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SKEW-PRODUCT DECOMPOSITIONS OF BROWNIAN MOTIONS

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1. INTRODUCTION. Every good probabilist knows examples of skew-product decompositions of Brownian motions on various manifolds, the most celebrated being the skew-product decomposition of Brownian motion in $\mathbb{R}^n$, BM($\mathbb{R}^n$). The aim of this paper is to show that all such decompositions can be considered as examples of a common phenomenon: we study this in a general setting, and obtain further skew-product decompositions of Brownian motions on certain manifolds of matrices, extending the results of Norris, Rogers and Williams [5].

In all these skew-product decompositions, we observe the following features:

(1.i) *the statespace of the diffusion* X *is a* C^∞ *Riemannian manifold* (M,g) *which has the product form*

$$M = R \times \Theta$$

where Θ and R are connected C^∞ manifolds;

(1.ii) for each $\xi \equiv (r,\theta) \in M$, the tangent space $T_\xi M$ is naturally isomorphic to $T_r R \oplus T_\theta \Theta$; *the subspaces* $T_r R$ *and* $T_\theta \Theta$ *of* $T_\xi M$ *are orthogonal with respect to* g.

Any linear operator V on $C^\infty(R)$ (in particular, any vector field, or second order differential operator on R) has a natural extension to $C^\infty(M)$, which will again be denoted by V, defined by

$$Vf(r,\theta) \equiv Vf_\theta(r) \qquad \forall\, f \in C^\infty(M),\ (r,\theta) \in M,$$

where

$$f_\theta(r) \equiv f(r,\theta).$$

Thus for each $r \in R$ there is a Riemannian metric tensor g_r^Θ on Θ defined by

$$g_r^\Theta(U,V)(\theta) \equiv g(U,V)(r,\theta)$$

where U,V are vector fields on Θ.

Evidently the roles of R and Θ in these definitions can be interchanged throughout. If we now assume that

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(1.iii) *There is a Riemannian metric tensor* g^R *on R such that*

$$g^R = g^R_\theta \qquad \forall \theta \in \Theta$$

and also that

(1.iv) *the Riemannian volume element on* (M,g) *is a product measure,*

then there is a skew-product decomposition of Δ, the Laplace-Beltrami operator of (M,g);

$$\Delta f(r,\theta) = \Delta^R f(r,\theta) + \mathrm{V} f(r,\theta) + \Delta_r^\Theta f(r,\theta), \tag{2}$$

where V is (the extension to $C^\infty(M)$ of) some vector field on R, and Δ_r^Θ (respectively, Δ^R) is the Laplace-Beltrami operator of (Θ, g_r^Θ) (respectively, (R, g^R)).

The proof of (2) under assumptions (1.i) - (1.iv) is quite straightforward, and is dealt with in §2. The probabilistic aspects of (2) are worth singling out; if in some coordinate neighbourhood one takes vector fields $\mathrm{V}_0, \ldots, \mathrm{V}_n$ on R and $U_0(r), \ldots, U_m(r)$ on Θ such that

$$\Delta^R = \mathrm{V}_0 + \sum_{j=1}^{n} \mathrm{V}_j^2,$$

$$\Delta_r^\Theta = U_0(r) + \sum_{j=1}^{m} U_j(r)^2,$$

which can always be done, then in this coordinate neighbourhood BM(M) is the solution $(r_t, \theta_t)_{t \ge 0}$ to the SDE

$$\partial r_t = \mathrm{V}_j(r_t)\, \partial B_t^j + \tfrac{1}{2} \mathrm{V}_0(r_t)\, \partial t + \tfrac{1}{2} \mathrm{V}(r_t)\, \partial t$$

(3)

$$\partial \theta_t = U_j(r_t)(\theta_t)\, \partial W_t^j + \tfrac{1}{2} U_0(r_t)(\theta_t)\, \partial t,$$

where ∂ denotes Stratonovich differential, and $B^1, \ldots, B^n, W^1, \ldots, W^m$ are independent BM(ℝ)'s. The interpretation (stated rather loosely) is that

(4.i) *the r-motion is* BM(R) *with drift* V;

(4.ii) *at time* t, *the* θ*-motion is Brownian motion on* $(\Theta, g^\Theta_{r(t)})$ *driven by a white noise independent of r.*

One problem which arises in most examples is that the manifold (R, g^R) may not be complete, and one has to ensure that the r-motion does not 'explode in finite time', which appears in the case of the skew-product decomposition of BM$(\mathbb{R}^n)$ as the possibility that the radial process reaches the degenerate point 0 in finite time. Techniques for handling this problem are discussed in §2.

This would be the end of the matter if we were content to consider no examples. However, the limitations of what we have done become readily apparent if we consider the eigenvalue/eigenvector decomposition of Brownian motion on $U(n)$. The eigenvalues of $U \in U(n)$ all lie on the unit circle, so U can be represented (if its eigenvalues are distinct) as

$$U = \mathrm{V}^* \Lambda \mathrm{V},$$

where $\Lambda \in R \equiv \{n \times n$ diagonal complex matrices with distinct eigenvalues$\}$, and $\mathrm{V} \in U(n)$. This representation is not unique, of course; laying aside the trivial permutations of eigenvalues and eigenvectors, it can only be made unique by making some arbitrary choice of the eigenvectors, since if x is an eigenvector of U, so is ωx, where $|\omega| = 1$. So the product decomposition of $U(n)$ *should* be $U(n) = R \times \Theta$, where

$$\Theta = U(n)/U_0(n), \qquad U_0(n) \equiv \{\mathrm{V} \in U(n) : \mathrm{V} \text{ is diagonal}\}.$$

The homogeneous space Θ is a much clumsier object than $U(n)$; there is no nice global chart, and, indeed, the only tidy way to express Θ is as $U(n)/U_0(n)$. This leads us to ask the natural question "Can we obtain a diffusion (r_t, V_t) on $R \times U(n)$ such that $\mathrm{V}_t^* r_t \mathrm{V}_t$ is Brownian motion on $U(n)$?" It turns out that the answer is "Yes", that the form of the diffusion on $R \times U(n)$ is extremely simple, and that the construction is a special case of a general skew-product decomposition. In more detail, we turn in §3 to a large class of examples with the additional structure:

(5) *there is a Lie group* G *acting transitively on the left on* Θ, *and the Riemannian structure of* M *is invariant under the action of* G.

(The case of Brownian motion in $\mathbb{R}^n \setminus \{0\}$ has $R = (0, \infty)$, $\Theta = S^{n-1}$, and $G = O(n)$, for example.)

We shall see that in examples satisfying (5), the conditions (1.iii) and (1.iv) are automatically satisfied, and there is a Lie subgroup H of G such that $\Theta = G/H$. A Riemannian structure can then be put on $R \times G$ in such a way that Brownian motion projects down to Brownian motion on $R \times \Theta$.

To finish with, we give in §4 some examples of such skew-product decompositions, notably to derive the eigenvalue/eigenvector representation of Brownian motion on various manifolds of matrices.

2. SKEW-PRODUCT DECOMPOSITIONS IN A GENERAL SETTING.

We shall suppose now that (1.i) - (1.iv) hold. In local coordinates, the Laplace-Beltrami operator Δ can be written

$$\Delta = (\det g)^{-1/2} D_i \left(g^{ij} (\det g)^{1/2} D_j \right) \tag{6}$$

where $(g^{ij}) \equiv g^{-1}$, $D_i \equiv \partial / \partial x^i$. In view of (1.ii) and (1.iii), the matrix representing g has the form

$$g_{ij}(r,\theta) = \begin{pmatrix} g^R_{\alpha\beta}(r) & \mathbf{0} \\ \mathbf{0} & (g_r^\Theta)_{\lambda\mu}(\theta) \end{pmatrix} \tag{7}$$

where α, β run from 1 to $n = \dim R$, λ, μ run from 1 to $m = \dim \Theta$. For a tidier notation, we write

$$\rho_{\alpha\beta}(r) \equiv g^R_{\alpha\beta}(r), \qquad \gamma_{\lambda\mu}(r,\theta) \equiv (g_r^\Theta)_{\lambda\mu}(\theta).$$

Because of (1.iv), we have also that

(8) $$\det \gamma(r,\theta)^{1/2} = \phi(r)\,\psi(\theta).$$

We can now state and prove the fundamental result.

THEOREM 4. *The Laplace-Beltrami operator* Δ *of* (M,g) *has the form*

(9) $$\Delta f(r,\theta) = \Delta^R f(r,\theta) + \mathrm{V} f(r,\theta) + \Delta_r^{\Theta} f(r,\theta),$$

where V *is the vector field on* R *which has the expression*

(10) $$\mathrm{V} = \rho^{\alpha\beta} D_\alpha(\log\phi) D_\beta$$

in local coordinates.

Proof. From (6),

$$\begin{aligned}\Delta &= (\det\rho)^{-1/2}\phi^{-1}\psi^{-1}\Big\{D_\alpha(\rho^{\alpha\beta}(\det\rho)^{1/2}\phi\psi D_\beta) \\ &\qquad + D_\lambda(\gamma^{\lambda\mu}(\det\rho)^{-1/2}\phi\psi D_\mu)\Big\} \\ &= (\det\rho)^{-1/2}\phi^{-1} D_\alpha(\rho^{\alpha\beta}\det\rho^{1/2}\phi D_\beta) \\ &\qquad + \phi^{-1}\psi^{-1} D_\lambda(\gamma^{\lambda\mu}\phi\psi D_\mu) \\ &= (\det\rho)^{-1/2} D_\alpha(\rho^{\alpha\beta}\det\rho^{1/2} D_\beta) + \rho^{\alpha\beta}\phi^{-1}D_\alpha\phi D_\beta \\ &\qquad + \phi^{-1}\psi^{-1} D_\lambda(\gamma^{\lambda\mu}\phi\psi D_\mu) \\ &= \Delta^R + \mathrm{V} + \Delta_r^{\Theta}.\end{aligned}$$

◊

We turn now to consider questions concerning the explosion of BM(M). In the cases which interest us later, where Θ is a homogeneous space, the important thing is to decide whether or not the r-motion explodes, so we now concentrate on this.

A few heuristic remarks will help to explain what is being done. The r-motion has generator

$$\mathcal{G} \equiv \tfrac{1}{2}(\Delta^R + \mathrm{V}),$$

so it is a Brownian motion on R with drift V. This calls out for the Cameron-Martin change of measure; if $(r_t)_{t\geq 0}$ is the canonical continuous R-valued process, then a few formal calculations show that the diffusion with generator $\mathcal{G}$ has density

$$(\phi(r_t)/\phi(r_0))^{1/2}\exp\Big(\int_0^t h(r_s)\,ds\Big) \qquad \text{on } \sigma(\{r_u : u \leq t\})$$

with respect to the law of Brownian motion of R, where

(11) $$h \equiv -\tfrac{1}{4}\Delta^R(\log\phi) - \tfrac{1}{8} g^{jk} D_j \log\phi\, D_k \log\phi.$$

Writing $\kappa \equiv \phi^{1/2}$, we can easily show that

(12) $$h = -\tfrac{1}{2}\kappa^{-1}\Delta^R\kappa.$$

This explains why the local martingale at (13) is, in fact, a completely natural thing to consider, even though (as with all real applications of Cameron-Martin) the details are too ugly to be worth trying to do anything rigorous with them. What we shall do is to obtain a condition which in many cases is sufficient to ensure that the diffusion with generator $\mathcal{G}$ does not 'explode' to the region where $\phi = 0$.

With $\kappa = \phi^{1/2}$, the generator of the r-motion can be written

$$\begin{aligned}\mathcal{G} &\equiv \tfrac{1}{2}(\Delta^R + \mathrm{V}) \\ &= \tfrac{1}{2}\kappa^{-2}(\det\rho)^{-1/2} D_\alpha(\rho^{\alpha\beta}\,.\det\rho^{-1/2}\,.\,\kappa^2 D_\beta).\end{aligned}$$

Of course, we do not yet know whether the r-motion can be constructed for all time; there are cases where it cannot. Nonetheless, it certainly can be constructed *locally*, as follows. Fixing the starting pont $r_0 \in R$, we can find compact $K_n \subseteq R$ with union R and such that $K_n \subseteq \operatorname{int}(K_{n+1})$, and $r_0 \in K_1$, and we can construct r up until the time

$$T_n \equiv \inf\{t : r_t \notin K_n\},$$

since K_n can be covered with finitely many coordinate neighbourhoods, and r can be constructed in each, using the local coordinate system. This way, r can be constructed up to the explosion time $\zeta \equiv \sup T_n$. Assuming this done, we define

$$A_t \equiv \tfrac{1}{2}\int_0^t (\kappa^{-1}\Delta^R\kappa)(r_s)\,ds \qquad (0 \leq t < \zeta).$$

We claim that

(13) $$Y_t \equiv \kappa(r_0)\,\kappa(r_t)^{-1}\exp(A_t) \qquad (0 \leq t < \zeta)$$

is a local martingale, in the sense that $Y(t\wedge T_n)$ is a martingale for each n. This follows easily from the calculation

$$\begin{aligned}\mathcal{G}(\kappa^{-1}) &= \tfrac{1}{2}\kappa^{-2}(\det\rho)^{-1/2} D_\alpha(\rho^{\alpha\beta}\det\rho^{1/2}(-D_\beta\kappa)) \\ &= -\tfrac{1}{2}\kappa^{-2}\Delta^R\kappa.\end{aligned}$$

If we now assume that for some $a \in \mathbb{R}$,

(14) $$\tfrac{1}{2}(\kappa^{-1}\Delta^R\kappa)(r) \geq -a \qquad \textit{for all } r \in R,$$

and if we let

$$S_\varepsilon \equiv \inf\{u : \kappa(r_u) < \varepsilon\},$$

then for $t > 0$ fixed

$$1 = Y_0 = E(Y(S_\varepsilon \wedge T_n \wedge t))$$
$$\geq E(Y(S_\varepsilon \wedge T_n \wedge t);\ S_\varepsilon < T_n \wedge t)$$
$$\geq \kappa(r_0)\,\varepsilon^{-1} e^{-at} P(S_\varepsilon < T_n \wedge t),$$

so that

$$P(S_\varepsilon < T_n \wedge t) \leq \varepsilon e^{at}/\kappa(r_0)$$

and letting $n \to \infty$,

$$P(S_\varepsilon < \zeta \wedge t) \leq \varepsilon e^{at}/\kappa(r_0)$$

so that, if $S \equiv \lim_{\varepsilon \downarrow 0} S_\varepsilon$, we conclude that

(15) $$P(S < \zeta \wedge t) = 0.$$

Thus condition (14) ensures that, if explosion *does* occur in finite time, *it cannot occur in such a way that* $\kappa(r_t) \to 0$ *as* $t \uparrow \zeta$.

To illustrate the use of this criterion, we consider the case of Brownian motion on $\mathbb{R}^n \setminus \{0\}$ $(n \geq 3)$, when $R = (0,\infty)$ with the usual Riemannian metric, and $\phi(r) = r^{n-1}$. Then $\kappa(r) = r^{(n-1)/2}$, and

$$(\kappa^{-1} \Delta^R \kappa)(r) = \tfrac{1}{4}(n-1)(n-3)\,r^{-2} \geq 0,$$

so that condition (14) holds, and explosion to zero is impossible. To prove that explosion does not occur requires a further argument; it appears to be simpler to use *ad hoc* arguments for each case than to devise some general criterion. Several examples will be analysed in §4.

3. THE CASE WHERE Θ IS A HOMOGENEOUS MANIFOLD.

Throughout this section we shall assume that there is a Lie group G and a smooth left action

$$\eta : G \times \Theta \to \Theta$$

of G on Θ, which is *transitive* (that is, if $\theta_1, \theta_2 \in \Theta$, there there exists $y \in G$ such that $\eta(y, \theta_1) = \theta_2$). As alternative notations, we write $\eta_y(\theta) \equiv y\cdot\theta \equiv \eta(y, \theta)$. The left action η induces a natural left action $\bar{\eta}$ on M defined by

$$\bar{\eta}(y, (r, \theta)) \equiv (r, \eta(y, \theta)).$$

Suppose we now assume the following condition:

(16) *The Riemannian structure of* M *is invariant under the action* $\bar{\eta}$;

in view of the transitivity of η, it is immediate that condition (1.iii) is then satisfied, but it is even true

that condition (1.iv) holds, as the following simple result shows.

PROPOSITION 1. *Assuming condition* (16), *the Riemannian volume element on* (M,g) *is a product measure.*

Proof. Fix some $r_0 \in R$, and let μ be the Riemannian volume of $(\Theta, g^\Theta_{r_0})$. In any chart, μ has a strictly positive continuous density with respect to Lebesgue measure, and, by (16), μ is invariant under η: for $f \in C_K(\Theta)$, $y \in G$,

$$\int_\Theta f(\theta)\,\mu(d\theta) = \int_\Theta f \circ \eta_y(\theta)\,\mu(d\theta).$$

If ν is any η-invariant measure with a continuus density with respect to Lebesgue measure in any chart, then the density $\rho \equiv d\nu/d\mu$ is continuous. In fact, ρ is *constant*; indeed, for Borel $A \subseteq \Theta$, $y \in G$,

$$\nu(A) = \int I_A(\theta)\,\rho(\theta)\,\mu(d\theta)$$
$$= \int I_A(\eta_y(\theta))\,\rho(\eta_y(\theta))\,\mu(d\theta)$$
$$= \nu(\eta_y^{-1}(A))$$
$$= \int I_A(\eta_y(\theta))\,\rho(\theta)\,\mu(d\theta),$$

from which $\rho = \rho \circ \eta_y$ and transitivity of η implies that ρ is constant. Fixing some non-negative $f \in C_K^\infty(R)$ and letting Vol be the Riemannian volume of (M,g), the measure ν on Θ defined by

$$\nu(A) \equiv \int f(r)\,I_A(\theta)\,\mathrm{Vol}(d\xi)$$

has continuous density and is invariant under η, so is a multiple of μ:

$$\nu(A) \equiv \int f(r)\,I_A(\theta)\,\mathrm{Vol}(d\xi) = C(f)\int I_A(\theta)\,\mu(d\theta).$$

The result follows. ◊

Thus we can invoke Theorem 1 and conclude that if (1.i), (1.ii) and (16) hold, then BM(M) has a skew product decomposition. However, this is far from satisfactory: for example, condition (16) is going to be difficult to check in general, for it involves knowledge of the Riemannian structure g_r^Θ on the clumsy homogeneous space Θ, which will often be hard to specify explicitly. The clean way to handle the problem is to build a process on $R \times G$ which has a skew-product decomposition, and then to drop this process down onto M.

To begin with, we establish some notation. Fix some $\theta_0 \in \Theta$ and let H denote the isotropy group at θ_0:

$$H \equiv \{y \in G : y\cdot\theta_0 = \theta_0\}.$$

This is a closed Lie subgroup of G, and G/H can be given a natural C^∞ structure such that the map $\beta : G/H \to \Theta$ defined by

$$\beta(yH) = y \cdot \theta_0$$

is a diffeomorphism; see Warner [9], pp. 120-123.

Let $N \equiv R \times G$ and define

$$\pi : N \equiv R \times G \to M \equiv R \times \Theta$$

by

$$\pi((r,x)) = (r, x \cdot \theta_0) .$$

We define the pull-back $\pi^* g$ of the Riemannian structure g of M by

(17) $$\pi^* g(U,V) \equiv g(\pi_* U, \pi_* V) ,$$

where U,V are vector fields on N. Thus $\pi^* g$ is a non-negative-definite bilinear form on the tangent bundle; it is *not* a Riemannian structure, because there are non-zero U such that $\pi_* U = 0$.

THEOREM 2. *Suppose that*

(18.i) $\pi^* g$ *is G-invariant;*

(18.ii) *the decomposition* $T_{(r,x)} N \equiv T_r R \oplus T_x G$ *of* $T_{(r,x)} N$ *is orthogonal with respect to* $\pi^* g$ *for each* $r \in R, x \in G$;

(18.iii) *Ad(H) has compact closure in* $G\,l(g)$.

Then the decomposition $T_{(r,\theta)} M \equiv T_r R \oplus T_\theta \Theta$ *is g-orthogonal, and the Riemannian structure of* g *of* M *is invariant under the action* $\overline{\eta}$ *of* G .

Moreover, there is a Riemannian structure g^N *on* N *with the properties*

(19.i) g^N *is G-invariant;*

(19.ii) $T_{(r,x)} N \equiv T_r R \oplus T_x G$ *is an orthogonal decomposition relative to* g^N ;

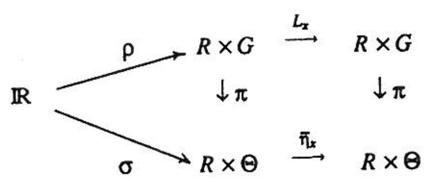
(19.iii) *the map* $\pi : (N, g^N) \to (M,g)$ *is a Riemannian submersion with totally geodesic fibres.*

Hence, in particular, if X *is Brownian motion on* (N, g^N) *then*

(20.i) X *has a skew-product decomposition;*

(20.ii) $\pi(X)$ *is Brownian motion on* (M,g) .

Proof. Consider a smooth curve $\sigma_t = (r_t, \alpha_t)$ in M. Since there are local smooth sections of G/H in G, then (at least locally) there is a smooth curve γ in G which projects down to $\alpha : \alpha_t = \gamma_t \cdot \theta_0$. Let $\rho_t \equiv (r_t, \gamma_t)$, and for $x \in G$ let L_x denote left multiplication by x (either as a map on G, or, by slight abuse of notation, as a map on $R \times G$). Fixing $x \in G$, we have the commuting diagram



so that

$$\begin{aligned} g(d\overline{\eta}_x\, \dot\sigma_0, d\overline{\eta}_x\, \dot\sigma_0) &= g(d\overline{\eta}_x\, d\pi\, \dot\rho_0, d\overline{\eta}_x\, d\pi\, \dot\rho_0) \\ &= g(d\pi\, dL_x\, \dot\rho_0, d\pi\, dL_x\, \dot\rho_0) \\ &= \pi^* g(dL_x\, \dot\rho_0, dL_x\, \dot\rho_0) \\ &= \pi^* g(\dot\rho_0, \dot\rho_0) \qquad \text{using (18.i)} ; \\ &= g(\dot\sigma_0, \dot\sigma_0) . \end{aligned}$$

This establishes condition (16); the Riemannian structure of M is invariant under the action of $\overline{\eta}$ of G. The proof that the decomposition $T_{(r,\theta)} M \equiv T_r R \oplus T_\theta \Theta$ is g-orthogonal is similar.

We turn now to the construction of a suitable Riemannian metric g^N on N. Recall (see, for example, Poor [7] p.213) that condition (18.iii) is equivalent to

(21) *the Lie algebra* g *of* G *admits an* Ad_H*-invariant inner product;*

let $\{\cdot, \cdot\}$ be one such. Denoting the Lie algebra of H by $\boldsymbol{h}$, let $\boldsymbol{h}^\perp$ be the orthogonal complement of $\boldsymbol{h}$ with respect to $\{\cdot,\cdot\}$. Now for each $r \in R$,

$$T_{(r,e)} N \equiv T_r R \oplus \boldsymbol{h}^\perp \oplus \boldsymbol{h} ;$$

we decompose $y, y' \in T_{(r,e)} N$ as

$$y = u + v + w , \qquad y' = u' + v' + w' ,$$

$u, u' \in T_r R$, $v, v' \in \boldsymbol{h}^\perp$, $w, w' \in \boldsymbol{h}$, and define g^N on $T_{(r,e)} N$ by

(22) $$g^N(y, y') \equiv \pi^* g(u + v, u' + v') + \{w, w'\}$$

(23) $$= \pi^* g(u, u') + \pi^* g(v, v') + \{w, w'\} ,$$

the last equality following from (18.ii). We define g^N elsewhere by left translation; for $y, y' \in T_{(r,x^{-1})} N$,

$$g^N(y, y') \equiv g^N(dL_x\, y, dL_x\, y') .$$

Thus (19.i) is immediate, and (19.ii) follows from (23).

To check that π is a Riemannian submersion, evidently π is onto (the action η is transient), and, fixing $\nu = (r, x^{-1}) \in N$, it is easy to show that $y \in T_\nu N$ is in $\ker \pi_*$ if and only if $dL_x\, y \in \boldsymbol{h}$, and y is in $(\ker \pi_*)^\perp$ if and only if $dL_x\, y \in T_r R \oplus \boldsymbol{h}^\perp$. Hence for $y, y' \in (\ker \pi_*)^\perp$,

$$\begin{aligned} g^N(y, y') &\equiv g^N(dL_x y, dL_x y') \\ &= \pi^* g(dL_x y, dL_x y') \qquad \text{from (22)} ; \\ &= \pi^* g(y, y') \qquad \text{by (18.}i) ; \end{aligned}$$

$$= g(\pi_* y, \pi_* y')$$

by definition, so π is a Riemannian submersion.

It remains to establish that the fibres of π are totally geodesic, that is, the second fundamental form of each fibre vanishes identically. See p.299 of Elworthy [3] for a resumé of relevant facts about the second fundemental form; the argument which follows extends the argument of J. Rawnsley which you will find on p.257 of Elworthy [3]. Take $\xi \equiv (r, \theta) \in M$, and let $P \equiv \pi^{-1}(\{\xi\})$ be the fibre above ξ. If we now take some $p \in P$ and vector fields U and V defined in a neighbourhood of p such that $V(p)$ is tangential $(V(p) \in T_pP)$ and $U(p)$ is normal $(U(p) \in (T_pP)^\perp)$, the aim is to prove that at p

$$g^N(\alpha_p(V,V), U) = 0, \tag{24}$$

where, if ∇^N (respectively, ∇^P) denotes covariant derivative on N (respectively, P) then

$$\alpha_p(V,V) \equiv (\nabla_V^N V - \nabla_V^P V)(p).$$

Since $(\nabla_V^P V)(p) \in T_pP$, (24) is equivalent to

$$g^N(\nabla_V^N V, U) = 0 \qquad \text{at } p. \tag{25}$$

Because of the G-invariance of g^N, we may as well left-translate θ back to θ_0, so that $P = \{r\} \times H$, and then left-translate by some element of H to ensure that $p = (r,e)$; this simplifies the notation somewhat. Now since $g^N(\nabla_V^N V, U) = g^N(\alpha_p(V,V), U)$, all that matters are the values $V(p), U(p)$ of the vector fields at p. So we may as well assume that the vector field V is obtained from $V(p) \in \boldsymbol{h}$ by firstly left-translating $V(p)$ around G to give a left-invariant vector field, and then extending this vector field to $R \times G$ in the canonical manner discussed in the introduction. Likewise, if we resolve $U(p)$ as $U(p) = u_1 + u_2$, $u_1 \in T_rR$, $u_2 \in \boldsymbol{h}^\perp$, we can extend u_2 to a vector field U_2 just as for V, and for u_1 we take a vector field U_1 on R which takes value u_1 at r, and then extend U_1 to a vector field (still denoted U_1) on $R \times G$. The point of doing this is that the well-known expression for the Riemannian connection yields

$$g^N(\nabla_V^N V, U) = 2V g^N(U,V) - U g^N(V,V) - 2 g^N(V, [V,U]), \tag{26}$$

and by extending $V(p), U(p)$ as described above, the terms on the right-hand side of (26) become easier to deal with. Indeed,

$$2V g^N(U_1, V) - U_1 g^N(V,V) - 2 g^N(V, [V,U_1]) = 0,$$

each term vanishing (the first because $V \perp U_1$ everywhere, the last because $[V,U_1] = 0$, V being the extension to $R \times G$ of a vector field on G, U_1 being the extension of a vector field on R, and the second term because $g^N(V,V) = \{V(e), V(e)\}$ is constant, by the way that g^N was defined, and the fact that $V(e) \in \boldsymbol{h}$).

Finally, each term of the expression

$$2V g^N(U_2, V) - U_2 g^N(V,V) - 2 g^N(V, [V,U_2]) \tag{27}$$

vanishes. Only the last needs comment. The point is that, since $\{\cdot, \cdot\}$ is Ad_H-invariant, the decomposition

$$\boldsymbol{g} = \boldsymbol{h}^\perp \oplus \boldsymbol{h}$$

is a *reductive* decomposition:

$$Ad_H(\boldsymbol{h}^\perp) \subseteq \boldsymbol{h}^\perp,$$

as is easy to verify. Hence if $h \in \boldsymbol{h}$, $h' \in \boldsymbol{h}^\perp$ then $[h,h'] \in \boldsymbol{h}^\perp$, implying that the last term in (27) vanishes. You will find the elements of this argument in more detail on pp.218-220 of Poor [7].

This establishes that $\alpha_p(V,V) = 0$ for all $V \in T_pP$, $p \in P$, and so, by the symmetry of α, $\alpha \equiv 0$.

The statement (20.i) follows from (19.i)-(19.ii) as in Proposition 1, and (20,ii) is a direct application of Theorem IX.10E of Elworthy [3]. ◊

REMARKS. (i) In the proof we saw that (18.i) implies that the Riemannian metric of M is G-invariant; in particular, by restricting the action to $\{r\} \times \Theta$, we see that g_r^Θ is a G-invariant Riemannian metric on Θ. Now if the action of G on Θ is *effective* (that is, $x \cdot \theta = \theta$ for all $\theta \Rightarrow x = e$), it can be shown (Poor [7] p.213) that the existence of a G-invariant Riemannian metric on Θ implies that $Ad(H)$ has compact closure in $Gl(\boldsymbol{g})$ - thus, (18.i) => (18.iii) if the action of G on Θ is effective. Thus the imposition of condition (18.iii) is not a very big assumption; (18.i) and (18.ii) on their own are almost enough!

(ii) The skew-product decomposition of BM(N) can be expressed (compare (9)) as

$$\Delta^N f(r,x) = \Delta^R f(r,x) + Vf(r,x) + \Delta_r^G f(r,x) \qquad (r \in R,\ x \in G,\ f \in C^\infty(N)). \tag{28}$$

Each Δ_r^G is the Laplace-Beltrami operator of a left-invariant Riemannian metric of g_r^G on G; for this decomposition to be really useful, we need to be able to make explicit the construction of Brownian motion on such a manifold. We describe now how this may be done.

Let us fix r, and drop it from the notation for the time being; thus we are going to construct a Brownian motion on $(G, g^G)\,(= (G, g_r^G))$. Take left-invariant vector fields $U_1, \ldots, U_n$ on G such that $\{U_1(e), \ldots, U_n(e)\}$ form an orthonormal basis for $\boldsymbol{g}$. Defining the structural constants $c_{\cdot\cdot}^{\cdot}$ by

$$[U_i, U_j] = c_{ij}^k U_k,$$

and taking independent real Brownian motions $B^1, \ldots, B^n$, the solution to the SDE

$$\partial X_t = U_j(X_r)\, \partial B_t^j + U(X_t)\, \partial t \tag{29}$$

is BM(G), where

(30) $$U \equiv -\frac{1}{2}\sum_{i=1}^{n} c_{ij}^{j}\, U_i\,;$$

see, for example, Rogers-Williams [8], p.236.

(iii) Do we really need n independent Brownian motions to drive the SDE (28)? After all, we intend to drop the solution down to the lower dimensional manifold Θ. To answer this, suppose that $\{U_1(e), \ldots, U_m(e)\}$ is an orthonormal basis for $h^{\perp}$, take $F \in C^{\infty}(\Theta)$, let $Y \equiv X \cdot \theta_0$, and let $f(x) \equiv F(x \cdot \theta_0)$, $x \in G$. then

$$\partial f(X) \equiv \partial F(Y)$$
$$= \sum_{j=1}^{m} U_j f(X)\,\partial B^j + Uf(X)\,\partial t$$
$$= \sum_{j=1}^{m} U_j f(X)\,dB^j + \frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{n} U_i U_j f(X)\,dB^i\,dB^j + Uf(X)\,dt\,,$$

so, using that fact that

(31) $$dB^i\,dB^j = \delta^{ij}\,dt \qquad (j = 1, \ldots, m,\ i = 1, \ldots, n)$$

we deduce that

$$\partial F(Y) = \sum_{j=1}^{m} U_j f(X)\,dB^j + \{\frac{1}{2}\sum_{i=1}^{m} U_i^2 f(X) + Uf(X)\}\,dt$$

(32) $$= \sum_{j=1}^{m} U_j f(X)\,dB^j + \frac{1}{2}\Delta^G f(X)\,dt\,.$$

But from (28) and (20.ii), $\Delta^G f(x) \equiv \Delta_r^G f(r,x) = \Delta_r^{\Theta} F(r, x\cdot\theta_0) \equiv \Delta^{\Theta} F(x\cdot\theta_0)$, displaying or suppressing the dependence on r as convenient. Thus (32) says

$$\partial F(Y) = \sum_{j=1}^{m} U_j f(X)\,dB^j + \frac{1}{2}\Delta^{\Theta} F(Y)\,dt\,,$$

so that Y solves the martingale problem for $\frac{1}{2}\Delta^{\Theta}$, and is therefore BM($\Theta$). But the same analysis is valid if $B^1, \ldots, B^m$ are independent Brownian motions, and $B^{m+1}, \ldots, B^n$ are *any* continuous semimartingales such that (31) is valid. In particular, $B^{m+1} = \cdots = B^n = 0$ is possible.

It is easy to see that we can find vector fields $U_1(r,\cdot), \ldots, U_n(r,\cdot)$ depending smoothly on r such that $\{U_1(r,e), \ldots, U_n(r,e)\}$ is an orthonormal basis for g with inner product g_r^G, with $\{U_1(r,e), \ldots, U_m(r,e)\}$ a basis for $h^{\perp}$. In terms of these, we can give a leaner version of Theorem 2.

THEOREM 3. *Assuming* (18.i)-(18.iii), *if* $(r_t)_{t\ge 0}$ *is a diffusion on* R *with generator* $\frac{1}{2}(\Delta^R + V)$ *independent of the real Brownian motions* $B^1, \ldots, B^m$, *and if* ξ *solves the* SDE

(33) $$\partial \xi_t = \sum_{j=1}^{m} U_j(r_t, \xi_t)\,\partial B_t^j + U(r_t, X_t)\,\partial t\,,$$

then $Y_t \equiv (r_t, \xi_t \cdot \theta_0)$ *is a Brownian motion on* $R \times \Theta$. *Here,*

(34) $$U(r,\cdot) = -\frac{1}{2}\sum_{i,j=1}^{m} g_r^G(U_j(r,\cdot), [U_i(r,\cdot), U_j(r,\cdot)])\, U_i(r,\cdot)\,.$$

Proof. The only thing needing any explanation is the fact that the sum in (34) is only over values of i,j at most m. But it is not hard to show from the definition (22) of g^N and the G-invariance of $\pi^* g$ that the restriction of g^G (suppressing the 'r' in the notation) to g is Ad_h-invariant. This being so, for each $i > m$,

$$g^G(U_j, [U_i, U_j]) = g^G([-U_i, U_j], U_j)$$

which must therefore be zero for all j. Finally, for $i \le m < j$, since $g = h^{\perp} \oplus h$ is a reductive decomposition, we have

$$g^G(U_j, [U_i, U_j]) = 0\,,$$

again using the form (22) of g^N which shows that h is orthogonal to $h^{\perp}$ in each g_r^G. ◊

4. SOME EXAMPLES. In this final section, we discuss a number of examples. A few of the most interesting or typical cases will be treated in detail, but for the remainder we give only a statement of the skew-product decomposition.

(i) $M = S^{n-1}$. We identify S^{n-1} (or, more strictly, S^{n-1} less the points $(0,0, \ldots, 0,1)$ $(0,0, \ldots, 0,-1)$) with the product manifold $R \times \Theta$, where $R = (0,\pi)$, $\Theta = S^{n-2}$, the identification being

$$(r,\theta) \longrightarrow (\theta \sin r,\ \cos r)\,.$$

The removal of the two poles from S^{n-1} is no real problem; the Brownian motion will never visit either. If g^{Θ} is the usual Riemannian metric of S^{n-2}, then the metric tensor g on $R \times \Theta$ has the form

$$g = \begin{pmatrix} 1 & \cdot \\ \cdot & \sin^2 r\, g^{\Theta} \end{pmatrix},$$

hence (recall (8))

$$\phi(r) = \sin^{n-2} r\,.$$

This gives the vector field (recall (10))

$$V = (n-2)\cot r\,\frac{\partial}{\partial r}\,,$$

so that the diffusion on $R = (0,\pi)$ has generator

$$\mathcal{G} = \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{1}{2}(n-2)\cot r \frac{\partial}{\partial r};$$

we can establish the sufficient criterion (14) for non-explosion provided $n \geq 4$, but since we are dealing with a one-dimensional diffusion, questions of accessibility of boundaries can be dealt with by the scale function; we find that there can be no explosion if $n \geq 3$, so removing the two poles does indeed cause no problem.

While this is a perfectly satisfactory skew-product decomposition, we can also regard $\Theta = S^{n-2}$ as a homogeneous space on which $SO(n-1)$ acts transitively. This puts us in the situation of §3, with $G = SO(n-1)$ and, taking $\theta_0 = (1, 0, \dots, 0)^T$, with isotropy group

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} : S \in SO(n-2) \right\}.$$

To check the conditions (18) of Theorem 2, consider a curve

$$t \to (b + ct,\ Ue^{tA}) \equiv \gamma(t)$$

in $R \times G$, where $b \in (0,\pi)$, $U \in G$, $A \in g$, a curve which maps under π to be the curve

$$t \to (\sin(b+ct)\ Ue^{tA}\,\theta_0,\ \cos(b+ct)) \equiv \sigma(t)$$

in M, and

$$\begin{aligned} \|\dot\sigma(0)\|^2 &\equiv \pi^* g(\dot\gamma_0, \dot\gamma_0) \\ &= \|(c\cos b\ U\theta_0 + \sin b\ UA\theta_0,\ -c\sin b)\|^2 \\ &= c^2 + \sin^2 b \sum_{j=2}^{n-1} (a_{j1}^2), \end{aligned}$$

as is easily verified. Thus (18.i) and (18.ii) are satisfied, and H is compact so $Ad(H)$ is compact, and (18.iii) holds. By Theorem 3, if r solves

$$dr = d\beta + \tfrac{1}{2}(n-2)\cot r\, dt$$

and U is a process in $SO(n-1)$ satisfying

$$\partial U = U\,\partial A,$$

where

$$\begin{aligned} da_{1j} &= \operatorname{cosec} r_t\, dW_j & 1 < j \leq n-1 \\ da_{j1} &= d a_{1j} & 1 < j \leq n-1 \\ da_{ij} &= 0 & \text{otherwise}, \end{aligned}$$

(and $\beta, W_1, \dots, W_{n-1}$ are independent BM($\mathbb{R}$)s), then $\pi(r_t, U_t)$ is Brownian motion on S^{n-1} (it is easy to check that the drift term in (33), given by (34), is zero in this example).

(ii) $M = \mathbb{R}^n \setminus \{0\} \equiv (0,\infty) \times S^{n-1}$. The form of the skew product decomposition in M is too well known to need comment. The skew-product decomposition in $R \times G = (0,\infty) \times \mathrm{SO}(n)$ is given by

$$\begin{aligned} dr &= d\beta + (n-1)(2r)^{-1}\, dt \\ dU &= U\,\partial A, \end{aligned}$$

where

$$\begin{aligned} da_{1j} &= r^{-1}\, dW_j & 1 < j \leq n \\ da_{j1} &= -da_{1j} & 1 < j \leq n \\ da_{ij} &= 0 & \text{otherwise}. \end{aligned}$$

(iii) In this example, we take M to be the manifold of non-degenerate *normal* matrices:

$$M = \{Y \in gl(n,\mathbb{C}) : YY^* = Y^*Y,\ \text{all eigenvalues of } Y \text{ are distinct}\}.$$

As is easily shown, each $Y \in M$ can be expressed in the form

$$Y = U \Lambda U^*$$

where $U \in U(n)$, $\Lambda \in R \equiv \{\Lambda \in gl(n,\mathbb{C}) : \Lambda \text{ is diagonal}\}$. Such a representation is not unique but (ignoring trivial permutations of columns of U) U is unique to within right multilication by elements of

$$H = \{U \in U(n) : U \text{ is diagonal}\}.$$

Thus we are in the situation of §3, with $G = U(n)$, $\Theta = G/H$. We give M the Riemannian structure it inherits as a submanifold of $gl(n,\mathbb{C}) \cong \mathbb{R}^{2n^2}$. If y is the curve in $R \times G$

$$y(t) = (\Lambda + t\Gamma,\ U e^{tA})$$

where $\Lambda \in R$, Γ is diagonal, $U \in G$, $A \in g$, the image of y under π is the curve

$$\sigma(t) = U e^{tA}(\Lambda + t\Gamma)e^{-tA} U^*.$$

Hence

$$\dot\sigma(0) = U\{A\Lambda - \Lambda A + \Gamma\}U^*$$

and

$$\pi^* g(\dot\gamma_0, \dot\gamma_0) \equiv \|\dot\sigma(0)\|^2$$
$$= \mathrm{Re}\, tr\, \dot\sigma(0)^T \dot\sigma(0)$$
$$(35) \qquad = 2\sum_{i<j} |a_{ij}|^2 |\lambda_i - \lambda_j|^2 + \sum_i |\gamma_i|^2$$

after a few elementary calculations. Thus conditions (18.i) and (18.ii) hold; compactness of H guarantees (18.iii), and we have a skew-product decomposition. By the definition (10) of ϕ, we have in this example that

$$\phi(r) = \prod_{i<j} |\lambda_i - \lambda_j|^2 .$$

(Yes, squared! This is because we are dealing with a real manifold, and, if $a_{ij} \equiv \alpha_{ij} + i\beta_{ij}$, the ij^{th} term in the sum is $(\alpha_{ij}^2 + \beta_{ij}^2)\,|\lambda_i - \lambda_j|^2$, contributing *two* factors of $|\lambda_i - \lambda_j|^2$ to the diagonal metric tensor.) Hence if B_i, W_{ij} are independent BM($\mathbb{C}$)s, we find that if

$$d\lambda_k = dB_k + \sum_{j\neq k} \frac{\lambda_k - \lambda_j}{|\lambda_k - \lambda_j|^2}\, dt$$

and $\partial U = U\,\partial A$ where

$$d a_{ij} = \frac{dW_{ij}}{\sqrt{2}\,|\lambda_i - \lambda_j|} \qquad (i<j)$$

$$d a_{ij} = -d\bar a_{ji} \qquad (i>j)$$

$$d a_{ii} = 0,$$

then $U\Lambda U^*$ is Brownian motion on M.

We just have to check non-explosion; but $\Delta^R(\log\phi) = 0$, so h defined at (11) is non-positive, and condition (14) is satisfied. To prove that the λ_j cannot explode to infinity in finite time, we observe that

$$d\,|\lambda_k - \lambda_j|^2 = 2(\lambda_k - \lambda_j).\,d(B^k - B^j) + 2\,dt + 2(\lambda_k - \lambda_j).\left\{\sum_{p\neq k}\frac{\lambda_k - \lambda_p}{|\lambda_k - \lambda_p|^2} - \sum_{p\neq j}\frac{\lambda_j - \lambda_p}{|\lambda_j - \lambda_p|^2}\right\}dt$$
$$= 2(\lambda_k - \lambda_j).\,d(B^k - B^j) + 2\,dt + 2(\lambda_k - \lambda_j).\sum_{p\neq j,k}\left(\frac{\lambda_k - \lambda_p}{|\lambda_k - \lambda_p|^2} - \frac{\lambda_j - \lambda_p}{|\lambda_j - \lambda_p|^2}\right)dt + 4\,dt .$$

Summing now over distinct j and k, the interesting piece is

$$\Sigma'\left\{\frac{(\lambda_k - \lambda_j).(\lambda_k - \lambda_p)}{|\lambda_k - \lambda_p|^2} - \frac{(\lambda_k - \lambda_j).(\lambda_j - \lambda_p)}{|\lambda_j - \lambda_p|^2}\right\}$$

where Σ' denotes the sum over distinct p, j, k. This sum is evidently twice S, where

$$S = \Sigma' \frac{(\lambda_k - \lambda_j)(\lambda_k - \lambda_p)}{|\lambda_k - \lambda_p|^2}$$
$$= \Sigma' \frac{(\lambda_p - \lambda_j)(\lambda_p - \lambda_k)}{|\lambda_k - \lambda_p|^2}$$
$$= -\Sigma' \frac{(\lambda_p - \lambda_j)(\lambda_k - \lambda_p)}{|\lambda_k - \lambda_p|^2} .$$

Hence

$$2S = \Sigma'\left(\frac{(\lambda_k - \lambda_j).(\lambda_k - \lambda_p)}{|\lambda_k - \lambda_p|^2} - \frac{(\lambda_k - \lambda_p).(\lambda_p - \lambda_j)}{|\lambda_k - \lambda_p|^2}\right) = n(n-1)(n-2) .$$

Thus

$$\sum_{j\neq k} d\,|\lambda_k - \lambda_j|^2 = d\,(\text{local martingale}) + \text{constant}\, dt$$
$$\equiv dM_t + c\,dt ,$$

say. Thus the continuous local martingale M satisfies $M_t \geq -ct$ for all t; M cannot explode to $-\infty$ in finite time, and therefore cannot explode to $+\infty$ in finite time either (since M is a time-change of Brownian motion). Hence the eigenvalue motion cannot explode to $+\infty$ in finite time, and, since (14) holds, neither can any pair of eigenvalues collide in finite time.

(iv) $M = U(n)$. In this case, M can be thought of as a submanifold of the manifold of non-degenerate normal matrices considered in the previous example. It is possible to deduce the skew-product decomposition for $U(n)$ from that example; it is easier just to mimic the argument given, *mutatis mutandis.* Either way, one arrives at the following result. Writing the eigenvalues of $U \in U(n)$ as $e^{i\gamma_k}$, $k = 1, \ldots, n$, and taking independent BM($\mathbb{C}$)s W_k, β_{ij}, then the solution to

$$d\gamma_k = dW_k + \tfrac{1}{2}\sum_{p\neq k} \cot\tfrac{1}{2}(\gamma_k - \gamma_p)\,dt ,$$

$$\partial U = U\,\partial A ,$$

where

$$da_{ij} = 8^{-1/2}\,\mathrm{cosec}\,\tfrac{1}{2}(\gamma_j - \gamma_i)\,dW_{ij} \qquad (i<j)$$
$$= -d\bar a_{ji} \qquad (i>j)$$

$$= 0 \qquad (i = j)$$

gives a Brownian motion Y on the unitary matrices defined by

$$Y \equiv U \exp(i\Gamma) U^* ,$$

where $\Gamma = \text{diag}(\gamma_j)$.

(v) McKean [4] discusses an example based on work of Dyson [2], in which $M = \{n \times n$ symmetric real matrices with distinct eigenvalues$\}$. The manifold M inherits the Riemannian structure of $\mathbb{R}^{n^2}$ in which it is an embedded submanifold. If the eigenvalues are $\lambda_1 < \cdots < \lambda_n$, then taking independent BM($\mathbb{R}$)s W_k, β_{ij}, the solution to

$$d\lambda_i = dW_i + \frac{1}{2}\left(\sum_{k \neq i} \frac{1}{\lambda_i - \lambda_k}\right) dt ,$$

$$\partial U = U\, dA ,$$

where

$$da_{ij} = \frac{1}{\sqrt{2}\,|\lambda_i - \lambda_j|}\, d\beta_{ij} \qquad (i < j)$$

$$= -da_{ji} \qquad (i > j)$$

$$= 0 \qquad (i = j)$$

gives a Brownian motion Y on M defined by

$$Y \equiv U \Lambda U^T ,$$

with $\Lambda \equiv \text{diag}(\lambda_i)$.

Although M is a submanifold of the manifold of normal matrices considered as example (iii), let the appearance of the factor $\frac{1}{2}$ in the drift serve to warn that some obvious conjectures are not true.

(vi) Let

$$S_+ = \{n \times n \text{ positive-definite symmetric matrices}\}$$

and let

$$M = \{V \in S_+ : \text{the eigenvalues of } V \text{ are distinct}\} ;$$

then M is a submanifold of the manifold considered in the previous example. However, there is another natural Riemannian structure which can be put on M which comes from regarding S_+ as the homogeneous space $Gl(n,\mathbb{R})/O(n)$, with the quotient map

$$X \longmapsto XX^T .$$

Thus $G \equiv Gl(n,\mathbb{R})$ acts on S_+, by

$$(X,V) \longmapsto XVX^T ,$$

and it is natural to put a Riemannian metric on S_+ which is invariant under this action. One way this can be done is to define

(36) $$\|\dot v(0)\|^2 \equiv \frac{1}{4} tr\,(\dot v(0)^T \dot v(0))$$

where v is a smooth curve in S_+, $v(0) = I$, and to shift the inner product round S_+ using the action of G (the inner product at I is invariant under the isotropy group $O(n)$, so this is a proper definition). The pullback of this Riemannian structure to G can be determined by taking a curve $\gamma(t) \equiv X e^{tA}$ in G ($X \in G$, $A \in gl(n,\mathbb{R})$), mapping down to S_+ and measuring its speed there. We find

$$\frac{d}{dt}\gamma(t)\gamma(t)^T \Big|_{t=0} = X(A + A^T)X^T ,$$

whose length is $\frac{1}{4} tr\,(A + A^T)^2$. Now, as in the proof of Theorem 2, we can put a metric g^G on G which is G-invariant, and project down the Brownian motion of (G,g^G) to get BM(S_+); we first had to choose some Ad_H-invariant inner product $\{\cdot,\cdot\}$ on g and here the choice

$$\{A,B\} \equiv tr\,A^T B$$

seems very natural. The Lie algebra h of the isotropy group $O(n)$ is just the set of skew-symmetric matrices, $h^\perp$ is the set of symmetric matrics, and from the definition (22) of g^N, we see easily that in this case

$$g^G(A,B) = \{A,B\} \equiv tr\,A^T B .$$

Using this Riemannian metric on G, the Brownian motion on (G,g^G) is the solution to

(37) $$\partial Z = Z\,\partial A ,$$

where A is a matrix of independent Brownian motions (the vector field (30) giving the drift is easily shown to be zero). Thus

(38) $Y \equiv ZZ^T$ *is Brownian motion on* S_+;

it was this characterisation of BM(S_+) which was used in Norris, Rogers and Williams [5]. Dynkin [1] and Orihara [6] also analysed BM(S_+) (incidentally, we remark in passing that this justifies the use of the term 'Brownian motion' in Norris, Rogers and Williams [5] to describe Y; the companion process $X \equiv Z^T Z$ is *not* in fact a Brownian motion relative to any Riemannian metric on S_+).

The paper of Norris, Rogers and Williams [5] also gave a skew-product decomposition for the eigenvalue/eigenvector motion; this also is a consequence of what we have done in this paper. To see this, take

$$R \equiv \{n \times n \text{ diagonal real matrices } \Gamma, \text{ with } \gamma_1 < \cdots < \gamma_n\}$$

$$\Theta = O(n),$$

and identify $R \times \Theta$ with M by the map

$$(\Gamma, H) \longrightarrow H \exp(2\Gamma) H^T.$$

(Although the map is not 1-1, it is locally a diffeomorphism; if $H_1 \Lambda H_1^T = H_2 \Lambda H_2^T$, then $H_2^T H_1$ is a diagonal orthogonal matrix). Thus the curve $\gamma(t) \equiv (\Gamma + tD, H e^{tA})$ maps to

$$\sigma(t) = H e^{tA} e^{2\Gamma + 2tD} e^{-tA} H^T$$

which, by the invariance of the Riemannian structure on S^+, has the same speed at $t = 0$ as

$$t \mapsto e^{-\Gamma} e^{tA} e^{2\Gamma + 2tD} e^{-tA} e^{-\Gamma} \equiv \nu(t).$$

Using the definition (36) of $\|\dot{\nu}(0)\|^2$ we obtain

$$\|\dot{\nu}(0)\|^2 = \sum_i d_i^2 + 2 \sum_{i<j} a_{ij}^2 \sinh^2(\gamma_i - \gamma_j) \tag{39}$$

after a few calculations (here, d_i is the i^{th} diagonal entry of D). Since the Riemannian structure of $R \times \Theta$ is invariant under the action of $O(n)$ (even under the action of $Gl(n, \mathbb{R})$), and, from (39), $T_r R \oplus T_\theta \Theta$ is an orthogonal direct sum decomposition of $T_{(r,\theta)} M$, we deduce the existence of a skew-product representation of BM(M): since

$$\phi(r) = \prod_{i<j} \sinh(\gamma_j - \gamma_i)$$

we obtain the SDE representation

$$d\gamma_k = dW_k + \frac{1}{2} \sum_{j \neq k} \coth(\gamma_k - \gamma_j)\, dt$$

and

$$\partial H = H \partial A,$$

where

$$d a_{ij} = \frac{d\beta_{ij}}{\sqrt{2} \sinh(\gamma_j - \gamma_i)} \qquad (i < j)$$

$$d a_{ij} = -d a_{ji} \qquad (i > j)$$

$$d a_{ii} = 0.$$

As usual, the W_k and β_{ij} are independent BM($\mathbb{R}$)s. See Norris, Rogers and Williams [5] for an elementary proof of this using only Itô calculus, starting from (38).

(vii) One may similarly treat the case of

$$M = \{\text{positive–definite Hermitian matrices with distinct eigenvalues}\},$$

insisting that the Riemannian structure on M is invariant under the action

$$(X, S) \mapsto X S X^* \qquad (X \in Gl(n, \mathbb{C}), S \in M)$$

of $Gl(n, \mathbb{C})$ on M.

The argument is very similar; if W_k, β_{ij} are independent BM($\mathbb{C}$), and if γ, U solve

$$d\gamma_k = dW_k + \sum_{j \neq k} \coth(\gamma_k - \gamma_j)\, dt$$

$$\partial U = U \partial A,$$

where

$$d a_{ij} = \frac{d\beta_{ij}}{\sqrt{2} \sinh(\gamma_j - \gamma_i)} \qquad (i < j)$$

$$= -d \bar{a}_{ji} \qquad (i > j)$$

$$= 0 \qquad (i = j),$$

and writing $\Gamma = \text{diag}(\gamma_i)$, then

$$Y \equiv U \exp(2\Gamma) U^* \text{ is Brownian motion on } M.$$

Note the factor of 2 difference between the drifts of this example and the last!

(viii) We finish with a pair of very interesting examples: $SO(2n)$ and $SO(2n+1)$. Firstly we study $SO(2n)$. The eigenvalues of $V \in SO(2n)$ are all of modulus 1 and appear in conjugate pairs. We shall restrict attention to

$$M = \{V \in SO(2n): \text{ the } n \text{ conjugate pairs of eigenvalues of } V \text{ are distinct}\}.$$

We count (1,1) and (-1,-1) as conjugate pairs. Defining

$$\rho(\theta) \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

then any $V \in M$ can be represented as

$$V = S \begin{pmatrix} \rho(\theta_1) & & \\ & \ddots & \\ & & \rho(\theta_n) \end{pmatrix} S^T,$$

where $S \in SO(2n)$, and $(\theta_1, ..., \theta_n) \in R$, where

$$R \equiv \{(\theta_1, ..., \theta_n) : -\theta_2 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n < 2\pi - \theta_{n-1}\}$$

if $n > 1$, and $R = \mathbb{R}$ if $n = 1$. If $G \equiv SO(2n)$ and

$$H = \left\{ \begin{pmatrix} \rho(\alpha_1) & & \\ & \ddots & \\ & & \rho(\alpha_n) \end{pmatrix} : \alpha_i \in \mathbb{R} \right\},$$

then M can be identified with $R \times \Theta$, $\Theta \equiv G/H$. We give M the Riemannian structure g it inherits from $\mathbb{R}^{n^2}$; to investigate $\pi^* g$, consider the curve

$$\sigma(t) = ((\theta_1 + t\gamma_1, ..., \theta_n + t\gamma_n), S\, e^{tA})$$

for $S \in G$, $A \in g$ and $(\theta_1, ..., \theta_n) \in R$. This maps under π to

$$y(t) \equiv S\, e^{tA} \begin{pmatrix} \rho(\theta_1 + t\gamma_1) & & \\ & \ddots & \\ & & \rho(\theta_n + t\gamma_n) \end{pmatrix} e^{-tA} S^T$$

for which

$$\dot{y}(0) = S\{AJ - JA + JD\} S^T,$$

where

$$J = \begin{pmatrix} \rho(\theta_1) & & \\ & \ddots & \\ & & \rho(\theta_n) \end{pmatrix}$$

and

$$D = \begin{pmatrix} \alpha(\gamma_1) & & \\ & \ddots & \\ & & \alpha(\gamma_n) \end{pmatrix},$$

defining

$$\alpha(\gamma) \equiv \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}.$$

Notice that $DJ = JD$. We now calculate

$$\begin{aligned} \pi^* g(\dot{\sigma}_0, \dot{\sigma}_0) &= \| \dot{y}(0) \|^2 \\ &= tr\{AJ^T - J^T A + D^T J^T\}\{AJ - JA + JD\} \\ &= tr[-2A^2 + 2AJ^T AJ + D^T D] \\ &= 2\sum_{i,j} a_{ij}^2 + 2\Sigma \gamma_j^2 + 2\, tr\, AJ^T AJ. \end{aligned}$$

To calculate the last term split A into 2×2 blocks;

$$c_{ij} \equiv \begin{pmatrix} a_{2i-1,2j-1} & a_{2i-1,2j} \\ a_{2i,2j-1} & a_{2i,2j} \end{pmatrix}$$

so that

$$\begin{aligned} tr\, AJ^T AJ &= tr \sum_{i,j} c_{ij}\, \rho(-\theta_j)\, c_{ji}\, \rho(\theta_i) \\ &= -tr \sum_{i,j} c_{ij}\, \rho(-\theta_j)\, c_{ij}^T\, \rho(\theta_i). \end{aligned}$$

It is elementary to verify that

$$tr\, c_{ij}\, \rho(-\theta_j)\, c_{ij}^T\, \rho(\theta_i) = \cos\theta_i \cos\theta_j\, tr\, (c_{ij}^T c_{ij}) + 2 \sin\theta_i \sin\theta_j \det c_{ij}$$

so that

$$\pi^* g(\dot{\sigma}_0, \dot{\sigma}_0) = 2\sum_{i,j} \{1 - \cos\theta_i \cos\theta_j)\, tr\, (c_{ij}^T c_{ij}) - 2 \sin\theta_i \sin\theta_j \det c_{ij}\} + 2\sum_j \gamma_j^2$$

$$= 4 \sum_{i<j} \{1 - \cos\theta_i \cos\theta_j)\, tr(c_{ij}^T c_{ij}) - 2 \sin\theta_i \sin\theta_j \det c_{ij}\}$$

$$+ 2 \sum_j \gamma_j^2 .$$

This immediately establishes conditions (18.i) and (18.ii); since H is compact, condition (18.iii) holds as well. Thus by Theorem 2 there is a skew product decomposition, and to calculate what it is, we need to find ϕ. To this end, observe that if

$$c_{ij} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

then

$$\alpha\, tr(c_{ij}^T c_{ij}) + 2\beta \det c_{ij} = (a\ b\ c\ d)\,\{\alpha \mathrm{I} + \beta L\}\,(a\ b\ c\ d)^T ,$$

where

$$L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\det(\alpha \mathrm{I} + \beta L) = (\alpha^2 - \beta^2)^2$$

$$= (\cos\theta_i - \cos\theta_j)^4 = (2 \sin \tfrac{1}{2}(\theta_i - \theta_j) \sin \tfrac{1}{2}(\theta_i + \theta_j))^4$$

if $\alpha = 1 - \cos\theta_i \cos\theta_j$, $\beta = \sin\theta_i \sin\theta_j$. Thus

$$\phi(r) = \left(\prod_{i<j} \sin \tfrac{1}{2}(\theta_j - \theta_i) \sin \tfrac{1}{2}(\theta_i + \theta_j) \right)^2 ,$$

from which we calculate V, and find that the SDE for the θ_k is

$$d\theta_k = \frac{1}{\sqrt{2}}\, dW_k + \frac{1}{4} \sum_{j \neq k} \{\cot \tfrac{1}{2}(\theta_k - \theta_j) + \cot \tfrac{1}{2}(\theta_k + \theta_j)\}\, dt .$$

As for the eigenvector motion, it can be shown that

$$\partial S = S\, \partial A$$

where A is a skew-symmetric matrix-valued process whose ij^{th} 2×2 sub block c_{ij} satisfies for $i < j$

$$dC_{ij} = 8^{-1/2} \left\{ \operatorname{cosec} \tfrac{1}{2}(\theta_i + \theta_j) \begin{pmatrix} dZ_{ij}^1 & dZ_{ij}^2 \\ dZ_{ij}^2 & dZ_{ij}^1 \end{pmatrix} + \operatorname{cosec} \tfrac{1}{2}(\theta_j - \theta_i) \begin{pmatrix} dZ_{ij}^3 & dZ_{ij}^4 \\ -dZ_{ij}^4 & dZ_{ij}^3 \end{pmatrix} \right.,$$

where Z_{ij}^1, Z_{ij}^2, Z_{ij}^3 and Z_{ij}^4 are independent BM(IR)s, independent of the BM(IR) process W_k.

Remark. Notice that the eigenvalues are repelled from each other, *except* that there is no repulsion between the two members of a conjugate pair. This allows θ_1, θ_n to leave $(0,\pi)$, but no other θ_j will be able to do so.

Finally, the case of $SO(2n+1)$ is considered; we merely state the results. Any $V \in SO(2n+1)$ can be written as

$$V = S \begin{pmatrix} 1 & & & & \\ & \rho(\theta_1) & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \rho(\theta_n) \end{pmatrix} S^T$$

for some $S \in SO(2n+1)$ and (assuming the eigenvalues distinct, except possibly for a repeated eigenvalue of -1) $0 < \theta_1 < \cdots < \theta_{n-1} < \theta_n < 2\pi - \theta_{n-1}$. For the motion of the eigenvalues we find this time

$$d\theta_k = \frac{1}{\sqrt{2}}\, dW_k + \frac{dt}{4} \left[\sum_{j \neq k} \{\cot \tfrac{1}{2}(\theta_k - \theta_j) + \cot \tfrac{1}{2}(\theta_k + \theta_j)\} + \cot \tfrac{1}{2}\theta_k \right],$$

which is as it was before, with repulsions between all the eigenvalues except members of a conjugate pair, with also a repulsion from the fixed eigenvalue +1. The motion of the eigenvectors is similar, except that now we have additionally

$$d\, a_{1,2l} = 8^{-1/2} \operatorname{cosec} \tfrac{1}{2}\theta_l\, d\beta_l$$

$$d\, a_{1,2l+1} = 8^{-1/2} \operatorname{cosec} \tfrac{1}{2}\theta_l\, d\beta_l'$$

where the β, β' are further independent BM(IR)s.

The special case of SO(3) was proved by other means in Rogers and Williams [8], Theorem V. 35.18. A different normalisation means that the solution of Rogers and Williams runs at twice the speed of ours.

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LOCAL STOCHASTIC DIFFERENTIAL GEOMETRY

or

What can you learn about a manifold by watching Brownian motion?

Mark Pinsky [1]

ABSTRACT. This is a five-year report on progress in the above-named area during the period 1982-87, including work by the author, A. Gray, L. Karp, Ming Liao and others.

CONTENTS

§0. BROWNIAN MOTION OF A MANIFOLD. This is most conveniently thought of as a weak limit of a piecewise geodesic process, as follows. We begin with a d-dimensional Riemannian manifold (M^d, g) with geodesics $t \to \gamma(t)$, where $\gamma(0) = m \in M$ and $\gamma'(0) = \xi \in M_m$, the tangent space at m. The infinitesimal generator of the geodesic motion is the first order differential operator D, defined as $Df(m,\xi) = (d/dt)f(\gamma(t),\gamma'(t))|_{t=0}$. (In geometry textbooks this is referred to as the "geodesic spray", or "geodesic flow field".) The normalized Laplacian is defined by either of the formulas

$$d^{-1}\Delta f \doteq \langle D^2 f\rangle = \lim_{\delta\downarrow 0}\{(\delta^{-1}Df + \delta^{-2}(\langle f\rangle - f))\}$$

(The bracket operator refers to the average of the function with respect to normalized Lebesgue measure on the surface of the unit sphere in the tangent space.)

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