want to use a finite lower stop. On the other hand, if the probability of decently positive values of μ is quite high, we will be emboldened to take part in the trade.

Once we allow that the value of μ may be random, there are several possible stories which one could tell. We might have

(A) Each time we come out of a trade, we go back into the *same* fund;
(B) Each time we come out of a trade, we pick an *independent* fund with the same probabilistic structure and invest in that;
(C) Each time we come out of a trade on the down side, we pick an *independent* fund with the same probabilistic structure and invest in that;
(D) We perform an optimal stopping analysis for the Bayesian learning model. We consider a learning process for the drift. Having observed the data up to some time τ_0, resulting in a prior μ_0, we assume the distribution of the drift to be $N(\mu_0, \sigma^2/\tau_0)$. Then, referring to [1], the estimation process of the drift evolves like

$$\hat{\mu}(t, X_t) = \frac{\tau_0 \mu_0 + X_t}{\tau_0 + t} \tag{4.5}$$

and the gain process is

$$\mathrm{d}X_t = \hat{\mu}(t, X_t)\mathrm{d}t + \sigma \mathrm{d}W_t. \tag{4.6}$$

Each time we reach the stopping region, we invest in an *independent* fund with the same probabilistic structure.

The value function to be optimized depends on which of these stories we choose. If we let E^μ denote expectation when the drift is μ, and let m be the distribution of μ, then the value $\bar{\varphi}$ to be optimized has the forms:

For (A)-(C), every time we reallocate to a new investment, we pick a fund with constant drift according to the distribution m. Then, $\bar{\varphi} = \int \varphi(\mu) m(\mathrm{d}\mu)$ with a different kind of $\varphi(\mu)$.

(A) As in (2.3), we have value $\varphi(\mu)$ given by

$$\varphi(\mu) = E^\mu\left[e^{-\rho T} U(X_T - c)\right] + E^\mu\left[e^{-\rho T}\right]\varphi(\mu),$$

so the overall value this time is given by

$$\bar{\varphi} = \int \frac{E^\mu\left[e^{-\rho T} U(X_T - c)\right]}{1 - E^\mu\left[e^{-\rho T}\right]}\ m(\mathrm{d}\mu); \tag{4.7}$$

(B) This time we have value $\varphi(\mu)$ given by

$$\varphi(\mu) = E^\mu\left[e^{-\rho T} U(X_T - c)\right] + E^\mu\left[e^{-\rho T}\right]\bar{\varphi},$$

so the overall value will be

$$\bar{\varphi} = \frac{\int E^\mu\left[e^{-\rho T} U(X_T - c)\right] m(\mathrm{d}\mu)}{1 - \int E^\mu\left[e^{-\rho T}\right] m(\mathrm{d}\mu)}; \tag{4.8}$$

(C) If we denote by H the event that the position closes out on the high side, then

$$\varphi(\mu) = E^\mu\left[e^{-\rho T} U(X_T - c)\right] + E^\mu\left[e^{-\rho T} : H\right]\varphi(\mu) + E^\mu\left[e^{-\rho T} : H^c\right]\bar{\varphi},$$

so the overall value is

$$1 - \int \frac{E^\mu[e^{-\rho T} : H^c]}{1 - E^\mu[e^{-\rho T} : H]}\, m(\mathrm{d}\mu) \tag{4.9}$$

For the above three stories, the main point is that once we are able to find an explicit expression for the Laplace transforms $L(\rho, \gamma|\mu) = E^\mu[\,\exp(-\rho T - \gamma X_T)\,]$ and $L_H(\rho, \gamma|\mu) = E^\mu\,[\exp(-\rho T - \gamma X_T) : H\,]$ for the different stopping rules, we are able to deduce the value $\bar{\varphi}$ just by doing at most two integrations. $L(\rho, \gamma|\mu)$ was already calculated in section 3 for all stopping rules. For Example 1, $L_H(\rho, \gamma|\mu)$ can be obtained in the same manner by simply changing the boundary conditions giving

$$L_H(\rho, \gamma|\mu) = \frac{(e^{\alpha a} - e^{-\beta a})e^{-\gamma b}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}},$$

where μ is hidden in α and β, respectively. For Example 3, $L_H(\rho, \gamma|\mu)$ is a byproduct of equation (3.17):

$$L_H(\rho, \gamma|\mu) = e^{-(\nu + \gamma)b}.$$

(D) This time, the value must satisfy

$$\bar{\varphi} = \sup_{\tau \geq 0} E\left[e^{-\rho\tau}\left\{U(X_\tau - c) + \bar{\varphi}\right\}\right]. \tag{4.10}$$

We propose to solve this by recursively solving

$$\bar{\varphi}_{n+1} = \sup_{\tau \geq 0} E\left[e^{-\rho\tau}\left\{U(X_\tau - c) + \bar{\varphi}_n\right\}\right]. \tag{4.11}$$

starting from $\bar{\varphi}_0 = 0$. We solve a Crank-Nicolson finite-difference scheme to obtain the answer. The calculations are given in Appendix B.

5 Numerical study.

We shall compare the stopping rules of Section 3 in several examples. In all cases, we shall assume that $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, and $\rho = 0.1$. We have explored various other examples, and the behavior which we report in these two appears to be quite typical. A further comparison we make is with a fixed-revision rule, where the investor chooses $T > 0$ fixed, and then revises his position at multiples of T, regardless of the performance of the fund. The objective is once again given by (2.3), though now of course T is constant. We find the investor's best choice of fixed T and compare the performance of this rule with the various rules determined by stops.

In the first example, we assume that the investor knows $\mu = 0.15$ with certainty. There are four[5] stopping rules to be considered now, and the results are given in table 1. Notice how with fixed stops or with a fixed upper stop and a trailing stop, the best

Table 1 Known μ. Example with $\mu = 0.15$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Objective	τ
Fixed stops	∞	0.0184	4.1553	0.1224
Trailing stop	0.0894		0.5314	0.0983
Trailing stop and fixed stop	∞	0.0184	4.1553	0.1224
Fixed exit time			0.8724	0.3780

choice of a is $a = \infty$; it always pays to push the lower stop all the way down. If this is done, then of course the two stopping rules amount to stopping at b, and so it is no surprise that the values, the optimal choices of b, and the mean time per trade all agree. The value φ as a function of a and b is displayed in Figure 1; for finite a, the pictures for Examples 1 and 3 are in principle different, but in this example they are not visibly different. Notice that the value for a fixed upper and trailing stop is substantially higher than for a trailing stop only; this is of course to be expected, as we have optimized over a larger set, but the magnitude of the improvement is

[5] The converging stops are left out in the following tables because we found that for this stopping rule it is optimal to mimic stopping rule 3 by setting $a = b^*\varepsilon$, where b^* is the optimal upper barrier for Example 3, and letting ε converge to 0.

noteworthy. The trailing stop example, Example 2, is quite different in character, with a much shorter mean time in trade. The corresponding value φ as a function of a is displayed in Figure 2. The fixed revision rule performs very poorly relative to the two-sided stops rules, Examples 1 and 3.

As we explained in Section 4, it is uncertainty in the μ which vindicates trading to stops, and to illustrate this we study the stories where we do not suppose that μ is known. For (A)-(C), we assume that the drift of the fund we pick is a random variable with prior $N(\mu_0, \sigma_\mu^2)$ distribution. We suppose that $\mu_0 = 0.15$ and we take two different values for the standard deviation: firstly, $\sigma_\mu = 0.3$; and secondly the more uncertain case $\sigma_\mu = 0.7$.

(A) The results obtained for the story where we go back into the same fund are reported in the following tables. Table 2 is for $\sigma_\mu = 0.3$ and table 3 is for $\sigma_\mu = 0.7$. For $\sigma_\mu = 0.3$, the calculated values of the three stopping examples are displayed

Table 2 Story (A). Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Objective	τ
Fixed stops	0.2159	0.0470	0.8416	0.0998
Trailing stop	0.0603		0.2397	0.0439
Trailing stop and fixed stop	0.2375	0.0464	0.8398	0.0984
Fixed exit time			0.5627	0.0670

Table 3 Story (A). Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.7$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Objective	τ
Fixed stops	0.0837	0.0559	0.4410	0.0475
Trailing stop	0.0411		-0.4264	0.0203
Trailing stop and fixed stop	0.1069	0.0523	0.4223	0.0452
Fixed exit time			0.0698	0.0290

in Figures 3, 4 and 5, respectively. Due to the risk aversion, the values of all the rules have dropped, particularly the stops trading rules. As with the certain growth rate, the two-stops rules do substantially better than either the trailing stop alone or the fixed time to revision. Mean times in trades have fallen in all cases. As before, there is no appreciable difference between Examples 1 and 3; the trailing stop has very little effect. Increasing the deviation of the drift to $\sigma_\mu = 0.7$ leads to even smaller objectives. In all cases, the parameter a has fallen to protect against huge losses.

(B) The next tables 4 and 5 show the results for $\sigma_\mu = 0.3$ and $\sigma_\mu = 0.7$, respectively, if we pick a new independent fund each time we come out of a trade. The values of the three stopping examples which were calculated with respect to $\sigma_\mu = 0.3$ are displayed in Figures 6, 7 and 8, respectively. Compared to story (A), the values of the stops trading rules have grown. It can be seen that the optimal lower

Table 4 Story (B). Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Objective	τ
Fixed stops	0.0191	0.3409	1.4071	0.0804
Trailing stop	0.0952		1.3354	0.1159
Trailing stop and fixed stop	0.0985	0.6558	1.3431	0.1236
Fixed exit time			0.5596	0.0662

Table 5 Story (B). Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.7$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Objective	τ
Fixed stops	0.0083	0.3426	4.8955	0.0338
Trailing stop	0.0728		4.1224	0.0720
Trailing stop and fixed stop	0.0767	0.7572	4.1695	0.0792
Fixed exit time			0.0698	0.0290

barriers a have fallen to much lower values, while the values for the upper barriers b are much larger. The reason for this is that we do not want to stop a *good* investment having a large positive drift, but we get rid of those investments with negative drifts quickly. The difference in value of Examples 2 and 3 is comparatively small because the gain process in Example 3 will only occasionally get stopped at b; most will be caught by the trailing stop. We see however that the fixed stops Example 1 performs better than Example 3 with a trailing stop, presumably because the trailing stop may prematurely close out a trade which might have turned out to be profitable. Interestingly, when we compare the values for Examples 1 and 3 in story (B) with the values for Examples 1 and 3 in the certain-drift case, we find that for the smaller value $\sigma_\mu = 0.3$ we do better if we know the drift, while for the larger value $\sigma_\mu = 0.7$ we do better if we have uncertainty in the drift. The reason is not hard to discern. For small σ_μ, risk aversion is the dominant effect, but for larger σ_μ we benefit from the wider spread of μ-values; the lower stop closes down the unprofitable trades, but we get more of an upside from the profitable trades.

(C) In story (B) we have seen that the two-stops examples have small a to shut down the unprofitable trades, and large b to let the gains accumulate when we have found a profitable trade. In contrast, when we use story (C), which only changes funds if we come out at the lower stop, the results in the tables 6 and 7 look quite different[6]. For $\sigma_\mu = 0.3$, the values are displayed in Figures 9 and 10, re-

Table 6 Story (C). Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Objective	τ
Fixed stops	0.3620	0.0239	6.8525	0.0370
Trailing stop and fixed stop	0.3735	0.0240	6.8539	0.0371

[6] Of course, it does not make any sense to consider the single stopping rules, trailing stop and fixed exit time, because there we can not distinguish between upper and lower outcomes

Table 7 Story (C). Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.7$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Objective	τ
Fixed stops	0.2020	0.0213	20.216	0.0125
Trailing stop and fixed stop	0.2119	0.0219	20.231	0.0129

spectively. Notice firstly that the values of the objectives are substantially higher, because we are allowed to shop around for good funds, and once we have found one, we are allowed to play that fund until we get stopped out at a lower stop. Because of this, we have to be careful not to stop out a fund at the lower stop unless we are quite confident that it is a poor performer; the loss of profit from killing a good fund too early would be considerable. So this explains why we see larger a values than for story (B). We also see much smaller b values, which we understand as a desire to book profits quickly and avoid discounting them away; if we think we are playing a good fund, we will gladly do this, because we can just return to playing the same good fund immediately, in contrast to the situation of story (B) where we would have to pick a new independent fund and take our chances on its quality.

For the first time, we see Example 3 outperforming Example 1 (but only very slightly). This seems to be because the trailing stop will allow a slightly quicker closing out of bad trades, and since the lower stop is initially quite far from 0, this difference matters.

Another way we could try to capture this advantage would be by adding a time dependent slope to the barriers, and this is examined in section 6. For the same reasons as in story (B), a larger standard deviation $\sigma_\mu = 0.7$ yields a better objective.

(D) The Bayesian story has similarities to story (B); the stochastic nature of the funds is identical, but we allow *any* stopping rule. As was recorded at (4.6), we can model the gain process in the observation filtration as the solution of a stochastic differential equation, and the optimal stopping problem for this is found by solving the recursive scheme (4.11) by Crank-Nicolson. To compare with our results for story (B) using $\sigma_\mu = 0.3$, we propose to take a prior distribution for μ with mean 0.15 as before, and precision $\tau_0 = \sigma^2/\sigma_\mu^2 = 1$. Table 8 compares the results from story (B) with the optimal solution obtained using story (D). Of course, we cannot report any fixed values for the optimal stopping solution as the stopping boundary is a curve, which can be see in Figure 11. The shape of the stopping region can be interpreted as time-dependent decreasing upper stop $\eta(t)$ and an increasing lower stop $\xi(t)$. The upper stop ξ begins at a high level to let *good* investments run and the lower stop η starts at a small negative value to immediately get rid of *bad* investments. As time goes by, the state of the gain process updates the drift estimator (4.5). The threshold $\eta(t)$ will decrease with t; if we decided to stop at time t when the gain value X_t was $y > 0$, then we would certainly want to stop at level y at any later time, because our estimate of μ would then be smaller and we would be more confident in that estimate. A corresponding argument applies to the lower stop $\xi(t)$.

Table 8 Story (D) and story (B). Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\tau_0 = 1$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Objective	τ
Optimal stopping			1.6770	
Fixed stops	0.0191	0.3409	1.4071	0.0804
Trailing stop	0.0952		1.3354	0.1159
Trailing stop and fixed stop	0.0985	0.6558	1.3431	0.1236
Fixed exit time			0.5596	0.0662

6 Time-dependent slope.

In this section, we examine what happens if some time dependent factors are added to the stops. Then, for a parameter q, the modified stopping times are as follows:
Example 1: fixed stops. We take $a > 0$, $b > 0$ and set

$$T \equiv \inf\{t : X_t = -a + qt \text{ or } X_t = b + qt\}.$$

Example 2: trailing stop. Fix $a > 0$ and define $\hat{X}_t = \sup_{0 \le s \le t}\{X_s - qs\}$. Then we use the stopping time

$$T \equiv \inf\{t : X_t = \hat{X}_t - a + qt\}.$$

Example 3: trailing stop and fixed stop. This time we fix $a > 0$ and $b > 0$, and set

$$T \equiv \inf\{t : X_t = \hat{X}_t - a + qt \text{ or } X_t = b + qt\}.$$

Regarding a process $Y_t \equiv X_t - qt$, then for Y, the above stopping rules correspond to the time-independent ones defined in section 2. Thus, for process X and the time-dependent barriers, the joint Laplace transforms can be computed from the Laplace transforms of the previous sections with respect to process Y.

$$\begin{aligned} L_X(\rho,\gamma) &= E[e^{-\rho T - \gamma X_T}] \\ &= E[e^{-\rho T - \gamma(Y_T + qT)}] \\ &= E[e^{-(\rho+\gamma q)T - \gamma Y_T}] \\ &= L_Y(\rho + \gamma q, \gamma) \end{aligned}$$

Having added an additional parameter q which defines the slope of the barriers, we can optimize over a, b and q to obtain the objectives for the stories (A)-(C). As $q = 0$ is a feasible choice, the objectives we will find can not be smaller than those found in section 5. For a better comparison, in the following tables, the optimal objectives for $q = 0$ are given as well. As we have seen that the upper barrier is of importance, we will concentrate on Examples 1 and 3 only. Furthermore, only the case $\sigma_\mu = 0.3$ will be considered. All other parameters are as in section 5.

(A) If we stick with the same fund forever, we will get the results given in table 9. The results are very close to those in section 5 because the optimal parameter q is close to 0. In other words, allowing the barriers to have a time-dependent drift does not yield a substantial improvement.

Table 9 Story (A) with slope q. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Best q	Objective	τ	$q = 0$
Fixed stops	0.2244	0.0447	0.0347	0.8426	0.1005	0.8416
Trailing stop and fixed stop	0.2450	0.0441	0.0346	0.8407	0.0991	0.8398

Table 10 Story (B) with slope q. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Best q	Objective	τ	$q = 0$
Fixed stops	0.0345	0.2944	0.4949	1.5976	0.0782	1.4071
Trailing stop and fixed stop	0.1102	0.5445	0.2789	1.3519	0.1251	1.3431

(B) Choosing a different investment from the marketplace on each side yields table 10. Just as in the case $q = 0$, we still get a large b to let *good* investments run and a small a to quickly stop *bad* investments. The improvement in the objective is much greater for Example 1 than for Example 3.

(C) If we pick an independent fund if we end up on the down side gives the results which are summarized in table 11. In this case there is a 1.7% improvement of the objective due to the slope q. As guessed above, the increasing lower barrier tackles below-average investments. The slope parameter q is considerably larger than 0 but it is not as high as in story (B) which reflects the risk to accidentally stop a *good* investment.

Table 11 Story (C) with slope q. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Best q	Objective	τ	$q = 0$
Fixed stops	0.4181	0.0192	0.1212	6.9719	0.0330	6.8525
Trailing stop and fixed stop	0.4274	0.0192	0.1209	6.9721	0.0330	6.8539

(D) In section 5, we found that the objective for the best fixed-stops rule is quite far from the optimum. However, adding a time-dependent drift yields the situation given in table 12. The result of the optimal stopping problem cannot be improved, so we get the same objective as in section 5. But the time-dependent slope pushes the fixed stop's objective up by 13.5%, bringing the value much closer to the

Table 12 Story (D) and story (B) with slope q. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\tau_0 = 1$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

	Best a	Best b	Best q	Objective	τ	$q = 0$
Optimal stopping				1.6770		
Fixed stops	0.0345	0.2944	0.4949	1.5976	0.0782	1.4071
Trailing stop and fixed stop	0.1102	0.5445	0.2789	1.3519	0.1251	1.3431

optimum, remarkably so given the very simple-minded nature of the stopping rule.

Acknowledgements We are grateful to seminar participants at Imperial College London, and at the Cambridge Finance seminar for comments and suggestions which have improved the paper.

References

1. Bawa, V.S., Brown, S.J., Klein, R.W.: Estimation risk and optimal portfolio choice. North-Holland, Amsterdam and New York (1979)
2. Crank, J., Nicolson, P.: A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. In: Proc. Cambridge Phil. Soc., vol. 43, pp. 50–67 (1947)
3. Glynn, P.W., Iglehart, D.L.: Trading securities using trailing stops. Managm. Sci. **41**(6), 1096–1106 (1995)
4. Itô, K.: Poisson point processes attached to markov processes. In: Proc. Sixth Berkeley Symp. Math. Statist. Probab., vol. 3, pp. 225–240. Univ. California Press, Berkeley (1971)
5. Lehoczky, J.P.: Formulas for stopped diffusion processes with stopping times based on the maximum. Ann. Probab. **5**(4), 601–607 (1977)
6. Rogers, L.C.G.: A guided tour through excursions. Bull. London Math. Soc. **21**(4), 305–341 (1989)
7. Rogers, L.C.G., Williams, D.: Diffusions, Markov processes and martingales. Vol. 2: Itô calculus. Cambridge Univ. Press (2000)
8. Taylor, H.M.: A stopped brownian motion formula. Ann. Probab. **3**(2), 234–246 (1975)
9. Williams, D.: Diffusions, Markov processes and Martingales, vol. 1. Wiley, Chichester (1979)
10. Wilmott, P.: Paul Wilmott on quantitative finance, vol. 2. John Wiley & Sons (2006)

A Proof of Proposition 4.1.

PROOF OF PROPOSITION 4.1. The case $\gamma > \alpha$ is easy; the dominant term in (4.1) is the term $k_1 e^{\gamma(a+c)}$ in the numerator, and this makes it obvious[7] that $\varphi(a) \to -\infty$ as $a \to \infty$.

The second case is more delicate. The limit of $\varphi(a)$ is easily seen to be

$$\varphi(\infty) = -1 + \frac{1 - B_2 e^{\gamma c}}{1 - B_3}.$$

If we now consider $\varphi(\infty) - \varphi(a)$, we find a rational expression whose denominator is positive, and whose numerator is (a positive multiple of)

$$H \equiv (1 - B_3)z - (B_2 e^{\gamma c} - B_3)y - (1 - B_2 e^{\gamma c}), \tag{A.1}$$

where we set $z \equiv e^{\gamma(a+c)}$, $y \equiv e^{-\beta a}$ for brevity. Thus it will be sufficient to prove that the expression H is non-negative.

Since $b > c$, we may write $\varepsilon = b - c > 0$, and then H becomes

$$\begin{aligned} H &= (1 - B_3)z + B_3(1 - e^{-\gamma-\varepsilon})y - (1 - B_3 e^{-\gamma\varepsilon}) \\ &= (1 - B_3)(z - 1) + B_3(1 - e^{-\gamma\varepsilon})y - B_3(1 - e^{-\gamma\varepsilon}) \\ &= (1 - B_3)(z - 1) - B_3(1 - e^{-\gamma\varepsilon})(1 - y). \end{aligned} \tag{A.2}$$

It is clear from the final equation that if we now hold $a > 0$ fixed, and consider H as a function of γ, then H is convex, and vanishes as $\gamma \downarrow 0$. To prove non-negativity of H, we now investigate the gradient of H with respect to γ, which is

$$\begin{aligned} \frac{\partial H}{\partial \gamma} &= (1 - B_3)(a + c)e^{\gamma(a+c)} - \varepsilon B_3(1 - y)e^{-\gamma\varepsilon} \\ &= e^{-\gamma\varepsilon}\left[(1 - B_3)(a + c)e^{\gamma(a+b)} - (1 - y)B_3(b - c)\right]. \end{aligned}$$

As $\gamma \downarrow 0$, we obtain the limit

$$\begin{aligned} \frac{\partial H}{\partial \gamma}(0) &= (1 - B_3)(a + c) - (1 - y)B_3(b - c) \\ &= (1 - e^{-\beta b})(a + c) - e^{-\beta b}(b - c)(1 - e^{-\beta a}) \\ &= e^{-\beta b}\left[\, (a + c)e^{\beta b} + (b - c)e^{-\beta a} - (a + b) \,\right] \\ &= (a + b)e^{-\beta b}\left[\, \frac{a + c}{a + b}\, e^{\beta b} + \frac{b - c}{a + b}\, e^{-\beta a} - 1 \,\right] \\ &\geq (a + b)e^{-\beta b}\left[e^{\beta c} - 1\right] \\ &> 0, \end{aligned}$$

where we have used convexity of the exponential function for the first inequality. Since H is convex, and its derivative at zero is positive, it follows that H is increasing, and therefore is everywhere non-negative, since it is zero at $\gamma = 0$.

□

B Crank-Nicolson finite-difference scheme for story (D).

In the sequel it will be shown how to calculate the value $\bar{\varphi}_{n+1}$ with $\bar{\varphi}_n$ given. The gain process satisfies the diffusion equation given in (4.6). The stopping reward process is of the form

$$\begin{aligned} Z(t, X_t) &= e^{-\rho t}(U(X_t - c) + \bar{\varphi}_n) \\ &\equiv e^{-\rho t} g(t, X_t), \end{aligned}$$

[7] The denominator is asymptotic to $e^{\alpha a}(1 - B_3)$ which is certainly positive.

where $\rho \geq 0$. We fix a final time $\bar{T}$ which should be large enough to be outside the continuation region and define the value function

$$V(t,x) \equiv \sup_{t \leq \tau \leq \bar{T}} E[e^{-\rho(\tau - t)} g(\tau, X_\tau) | X_t = x] \tag{B.1}$$

then we have that $V \geq g$ everywhere, and that

$$\mathscr{L}V + V_t - \rho V \leq 0 \tag{B.2}$$

which holds with equality when $V > g$, where $\mathscr{L}$ is the generator of the diffusion,

$$\mathscr{L} \equiv \tfrac{1}{2}\sigma^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \hat{\mu}(t, X_t)\frac{\mathrm{d}}{\mathrm{d}x}.$$

This problem can be solved numerically by using the Crank-Nicolson finite-difference scheme. See [2] for the original paper or [10] for a more general description. We set down a grid of x-values and a grid $0 = t_0 < t_1 < \ldots < t_N = \bar{T}$ of time values, and let $L^{(n)}$ be a discrete approximation of the diffusion generator at $t = t_n$. If $v^{(n)}$ denotes the approximation of the value function at time t_n, then the Crank-Nicolson method approximates (B.2) by

$$\tfrac{1}{2}\{L^{(n)}v^{(n)} + L^{(n+1)}v^{(n+1)}\} - \tfrac{1}{2}\rho(v^{(n)} + v^{(n+1)}) + \Delta t_n^{-1}(v^{(n+1)} - v^{(n)}) \leq 0 \tag{B.3}$$

with equality where it is optimal to continue. Here, we write $\Delta t_n = t_{n+1} - t_n$. The unknown in this equation is $v^{(n)}$; we start with $v^{(N)}(x) = g(\bar{T}, x)$, since, by assumption, the final time point $\bar{T}$ is outside the continuation region, and we work recursively back through the grid in the usual dynamic-programming fashion. Rewriting (B.3) to make the unknown the subject, we have

$$\begin{aligned}(L^{(n)} - \rho - 2\Delta t_n^{-1})v^{(n)} &\leq -(L^{(n+1)} - \rho + 2\Delta t_n^{-1})v^{(n+1)} \\ &\equiv -\alpha^{(n)} \end{aligned} \tag{B.4}$$

say, with equality at all places where it is optimal to continue, and with $v^{(n)}(x) = g(t_n, x)$ in places where it is optimal to stop.

However, the problem (B.4) is an optimal stopping problem for the Markov chain with generator[8] $L^{(n)}$, with discount rate $\rho + 2\Delta t_n^{-1}$, and running reward $\alpha^{(n)}$. It is quite straightforward (and very fast) to solve this by policy improvement. Probably the simplest thing to do at the boundaries is to insist that the process gets absorbed there, so in the original stopping problem, we have to stop when we reach one end or the other end of the x-grid.

[8] With a three-point finite difference scheme, the matrix $L^{(n)}$ will usually be a Q-matrix; the calculations need to check this.

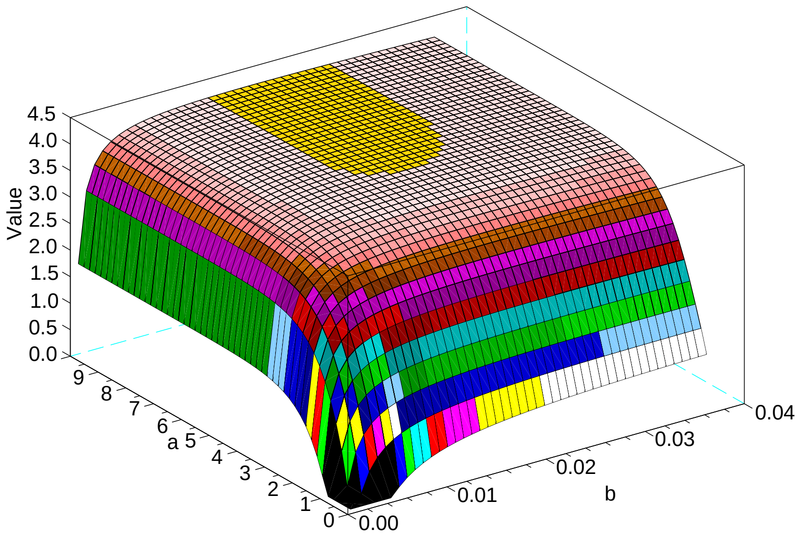

Fig. 1 Known μ. Two stops rules. Example with $\mu = 0.15$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

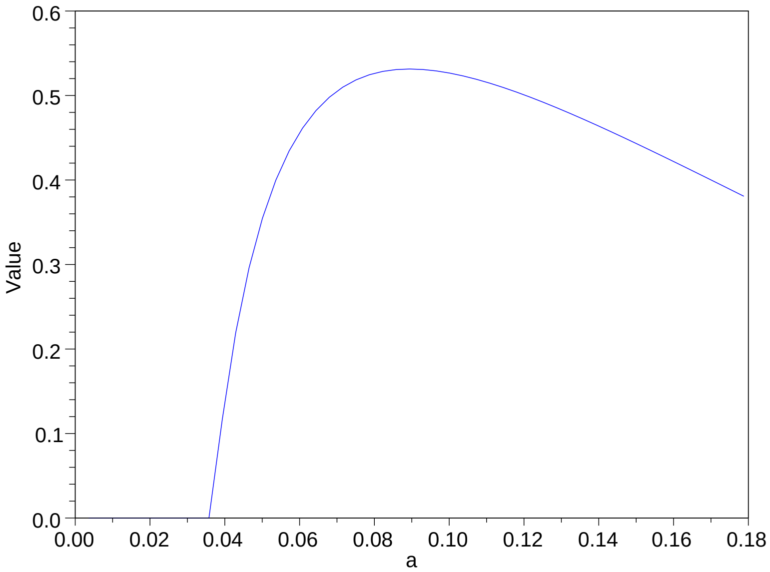

Fig. 2 Known μ. Trailing stop. Example with $\mu = 0.15$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

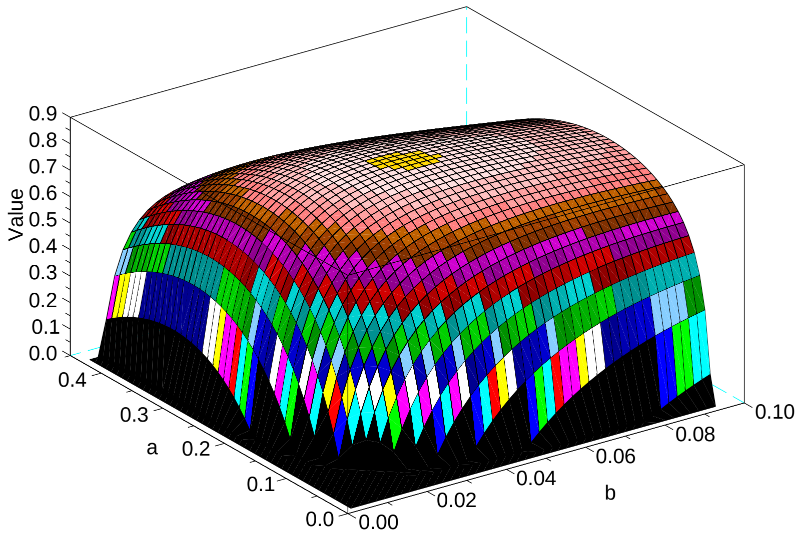

Fig. 3 Story (A). Fixed stops. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

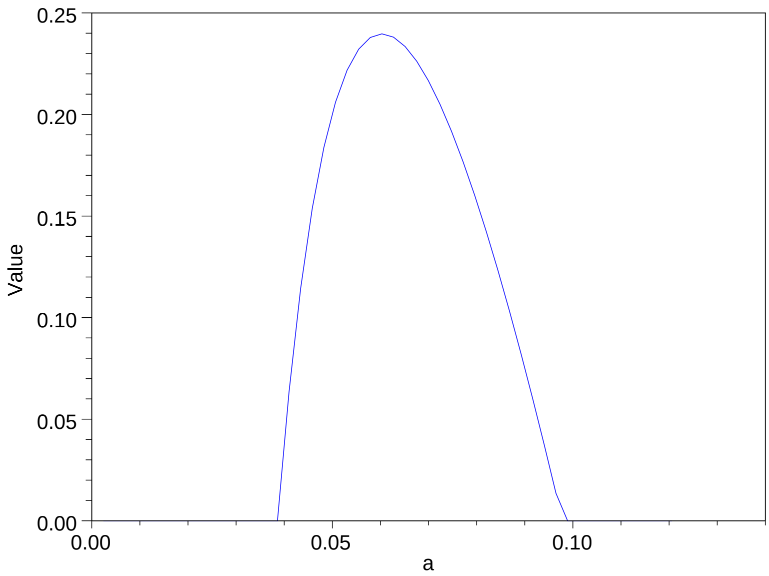

Fig. 4 Story (A). Trailing stop. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

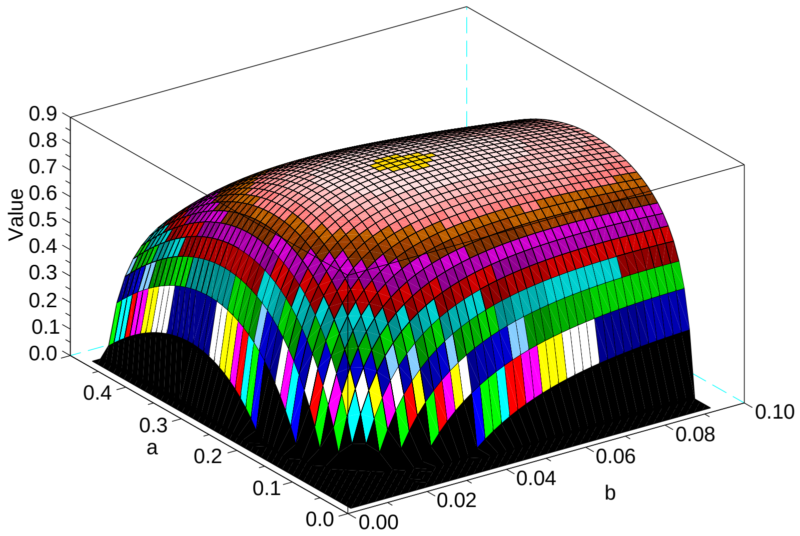

Fig. 5 Story (A). Trailing stop and fixed stop. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

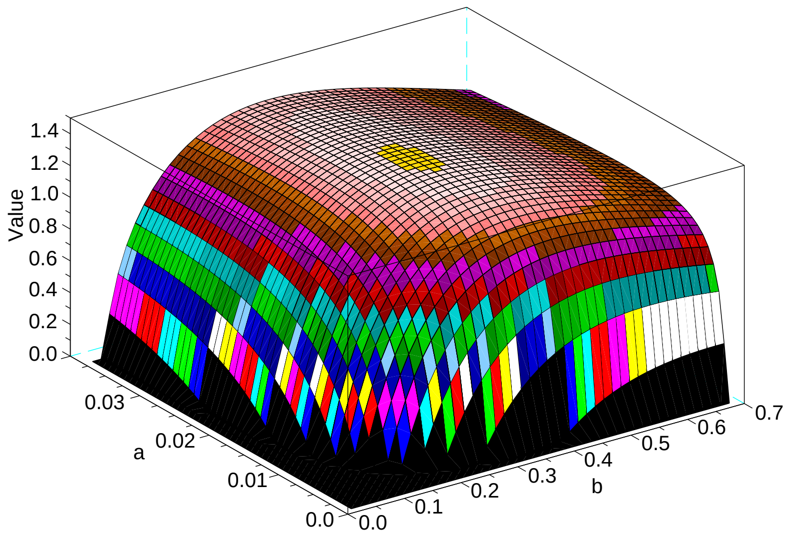

Fig. 6 Story (B). Fixed stops. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

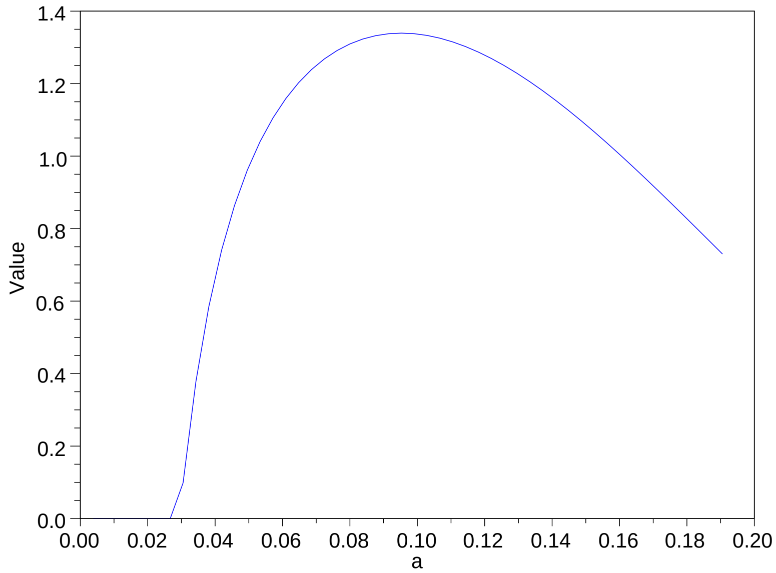

Fig. 7 Story (B). Trailing stop. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

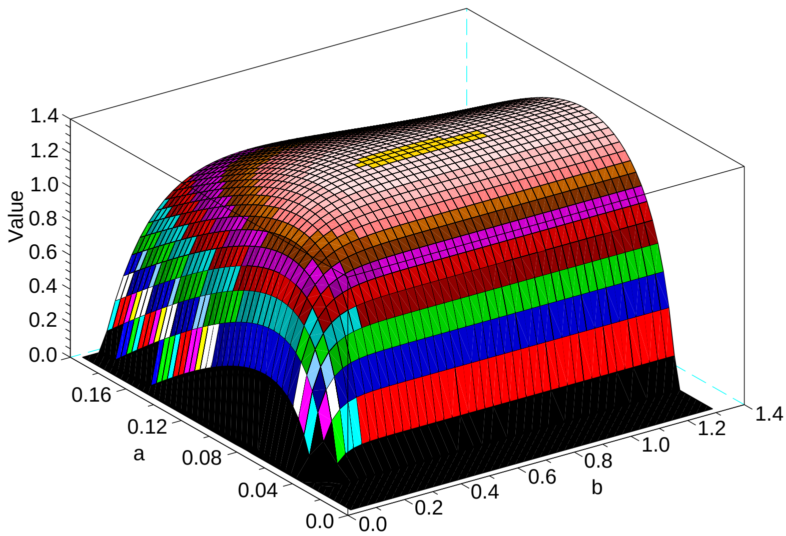

Fig. 8 Story (B). Trailing stop and fixed stop. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

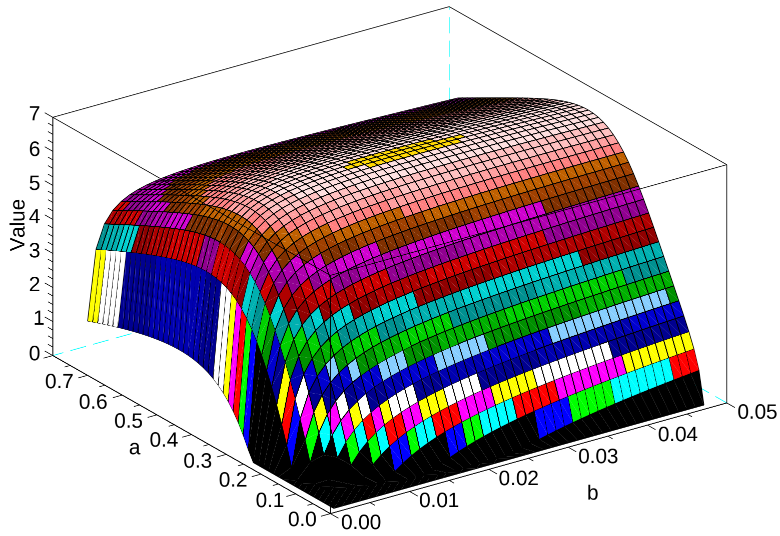

Fig. 9 Story (C). Fixed stops. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

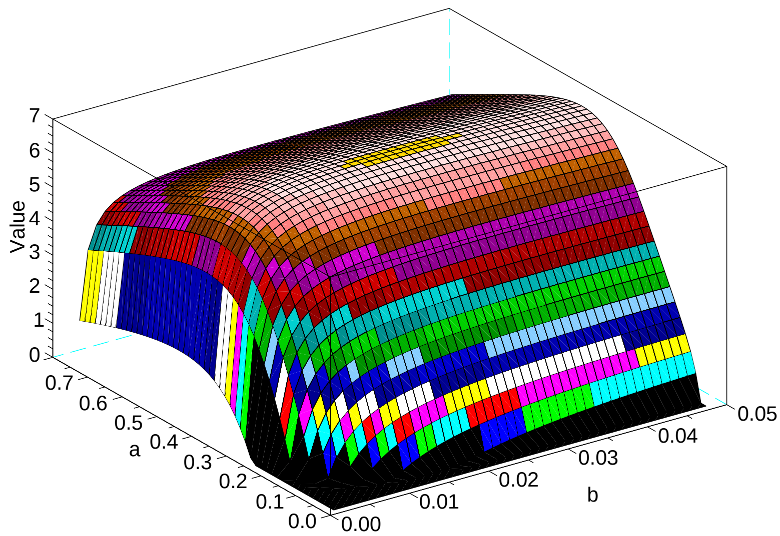

Fig. 10 Story (C). Trailing stop and fixed stop. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$

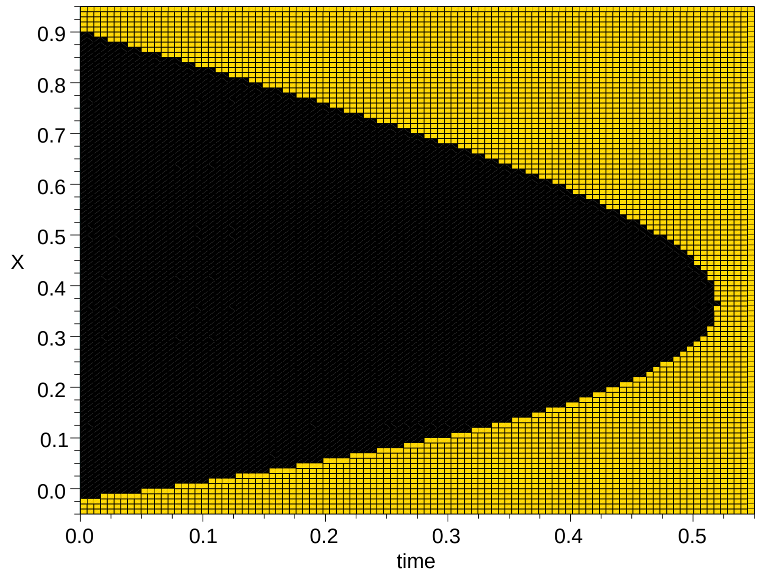

Fig. 11 Story (D). Optimal stopping problem. Example with $\mu_0 = 0.15$, $\tau_0 = 1$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$