Firms, Banks and Households

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Abstract

This paper sets up and analyses a continuous-time equilibrium model with firms, households and a bank. The model allows us to study the inter-relation of production, consumption, levels of working, interest rates, debt, inflation and wage levels.

1 Introduction.

As the contents of this volume testify, what constitutes systemic risk, and how this may be modelled and analyzed, is open to many different interpretations. In general terms, we are concerned with how financial markets affect the real economy; and with the possibility that many different assets might fall in value at the same time, with consequent loss of confidence and further losses to follow. One simple approach would be to treat the returns of a set of assets of interest as correlated time series; but such a view would only address correlation, without making any statements about causation. At the next level, we could investigate models where changes in the price of one asset may impact the price of other assets, establishing a causal route, but not giving much guidance on the form of such causation, nor on its origins; what is the chicken, what the egg? The key issues for modelling revolve around how shocks to the system occur, and how the components of the system respond to those shocks *and to the responses of other components.* It is the second of these issues which is the most problematic; a simple-minded imposition of some rules by which the components of the system respond together is liable to appear *ad hoc*, and may lead to inconsistencies and

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calibration issues not apparent at the outset. Our attempt to deal with this is to work within an equilibrium framework, where the principles by which prices form and evolve are at least well established, even if quite hard to analyse.

What we attempt to do in this paper is to offer as simple a model as possible in which to try to understand how the financial markets affect the real economy. Such a model has to contain a financial sector; the real economic sector; and the households who consume the output of the real economy, supply labour to its workings, who lend to banks which in turn lend to the real economy. These elements are the minimal possible to begin to address the question 'How do financial markets affect the real economy?' As we shall see, even with such a sparsely-populated model, the analysis becomes remarkably involved. We were inspired in this attempt by the paper of Bernanke *et al.* [1], who study a discrete-time model with many firms subject to idiosyncratic shocks and to aggregate economy-wide shocks. The financial sector makes loans to the firms, who accept all the risk by agreeing to outcome-dependent[1] repayment terms; these terms are set so that the bank is not exposed to net default risk. This model is rich and interesting, though its analysis is non-trivial. In the end, Bernanke *et al.* end up log-linearising the model and deal with that approximation. In this paper, we propose to work in a continuous-time setting, where all processes have continuous sample paths. This brings in methodological and conceptual simplifications, and allows us to make progress more easily than in a discrete-time model, where random shocks are not infinitesimal. The preprint of Brunnermeier *et al.* [2] presents a model of an economy with a financial sector set in continuous time, differing in so many ways from what we do here that it is hard to compare the two.

Many important factors have been omitted from this story, such as international trade and the role of government. Heterogeneity of agents and goods is ignored. Nevertheless, we believe that the model here may serve some useful purpose in clarifying the linkages between the different elements in an oversimplified economy. Though there are only three components to our story, more than a dozen processes have to be determined in equilibrium, so the picture is much more complex than the initial description might suggest. Analysis takes us some way, but we find ourselves carrying out simulations at the end in order to understand better how the model works. This is not a surprise, nor a defeat; models which can be solved in closed form can only ever be a parody of a caricature of reality, and we have to expect to go numerical if what we do is ever to become relevant.

In Section 2 we set out the model, and the various dynamic and static relations between the processes featuring in it. Section 3 gathers together the conclusions of Section 2, and next in Section 4 we explore what happens in a simple situation with a Cobb-Douglas production function, and a multiplicatively-separable utility for the households. Numerical results are reported and commented on in Section 5.

[1]This we find somewhat unrealistic; a bank loan is only outcome-dependent insofar as default of the firm may prevent the due repayments being made.

2 Modelling assumptions

We will present here a model set in continuous time of a banking sector, a productive sector and a household sector. Each sector is constituted of a continuum of identical agents, so we will feel free to speak in the singular of *the* bank, firm, or household. There is a single consumption good, which can either be consumed or used to produce capital, and there is money. We shall suppose that the speed of money is infinite, so that neither the household nor the firm retains a stock of cash; physical cash resides at all times in the bank vault.

All processes will be taken to be continuous and adapted to some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ which supports a Brownian motion W. We shall require some notation for various different processes in the model:

$$\begin{array}{llll}
C_t &= \text{consumption rate} & L_t &= \text{rate of working} \\
K_t &= \text{firms' capital} & D_t &= \text{firms' nominal debt} \\
p_t &= \text{price level} & \Delta_t &= \text{households' bank deposits} \\
w_t &= \text{wage rate} & R_t &= \text{interest rate on loans} \\
I_t &= \text{investment rate} & r_t &= \text{interest rate on deposits} \\
Z_t &= \text{scaling of net output} & \ell_t &= \text{rate of new lending} \\
S_t &= \text{value of equity} & a_t &= \text{dividend rate} \\
Q_t &= \text{bank equity} & x_t &= \text{bank reserves}
\end{array}$$

Let us now describe the various elements of the model and their interrelations.

The firm. The firm employs labour from the households at rate L_t, for which it pays $w_t L_t$; it operates capital K_t, and from these inputs generates net[2] output at rate $Z_t f(K_t, L_t)$, where f is homogeneous of degree one, smooth, increasing in both arguments, and strictly concave. We therefore can represent f as

$$f(K, L) = Kh(L/K), \tag{2.1}$$

where h is increasing and strictly concave; we shall write $\tilde{h}$ for the convex dual function

$$\tilde{h}(y) \equiv \sup_x \{h(x) - yx\}. \tag{2.2}$$

The process Z is strictly positive, and scales the output of production; this is the *only* source of randomness in the model, and could be motivated by variations in the cost of the raw materials for production. For concreteness[3], we shall assume that the random factor Z in the output is a geometric Brownian motion:

$$dZ_t = Z_t(\sigma dW_t + \mu dt). \tag{2.3}$$

[2]We imagine that the production of goods may require some goods as input raw materials, but that the production function accounts for the difference between the quantity produced and the quantity required for production.

[3]More general stories could be told here, and much of what follows goes through without this assumption, but the examples always assume this.

The firm's output is split between consumption and investment:

$$Z_t f(K_t, L_t) = C_t + I_t. \tag{2.4}$$

Ownership of the firm's capital K_t is divided between the shareholders and the bank:

$$p_t K_t = S_t + D_t, \tag{2.5}$$

expressing the monetary value of the capital K_t as the sum of the value of the debt owed to the bank, and the equity owed to the share holders. We shall define the *gross profitability* of the firm as

$$q_t = \big[\, p_t Z_t f(K_t, L_t) - w_t L_t - R_t D_t) \,\big]/S_t. \tag{2.6}$$

This is the gross rate of output of the firm, net of labour costs and interest costs, expressed as a fraction of the total shareholder capital available; notice that gross profitability is a dimensionless quantity, not expressed in either goods or money, but simply a number.

The firm's borrowing is constrained by certain natural inequalities, involving constants $\alpha, \kappa > 0$, $b \in (0, 1)$, namely

$$\begin{aligned} 0 \le D_t &\le b\, p_t K_t && (2.7) \\ D_t &\le \alpha\, x_t && (2.8) \\ D_t &\le \kappa\, Q_t. && (2.9) \end{aligned}$$

The first inequality (2.7) expresses a leverage constraint on the firms' borrowing; they may not borrow more than some fraction of the current value of their assets. The second inequality (2.8) expresses the fact that the amount lent out to firms cannot exceed some multiple of the bank's reserves, and therefore D/x must be bounded. The third inequality is a capital adequacy requirement, imposing the constraint on the banks not to lend out more than some multiple of their equity base.

We shall suppose that the processes D and K are finite-variation processes, but that the price-level process p has some martingale part. Most of the time the inequality (2.7) will be strict, and so does not constrain the dynamics, but occasionally it will attain equality, and at such time there will have to be forced selling in the nature of local time, that is to say, continuous and finite-variation, but singular with respect to Lebesgue measure.

Default forced sales. We envisage two types of forced sales, one when the leverage constraint is hit, and the other as defaults hit the firms. If a quantity ε of capital has to be sold, then a fraction[4] $\gamma = \gamma(q) \in (0, 1]$ will find no buyer and has to be scrapped. In the case of a default, the fractions γ^S of the loss borne by the shareholders and γ^B of the loss borne by the bank are given by

$$\gamma_t^S = \min\left\{ \gamma(q_t),\ \frac{S_t}{S_t + D_t} \right\}, \qquad \gamma_t^B = \left(\gamma(q_t) - \frac{S_t}{S_t + D_t} \right)^+, \tag{2.10}$$

[4] We allow that the loss rate γ may vary negatively with gross profitability, though in a simple model we would assume γ constant.

as would be expected from the rule that equity takes the first losses. We shall suppose that firm capital gets into difficulty at rate $\psi(q_t)K_t dt$, where ψ is positive and decreasing. Thus default losses occur at rate $\varphi_t \equiv \varphi(q_t) = \gamma(q_t)\psi(q_t)$ per unit of capital, and these are split between the firm and the bank as

$$\varphi_t = \varphi_t^S + \varphi_t^B \equiv \gamma_t^S \psi(q_t) + \gamma_t^B \psi(q_t). \tag{2.11}$$

Leverage forced sales. In the case of a sale forced by leverage constraints, the resolution is different, and needs to be carefully analyzed. We envisage that the firm capital process K and the debt process D will both be continuous finite-variation, but that the price level p and the equity S will both have a martingale part. Looking at the inequality (2.7), we expect that the times when the inequality becomes an equality will have the character of the zero set of a Brownian motion, so we shall suppose that *the leverage forced sales come as a singular increasing process A.*

Accordingly, when the firm has to make an infinitesimal sale dA of capital as a result of hitting the leverage constraint, the loss $\gamma(q)dA$ is borne entirely by the shareholders, as the firm is not in default, it merely has to reduce part of its loans to rein in leverage. The surviving capital $(1-\gamma(q))dA$ is transferred from the ownership of the bank to the ownership of new shareholders, who withdraw money from their deposits to finance the purchase. Thus we can deduce the evolution of the firm capital:

$$dK_t = (I_t - \delta K_t)dt - \varphi_t K_t dt - \gamma(q_t)dA_t, \tag{2.12}$$

where $\delta > 0$ is the fixed depreciation rate.

To proceed, we need to assume that $\pi_t \equiv 1/p_t$ evolves as a continuous semimartingale[5]

$$d\pi_t = \pi_t(\sigma_\pi\, dW_t + \mu_\pi\, dt - \aleph_t dA_t), \tag{2.13}$$

and introduce the processes $\tilde{S}_t \equiv \pi_t S_t$ and $\tilde{D}_t \equiv \pi_t D_t$ for the equity and debt expressed in units of goods, in terms of which the basic identity (2.5) becomes

$$K = \tilde{S} + \tilde{D}. \tag{2.14}$$

Notice that when a leverage forced sale happens, the total firm capital falls by $\gamma(q)dA$, and the ownership of the remaining $(1-\gamma(q))dA$ that was sold passes from the bank, $\tilde{D}$, to the shareholders, $\tilde{S}$. Hence we deduce the part of the dynamics of $\tilde{S}$ relating to leverage-forced sales:

$$\begin{aligned} d\tilde{S} &= dK - d\tilde{D} \\ &= -\gamma dA + (1-\gamma)dA + \ldots \\ &= (1-2\gamma)dA + \ldots, \end{aligned} \tag{2.15}$$

where $\ldots$ simply denotes various terms in dt and dW that we do not care about for the moment.

[5]The coefficients σ_π and μ_π are processes, not constants.

A simple thought experiment *allows us to determine* $\aleph$. Indeed, the process A is previsible[6]; the times when some leverage-forced sales happen can be seen coming, and at those times the value of equity will be reduced by the losses γdA caused by a forced sale. Why then do we not find that the households move their money completely out of equity into the bank account to avoid the equity losses? The answer has to be that *the bank account suffers exactly the same proportional loss.* Of course, the cash value of the bank deposits is not changed, so what must be happening is that *the price level rises* to compensate for this. Equating the proportional drop in equity value (expressed as the fall in the quantity of capital divided by the initial quantity of capital) to the proportional drop in purchasing power of cash, we discover that

$$\frac{\gamma dA}{\tilde{S}} = \frac{dp}{p} + \ldots = \aleph\, dA. \tag{2.16}$$

Hence we find that

$$\frac{dp}{p} = \frac{\gamma dA}{\tilde{S}} + \ldots$$

so we are able to express the dynamics of D as

$$dD_t \equiv d(p_t \tilde{D}_t) = (\ell_t - \varphi_t^B p_t K_t)\, dt - \frac{p_t dA_t}{S_t} \{ \ (1-\gamma_t) S_t - \gamma_t D_t \ \}, \tag{2.17}$$

where we write $\gamma_t \equiv \gamma(q_t)$ for short. The terms in dt here come from the new loans and the default losses, and the final term is coming from the leverage-forced sales. In this final term, we understand the first part, $(1-\gamma)p\, dA$, as the cost of the purchase of the additional capital *if there was no change in the price level.* However, the price level is also changing, so there is the correction term $\gamma pD\, dA/S$ to account for that.

The other part of the story on leverage-forced sales which we have to understand is the *dilution effect*. Someone who holds a fraction λ of equity just before a leverage-forced sale will hold some smaller fraction afterwards, because new stock was issued to raise the money required for the buyback of bank capital. The fraction of the total stock held will therefore change to

$$\frac{\lambda(\tilde{S} - \gamma dA)}{\tilde{S} + (1-2\gamma) dA} = \lambda(1 - (1-\gamma) dA/\tilde{S}) + O(dA^2),$$

so we find that the fraction λ_t of stock held by the original stockholders evolves as

$$d\lambda_t = -\ \frac{\lambda_t(1-\gamma_t)}{\tilde{S}_t}\, dA_t. \tag{2.18}$$

This matters because the fraction of total dividends paid to the original stockholders will be reduced by dilution, and this has to be taken into account.

Remarks. Notice that by selling $(1-\gamma)dA$ units of capital, the gap $\tilde{D}_t - bK_t$ is reduced by

$$\{(1-\gamma(q)) - b\gamma(q)\ \} dA = (1 - (1+b)\gamma(q)) dA, \tag{2.19}$$

[6] A will be continuous and adapted.

and this is only positive if

$$\gamma(q) < \frac{1}{1+b}. \tag{2.20}$$

Thus if this inequality is *not* satisfied, we can expect that it will be impossible to restore the leverage inequality (2.7); and the closer $\gamma(q)$ is to $(1+b)^{-1}$, the more selling will be required to restore the inequality. Notice that if the gross level of profitability falls too far, we may find $\gamma(q)$ getting dangerously close to its upper bound, with the risk of substantial forced sales.

We now detail the cashflows faced by the firm. The firm receives in a cashflow $p_t C_t$ from the households to pay for consumption, it pays out cashflows $w_t L_t$ to pay for labour, $R_t D_t$ in interest payments to the bank, and a_t as dividends. The remaining cashflow arises because of new bank lending to the firm. We interpret this flow ℓ_t as the purchase by the banks of capital which the firm is then permitted to operate in return for interest payments. We suppose that there is no cash accumulation at the firm or at the household, so the net inflows to the firm equal the net outflows:

$$w_t L_t + a_t + R_t D_t = p_t C_t + \ell_t. \tag{2.21}$$

The households. The households supply labour at rate L_t to the firms, purchase consumption goods at rate C_t at prevailing price p_t, and wish to maximize the objective

$$E \int_0^\infty e^{-\rho t}\, U(C_t, L_t)\; dt, \tag{2.22}$$

where U is smooth, concave, increasing in the first argument[7], and satisfies the Inada condition

$$\lim_{C \downarrow 0} U_C(C, L) = \infty, \qquad \lim_{C \uparrow \infty} U_C(C, L) = 0 \qquad \forall L. \tag{2.23}$$

The households deposit any surplus cash in a deposit account at the bank, which generates interest at rate r_t. They also receive dividends on the bank equity Q_t that they hold. *We shall make the simplifying assumption that*

the process Q is finite-variation[8].

The dividend rate d_t to bank equity must be such that the households are indifferent between bank deposits and bank equity, otherwise they would not be willing to hold both. Thus it has to be that

$$\dot{Q}_t + d_t = r_t Q_t. \tag{2.24}$$

[7] We do not assume that U is decreasing with L, though it may be. This is to allow for the modelling possibility that extremely low values of L would correspond to high unemployment and would therefore not be desired.

[8] In the story as we tell it here, there is a single source Z of randomness, but three securities that the households can invest in: firm equity, bank equity and bank deposits. One of those securities must be redundant, and the choice we make is simple enough to work with.

The households' deposits change with the saving of their surplus income, and with any withdrawals needed to purchase capital at times when the firm's borrowing hits its leverage bound.

$$\begin{aligned} d\Delta_t &= (w_t L_t + a_t + r_t \Delta_t + d_t - p_t C_t)dt - \frac{p_t dA_t}{S_t}\{\ (1-\gamma_t)S_t - \gamma_t D_t\ \} \\ &= (w_t L_t + a_t + r_t(\Delta_t + Q_t) - \dot{Q}_t - p_t C_t)dt - \frac{p_t\, dA_t}{S_t}\{\ S_t - \gamma_t p_t K_t\ \}. \end{aligned} \tag{2.25}$$

The bank. The bank takes deposits and makes loans. The balance-sheet identity

$$\Delta_t = x_t + D_t \tag{2.26}$$

equates the bank's liabilities to depositors to its assets (in the form of reserves plus loans.) When default losses occur, at rate $\varphi_t K_t$, these are split proportionally between the bank and the equity of the firm; the face value D of the debt falls by $\varphi_t^B p_t K_t$ and the banks pay the same amount out of equity into the reserves x so as to maintain the balance-sheet identity (2.26). Additionally, when the leverage constraint is hit, the face value of debt is reduced due to shareholder purchase of additional capital. Thus we have

$$dD_t = (\ell_t - \varphi_t^B p_t K_t)dt - \frac{p_t\, dA_t}{S_t}\{\ S_t - \gamma_t p_t K_t\ \}. \tag{2.27}$$

Bank equity receives interest payments at rate $R_t D_t$ on the loans, pays out dividends at rate d_t to their shareholders, pays out interest to depositors at rate $r_t \Delta_t$, and compensates the depositors' reserves for default losses on the loan book. Thus in total we find the evolution equation for bank equity Q:

$$\dot{Q}_t = R_t D_t - r_t \Delta_t - d_t - \varphi_t^B p_t K_t, \tag{2.28}$$

from which (using (2.24)) we draw the simple conclusion that

$$R_t D_t = r_t(\Delta_t + Q_t) + \varphi_t^B p_t K_t. \tag{2.29}$$

The evolution for the bank reserves x is

$$dx_t = d\Delta_t - dD_t = (r_t(\Delta_t + Q_t) - \dot{Q}_t - R_t D_t + \varphi_t^B p_t K_t)dt = -\dot{Q}_t dt \tag{2.30}$$

using (2.21) and (2.25) to rework the first expression. This accords with what would be expected; there is no cash anywhere in the system except in the reserves x and the bank equity Q, and the sum of these two must be constant:

$$x + Q = M, \tag{2.31}$$

where M denotes the total cash in the system. It will turn out that the notional split of the bank cash M is indeterminate (because deposits and bank equity both deliver an identical return). In view of this, we may nominate to split M in whatever way is convenient, and by inspection of (2.8) and (2.9) we see that we make the bound on D implied by these two inequalities as generous as possible if we take $x \propto \kappa$, $Q \propto \alpha$, allowing us to replace the two bounds (2.8) and (2.9) by the one bound

$$D_t \leq \frac{\alpha\kappa M}{\alpha+\kappa}. \tag{2.32}$$

To summarize the cashflow rates between the four entities in the story, we have the following table:

From/To	Household	Firm	Bank reserves	Bank equity
Household		$\frac{p_t\,dA_t}{S_t}\{\ S_t - \gamma_t p_t K_t\ \} + p_t C_t dt$	$d\Delta_t$	
Firm	$(a_t + w_t L_t)dt$		$\frac{p_t\,dA_t}{S_t}\{\ S_t - \gamma_t p_t K_t\ \}$	$R_t D_t\,dt$
Bank reserves		$\ell_t\,dt$		
Bank equity	$(r_t\Delta_t + d_t)\,dt$		$\varphi_t^B p_t K_t dt$	

Optimality conditions. Further relations follow from optimality considerations. We have:

$$\theta_t U_C + U_L = 0, \tag{2.33}$$
$$Z_t f_L = \theta_t, \tag{2.34}$$

where we have introduced the notation

$$\theta_t \equiv \frac{w_t}{p_t} \equiv w_t \pi_t \tag{2.35}$$

for the real wage rate. The first equation comes by considering a marginal increase in working being used to pay for a marginal increase in consumption; at optimality, no such change would improve the objective of the agent. The second comes from considering the marginal increase in the value of output which the firm would achieve by employing more labour.

We shall assume that

$$\textit{the nominal wage rate } w \textit{ is finite-variation,} \tag{2.36}$$

and that

$$\textit{the labour deployed in production, } L, \textit{ is finite-variation,} \tag{2.37}$$

These assumptions result in a relatively stable labour market, justified by the casual observation that the prices of goods in the shops change far more frequently than the wages[9] an individual earns, or that individual's employment status. These assumptions have important consequences: from (2.35) we learn that *the volatility of θ equals the volatility of π*, since w is assumed to be finite-variation; and from (2.34) we likewise deduce that *the volatility of θ equals the volatility of Z*, which by assumption is the constant σ.

Further consequences of optimality can be deduced by considering the marginal expected changes which occur on switching small quantities of capital or cash between possible uses.

[9]There are echoes here of Keynes's [3] assertion in Chapter 2 that labour stipulates a money wage, rather than a real wage.

Considering the switching of household consumption into bank deposits; and a small increase in firm capital funded by borrowing, respectively, we may argue the following consequences:

(i) One property can be deduced from the assumption that the firm is a simple expectation-maximizer. The firm operates capital of two types, bank capital, and shareholder capital. The firm incurs costs at rate $R_t D_t - \varphi_t^B p_t K_t$ on the bank capital[10], and at rate $a_t + \varphi_t^S p_t K_t$ on equity; these costs, expressed as a fraction of the capital involved must be equal, otherwise the firm would switch immediately to all of one sort of capital or the other. This leads to the relation

$$\frac{a_t + \varphi_t^S p_t K_t}{S_t} \geq \frac{R_t D_t - \varphi_t^B p_t K_t}{D_t}, \tag{2.38}$$

with equality whenever the inequalities (2.7), (2.32) are strict. Of course, the inequality (2.38) does not apply when $D = 0$, since in that situation we are not able to reduce bank funding. The corresponding inequality when $D = 0$ would be

$$\frac{a_t + \varphi_t^S p_t K_t}{S_t} \leq R_t \tag{2.39}$$

(ii) If the consumption now is transferred into the bank account, we learn that

$$\exp\Big\{-\rho t + \int_0^t r_s \, ds\Big\} \, U_C(C_t, L_t)/p_t \quad \text{is a martingale.} \tag{2.40}$$

This we see by considering a situation where the households reduce their consumption in $(t, t+dt)$ by ε, place the $p_t \varepsilon \, dt$ thus saved into the bank account, and at later time T withdraw the value $p_t \varepsilon \, dt \exp(\int_t^T r_s \; ds)$ and use it to consume some more during $(T, T+dt)$. At optimality, this trade cannot change the expectation of the objective, from which the condition (2.40) follows. As a consequence, we learn that the household's state-price density process is

$$\zeta_t \equiv e^{-\rho t} U_C(C_t, L_t)/p_t. \tag{2.41}$$

(iii) We now consider the situation of the original shareholders' stake in the firm and how it is valued. At time t, the fraction of the firm capital owned by the original shareholders (who have *not* input any further capital into the firm) is λ_t, where the evolution of λ is given by (2.18). This stake entitles them to dividend flow $\lambda_s a_s ds$ at all later times s, so we deduce that

$$\zeta_t \lambda_t S_t + \int_0^t \zeta_s \lambda_s a_s \, ds \quad \text{is a martingale;} \tag{2.42}$$

the current value of the stock has to be the net present value of all future payouts.

[10] The sign of the term $\varphi_t^B p_t K_t$ is correct; the default losses suffered by the bank are a writedown of the firm's debt, therefore a credit for the firm.

(iv) We propose a *shadow firm* story, where the household foregoes a small amount ε of consumption for a period of length dt at time t, thereby saving goods εdt which is invested into productive capital. We shall suppose that the time interval $(t, t+dt)$ contains no point of increase of A, thereby excluding a Lebesgue-null set of times. The capital saved and invested into a shadow firm has decayed by time $u > t$ to

$$\Delta k_u \equiv \varepsilon dt \exp\left(-\int_t^u (\delta + \gamma(q_t)\psi(\bar{q}_s))\, ds \right),$$

where $\bar{q}_s$ is the profitability of the shadow firm at time s. Labour is employed at the going wage rate so as to maximize the output[11], so that at time u we find the maximized output to be

$$Z_u\, \tilde{h}(\theta_u/Z_u)\, \Delta k_u = \varepsilon dt\, Z_u\, \tilde{h}(\theta_u/Z_u) \exp\left(-\int_t^u (\delta + \gamma(q_s)\psi(\bar{q}_s))\, ds \right), \tag{2.43}$$

and in the process we learn that

$$\bar{q}_s = Z_s \tilde{h}(\theta_s/Z_s). \tag{2.44}$$

This additional output generates additional consumption and so utility; at time t, the expected increase in the future objective must exactly balance the loss of objective due to the sacrifice of consumption εdt now, resulting in

$$\varepsilon dt\, U_C(C_t, L_t) = \varepsilon dt\, E_t\left[\int_t^\infty e^{-\int_t^u (\rho+\delta+\gamma\psi(\bar{q}_s))\, ds}\, U_C(C_u, L_u) Z_u \tilde{h}(\theta_u/Z_u)\, du \right]. \tag{2.45}$$

For notational convenience, we write

$$\bar{\psi}_t \equiv \delta + \gamma(q_t)\psi(\bar{q}_t) = \delta + \gamma(q_t)\psi(Z_t\tilde{h}(\theta_t/Z_t)), \qquad \Psi_t \equiv \int_0^t \bar{\psi}_s\, ds. \tag{2.46}$$

From (2.45) we conclude that

$$d\left(e^{-\rho t - \Psi_t} U_C(C_t, L_t) + \int_0^t e^{-\rho u - \Psi_u}\, U_C(C_u, L_u) Z_u \tilde{h}(\theta_u/Z_u)\, du \right) \tag{2.47}$$

is the differential of a martingale, at least off the set of times of increase of A. We make no statement about what happens on the (null) set of times where A increases; the objective of the households does not in any case care about what happens on a Lebesgue-null set of times, so it cannot matter.

The argument given assumes that it is marginally viable for households to set up a shadow firm from their own resources in this way. This is not to say that there is no point in having a banking sector, rather that in equilibrium there is no point in having any *more or less* of the banking sector.

[11] The optimization problem to be solved is $\max_L \{pZf(K, L) - wL\} = \max_L pZK\{h(L/K) - (\theta/Z)(L/K)\} = pZK\, \tilde{h}(\theta/Z)$.

Conditions from the martingales. We have just deduced from optimality considerations that there are three martingales in this story, (2.40), (2.42), and (2.47). We exploit this information by working out what the drifts of these three processes are, and then setting them equal to zero. Since the process $U_C(C_t, L_t)$ appears in all the martingales, it will be helpful to have a shorter notation for it, so we shall write

$$dU_C = U_C\,(\,\sigma_U\,dW + \mu_U\,dt + \frac{\gamma dA}{\tilde{S}}), \tag{2.48}$$

and later make the processes σ_U and μ_U more explicit. The martingales are analyzed as follows.

(i) From (2.40), by taking an Itô expansion and setting the drift equal to zero, we learn that

$$r = \rho - \mu_U - \mu_\pi - \sigma_U\sigma_\pi. \tag{2.49}$$

(ii) From (2.47) we learn that

$$\mu_U = \rho + \delta + \gamma(q)\psi(\bar{q}) - \bar{q}, \tag{2.50}$$

where we recall that $\bar{q}_s = Z_s\tilde{h}(\theta_s/Z_s)$.

(iii) The final martingale (2.42) needs more work. Using (2.41) and (2.5), we express the martingale as

$$N_t = \lambda_t e^{-\rho t} U_C(C_t, L_t)(K_t - \tilde{D}_t) + \int_0^t \lambda_u e^{-\rho u} U_C(C_u, L_u)\pi_u a_u\,du \tag{2.51}$$

where $\tilde{D}_t \equiv \pi_t D_t$ evolves according to

$$d\tilde{D} = \sigma_\pi \tilde{D} dW + (\pi\ell - \varphi^B K + \mu_\pi \tilde{D})\,dt - (1-\gamma)dA. \tag{2.52}$$

Expanding the expression (2.51) for N with Itô's formula, the terms in dA cancel, leaving

$$\frac{dN}{\lambda_t e^{-\rho t} U_C} \doteq \big[\,(I - (\delta+\varphi)K) - (\pi\ell - \varphi^B K + \mu_\pi\tilde{D}) + (K - \tilde{D})(\mu_U - \rho) - \sigma_U\sigma_\pi\tilde{D} + \pi a\,\big]\,dt$$

where $\doteq$ signifies that the two sides differ by a (local) martingale. Since the drift term must vanish, we are able to develop this as

$$\begin{aligned}
0 &= I - (\delta+\varphi)K - (\pi\ell - \varphi^B K + \mu_\pi\tilde{D}) + (K - \tilde{D})(\mu_U - \rho) - \sigma_U\sigma_\pi\tilde{D} + \pi a \\
&= I - \delta K + \tilde{D}(\rho - \mu_U - \mu_\pi - \sigma_U\sigma_\pi) + K(\mu_U - \rho - \varphi^S) + \pi(a - \ell) &(2.53)\\
&= I - \delta K + r\tilde{D} + K(\mu_U - \rho - \varphi^S) + (C - \theta L - R\tilde{D}) &(2.54)\\
&= Zf - \theta L + K(\mu_U - \rho - \delta - \varphi^S) + \tilde{D}(r - R) &(2.55)\\
&= K(Z\tilde{h}(\theta/Z) + \mu_U - \rho - \delta - \varphi^S) + \tilde{D}(r - R) &(2.56)\\
&= K\big\{\,Z\tilde{h}(\theta/Z) + \gamma(q)\psi(\bar{q}) - \bar{q} - \varphi^S\,\big\} + \tilde{D}(r - R) &(2.57)\\
&= K\big\{\gamma(q)\psi(\bar{q}) - \varphi^S\,\big\} + \tilde{D}(r - R), &(2.58)
\end{aligned}$$

where we used (2.11) to arrive at (2.53); (2.21) and (2.49) to arrive at (2.54); (2.4) to arrive at (2.55); the optimality of θ to arrive at (2.56); (2.50) for (2.57); and finally the definition (2.44) of $\bar{q}$ to arrive at (2.58).

The conclusion therefore is

$$K\gamma(q)\psi(\bar{q}) + r\tilde{D} = K\varphi^S + R\tilde{D}, \tag{2.59}$$

which bears a neat interpretation: the left-hand side is the rate at which the shadow firm[12] runs up costs of defaults and (lost deposit) interest, the right-hand side is the rate at which the actual firm runs up costs of defaults and (actual) interest, and the result (2.59) says that these two are equal. Bearing in mind the assumption that in equilibrium there was no barrier to market entry for shadow firms, this result makes perfect sense.

3 Summary.

We shall gather here all the equations governing the economy. To begin with, let us notice that because of (2.34), and the relation $\bar{q} = Z\tilde{h}(\theta/Z)$, we have from the definition (2.6) that

$$q = \left\{ \bar{q} - \frac{RD}{D+S} \right\} \frac{pK}{S}. \tag{3.1}$$

We can rearrange this more informatively to read

$$\bar{q} = q\,\frac{S}{S+D} + R\,\frac{D}{S+D}. \tag{3.2}$$

This tells us that the profitability $\bar{q}$ of the shadow firm, which has no loan capital and therefore pays no interest, is a convex combination of the profitability of the firm, and the rate of return on bank loans. This makes good sense[13]; if $R > \bar{q}$, then the cost of loans is too high and it would be better for the firm not to take any, while if $\bar{q} > q$ it would again be better for the households to take all their capital out of the firm and start their own shadow enterprise.

Now we summarize the equations governing the economy. The state variables of the economy will be taken to be[14] (Z, K, D), and we will work also with auxiliary variables L and $\pi \equiv 1/p$, supposed to satisfy[15]

$$dL = L(\mu_L dt + \eta_L dA) \tag{3.3}$$

[12]... funded by borrowing in the same proportions to the actual firm ...

[13]We expect that $q > R$ in any case, otherwise the firm would do better to pay back its loans rather than produce.

[14]In view of the discussion at (2.32), we see that x and Q are held at fixed fractions of the constant money supply M, and therefore $\Delta = x + D$ is determined once D is known. Therefore there is no need to include Δ in the state variables; it can be deduced.

[15]Recall that L is supposed to be finite variation.

and (2.13) respectively. From (2.34), (2.33), (2.4), (2.5), (2.44), (3.2), (2.10) and (2.11), (2.29), (2.38), and (2.21) we obtain respectively:

$$
\begin{aligned}
Zf_L(K,L) &= \theta && (3.4)\\
0 &= \theta U_C(C,L) + U_L(C,L) && (3.5)\\
Zf &= C + I && (3.6)\\
pK &= S + D && (3.7)\\
q &= \frac{pZf - wL - RD}{S} && (3.8)\\
\bar{q} &= Z\tilde{h}(\theta/Z) && (3.9)\\
\bar{q} &= q\,\frac{S}{S+D} + R\,\frac{D}{S+D}. && (3.10)\\
\varphi^B &= \left(\gamma(q_t) - \frac{S_t}{S_t + D_t}\right)^+ \psi(q_t) && (3.11)\\
RD &= r(D+M) + \varphi^B pK && (3.12)\\
a &\geq RS - (\, S\varphi^B/D + \varphi^S\,)\, pK && (3.13)\\
pC + l &= a + wL + RD && (3.14)
\end{aligned}
$$

(where equality obtains in (3.13) when D is unconstrained), the relation $\varphi^S = \gamma\psi(q) - \varphi^B$, along with the dynamic equations

$$
\begin{aligned}
dK &= (I - (\delta + \varphi)K)dt - \gamma dA && (3.15)\\
dD &= (\ell - \varphi^B pK)dt - \frac{p\,dA}{S}(S - \gamma pK), && (3.16)
\end{aligned}
$$

the martingales (2.40), (2.47), (2.42), and the inequalities (2.7), (2.8), (2.9).

Intriguingly, the use of (3.12) in (2.59) gives us the surprisingly compact equation

$$p\gamma(q)K(\psi(\bar{q}) - \psi(q)) = rM. \qquad (3.17)$$

Notice that since $\bar{q} < q$ and ψ is decreasing, the left-hand side is indeed positive. The equation (3.17) has the simple interpretation that $rM + p\gamma(q)K\psi(q)$ is the net rate at which the firm pays out to the bank and to default costs, and that this should equal the rate of default costs if the fully-funded shadow firm was set up instead.

Let us now detail the route through these equations, bearing in mind that we have state variables (Z, K, D, L, π) at any time.

- We use (3.4) to find θ; (3.5) to find C; (3.6) to find I; (3.7) to find S; (3.9) to find $\bar{q}$;
- The implicit equation for q is solved by choosing a value for q; using (3.17) to deduce r; (3.10) to deduce R; (3.11) to find φ^B; and (3.12) to obtain another value for R, which has to agree with the earlier value, and this is done by adjusting the choice of q;
- We use (3.13) and (3.14) to deduce a and ℓ.

The one last thing to be done is to determine what may happen when the debt has reached its maximal allowed value (2.32), at which point all we are able to conclude is that (3.13) is an inequality. This inequality arises by comparing the cost of financing by equity with the cost of financing by bank borrowing. If the inequality is strict, then the demand for bank borrowing would be high; *we shall suppose that the rate ℓ of new lending is as high as possible while keeping within the bound* (2.32), that is, that $\ell = \varphi^B pK$ when (2.32) is an equality. This will mean that we now go to (3.14) to determine what the rate a of dividend payments should be. If we ever reached a time when $D = 0$, then the inequality (3.13) is replaced by the inequality (2.39). If this were to happen, then there is no point in the banks maintaining the lending rate R at any level higher than $(a + \varphi^S pK)/S$, since there are no loans anyway, so we will assume that the inequality (2.39) is an equality when $D = 0$.

This explains how the current values of all variables are to be found. Deriving the dynamical equations is in principle possible from this, but is rather cumbersome at a general level, so we shall study those questions in a simple example.

4 Examples.

We propose that the production function takes the standard (homogeneous of degree 1) form

$$f(K, L) = Kh(L/K) \tag{4.1}$$

for some positive increasing concave h. We shall try to deduce from the preceding equations and relations as much as we can about the unknowns, supposing that we know Z, K, D, L and π.

The first relation to work on is (3.4), which is stated as

$$Zh'(L/K) = \theta, \tag{4.2}$$

giving θ in terms of the known processes. In view of the form of this, it will be helpful to introduce the process $y \equiv L/K$ with evolution

$$\begin{aligned} dy &= y\big\{ \mu_L dt + \eta_L dA - (I/K - \delta - \varphi)dt + \gamma dA/K \big\} &(4.3)\\ &\equiv y(\mu_y\, dt + \eta_y\, dA). &(4.4)\end{aligned}$$

In terms of y, we have that $\theta = Zh'(y)$ and the evolution of θ is

$$\frac{d\theta}{\theta} = \frac{dZ}{Z} + \frac{h''(y)dy}{h'(y)} = \sigma dW + \mu dt + \frac{yh''}{h'}\,(\mu_y dt + \eta_y dA). \tag{4.5}$$

The second equation (3.5) allows us to find C as a function of θ and L, and from that we may find U_C as a function of θ and L. We shall suppose that we have

$$U_C = F(\theta, L) \tag{4.6}$$

for some function F which will need to be made explicit in any particular example. However, assuming that we have F, we may develop

$$\begin{aligned}
\frac{dU_C}{U_C} &= \frac{F_\theta\, d\theta + \frac{1}{2}F_{\theta\theta}\, d\theta d\theta + F_L\, dL}{F} \\
&= \frac{\theta F_\theta}{F}\Big\{\, \sigma dW + \mu dt + \frac{yh''}{h'}\,(\mu_y dt + \eta_y dA)\,\Big\} + \frac{\theta^2\sigma^2 F_{\theta\theta}}{2F}\, dt + \\
&\qquad\qquad + \frac{LF_L}{F}(\mu_L\, dt + \eta_L\, dA) \\
&= \frac{\sigma\theta F_\theta}{F}\, dW + \left\{\, \frac{\mu\theta F_\theta}{F} + \frac{\theta^2\sigma^2 F_{\theta\theta}}{2F} + \frac{LF_L}{F}\,\mu_L + \frac{\theta F_\theta y h''}{Fh'}\,\mu_y \,\right\} dt + \\
&\qquad\qquad + \left\{\, \frac{\theta F_\theta y h''}{Fh'}\,\eta_y + \frac{LF_L}{F}\,\eta_L \,\right\} dA.
\end{aligned}$$

Using the fact that we know (2.48) the component of dU_C involving dA, we are able to compare coefficients to learn that

$$\sigma_U = \frac{\sigma\theta F_\theta}{F} \tag{4.7}$$

$$\mu_U = \frac{\mu\theta F_\theta}{F} + \frac{\theta^2\sigma^2 F_{\theta\theta}}{2F} + \frac{LF_L}{F}\,\mu_L + \frac{\theta F_\theta y h''}{Fh'}\,\mu_y \tag{4.8}$$

$$\left\{\, \frac{\theta F_\theta y h''}{Fh'} + \frac{LF_L}{F} \,\right\}\eta_L = \gamma\left\{\, \frac{1}{\tilde{S}} - \frac{\theta F_\theta y h''}{KFh'} \,\right\}. \tag{4.9}$$

Using (2.50) and (4.4) gives μ_L.

4.1 Possible choices for h and U.

To apply the preceding analysis, we need to make choices for h, and for U, in such a way that the function F can be obtained in a reasonably simple form. Here are a few possibilities for U.

1. We could use
$$U(C, L) = -\,\frac{C^{-\varepsilon}}{(L^* - L)^\nu} \tag{4.10}$$
where $\varepsilon, \nu > 0$. Routine calculation leads to the conclusion
$$U_C = F(\theta, L) = (\varepsilon^{-\varepsilon'}\nu^{\varepsilon'})(L^* - L)^{-\nu-\varepsilon'} \tag{4.11}$$
where $\varepsilon' \equiv 1 + \varepsilon$.

2. Another product form choice for U would be to take
$$U(C, L) = -C^{-\varepsilon}(A + L^\nu), \tag{4.12}$$
where $A, \varepsilon > 0$ and $\nu > 1$. In this case we find
$$U_C = F(\theta, L) = \nu(\nu/\varepsilon)^\varepsilon\, \theta^{-\varepsilon-1}(A + L^\nu)^{-\varepsilon}\, L^{(1+\varepsilon)(\nu-1)} \tag{4.13}$$

All the examples we studied used the Cobb-Douglas choice

$$h(y) = y^{1-\beta} \tag{4.14}$$

for some $\beta \in (0,1)$ gives $yh''/h' = -\beta$.

4.2 Simulation.

The simulation story is largely straightforward, but a little care is needed over the treatment of the leverage forced sales. If we take a time step dt for the discretization, and we generate a Gaussian increment $dW \sim N(0, dt)$ for the Brownian driver of Z, then in the absence of any leverage forced sales we see the updating

$$\begin{array}{rcl} Z & \mapsto & Z' \equiv Z \exp(\,\sigma dW + (\mu - \frac{1}{2}\sigma^2)dt\,) \\ K & \mapsto & K' \equiv K + (\,I - \delta K - \varphi(q)K\,)\,dt \\ D & \mapsto & D' \equiv D + (\,\ell - \varphi^B pK\,)\,dt \\ \pi & \mapsto & \pi' \equiv \pi \exp(\,\sigma_\pi dW + (\mu_\pi - \frac{1}{2}\sigma_\pi^2)dt\,) \\ L & \mapsto & L \exp(\mu_L dt) \end{array}$$

If $\pi' D'/K' \leq b$, then there is no violation of the leverage constraint (2.7), and nothing more needs to be done. However, if this is not the case then there is need for some forced sales of magnitude dA, which (in view of (2.17), (2.12), (2.13)) gives us

$$\frac{\pi' D'}{K'} \mapsto \frac{\pi' D'}{K'}\; \frac{\exp(-\gamma dA/\tilde{S})(1 - \{(1-\gamma)S - \gamma D\}pdA/SD)}{1 - \gamma dA/K'}. \tag{4.15}$$

We also have that $dA \leq K'$, since we cannot sell more that the total capital; this leaves us with a numerical search for the value of dA.

5 Numerical results.

We now present some of the numerical results we have obtained by simulating evolutions of the economy, and comment on the most interesting features of our plots. We do not attempt to give an exhaustive list of relationships holding between variables in our economies; indeed, we shall be presenting 15 plots characterizing each run of an economy and in theory we could tell a story about a relationship holding between any pair of the plots. We will therefore concentrate on the features that we find most interesting, and will comment on those instead.

We shall refer to the unit of time as one year and, as we will see, a lot of our resulting variables make sense viewed on this time scale. Nevertheless, it is worth mentioning that we are not attempting any parameter estimation; in particular, we refer to units of capital or labour. Estimating these values and making them sensible when used as parameters in the utility function is a separate question which we are not attempting to answer in this paper.

In what follows, we shall take the default function $\psi(q)$ to be

$$\psi(q) = \lambda_\psi \left(\sqrt{q^2 + \psi_0^2/\lambda_\psi^2} - q \right), \tag{5.1}$$

where $\psi_0, \lambda_\psi > 0$.

Steady-state economy. We shall begin with considering a steady- state economy where the productivity shocks process Z_t is an exponential of an OU process. The driving process is then ergodic, and we obtain a steady-state solution. As we will see, even in this simple set up there are many interesting features visible.

We use Cobb-Douglas production function (4.14) and a log-separable utility function (4.12). The results of a sample simulation are shown on Figure 1.

Looking at the behaviour of capital, we see that the economy is experiencing moderate growth for about 1.5 years, subsequently seeing a contraction for a year, followed by a period of a rapid growth and an eventual slowdown. This is remarkably similar to what we see while observing real life economies!

Let us analyse the period when the economy is troubled around year 2. At first, the spike in the household's consumption there seems counterintuitive, noting that both labour and wages fall. A careful look at the graphs reveals what happened. The period between year 1 and year 2 shows high productivity Z, accompanied by high real wages and low commodity prices. This of course leads households to consume more, which we see by a buildup in consumption in that period. The thirst for consumption leads to withdrawal of savings from the banks to finance the purchase of goods, which in turn leads to banks having to decrease lending and lower the interest rates. This higher consumption reduces investment, leading to falling capital values and recession which we see dipping around year 2.5.

It is also interesting to note that the the fall in capital prices leads to firms being pushed to leverage forced sales, which we see by looking at the dA term on the graphs. This plays a role of a financial accelerator in our setting, similar to ideas found in Bernake *et al.* [1].

The recession brings an eventual drop in consumption, resulting in higher investment, and the economy goes back to a growth path. We see a period of fast growth in years $3-5$. This coincides with higher employment and an extremely rapid buildup of debt, even in a growing interest rates environment. Companies are so keen on taking on debt in this bull market that they quickly leverage up to their limit, which we see by noting the high leverage ratio in years $4-9$, causing periodical leverage forced sales.

When debt reaches its maximal regulated value 10, the nominal economy goes into a steady state. Prices and nominal wages, as well as interest rates, are oscillating around and do not move much.

The real economy, however, does not stay put. Increasing productivity Z_t, together with a buildup of capital and real wages, yet again lead to consumption gradually rising. Yet again, around year 8, higher consumption is choking off investment, capital levels fall, as does firm debt and employment; a new recession begins. . .

Effects of regulation. Having described our basic economy, we now proceed to investigate effects of regulation. We keep the exogenous path of process Z_t the same as in Figure 1, but change other input parameters in the model. This allows us to conduct a 'what if' study on the economy from Figure 1. We shall keep fixed axis scalings for all the scenarios we consider for easier comparison.

Firm's leverage. We consider a situation where firms are more tightly regulated, and reflect that in parameter $b = 0.3$, see Figure 2. We keep all other parameters fixed as before.

Let us now compare Figures 1 and 2. Firstly notice that tighter regulation results in much increased leverage forced sales activity, as would be expected. This in turn means that more capital in the economy gets scrapped, and hence we see that the total level of capital in the constrained economy is generally lower.

The effects of stricter regulation are particularly visible in years $3 - 8$ when the base economy experiences rapid expansion. Firms want to leverage up as much as they can, but they soon fall into a trap of leverage-forced sales. The regulation is so tight that the debt never reaches its maximal allowed value $D = 10$. Thus, a lot of capital is dissipated which makes the whole economy worse off. Output and profitability are lower; and hence so is consumption. We also see an increase in prices, labour and interest rates, and a decrease in real wages.

Perhaps the only positive outcome is that labour comes out around $0 - 1\%$ higher than in the less-regulated economy from Figure 1. This is because the drop in real wages lets the firms employ more people in an attempt to keep the productivity up.

Money supply. We have thus seen that over-regulating firms in the base economy from Figure 1 had generally negative effects. Let us now consider a scenario when banks are more tightly controlled through squeezing the monetary supply. An economy with $M = 5$ is presented on Figure 3.

The first striking feature of the economy with tighter money supply is that we see fewer leverage-forced sales, especially in years $3 - 8$ when the economy is growing. What happens is that tighter regulation on banks caps the firm's borrowing; the firms would like to borrow more, but the banks cannot lend it. This leads to firms being sheltered from leverage sales and hence the net capital actually grows faster than in the base economy from Figure 1.

The restrictions to debt are decreasing firm's profit rates, and also default forced sales $\varphi(q_t)$, yet the economy is doing better; who benefits then? The households do. We see that through increased consumption rates, as well as higher real wages. The households also have to work less and receive higher interest on their savings.

We therefore conclude that, in our simple economy, it is far more efficient to put tighter restrictions on banks than on the firms. This way the productive firms do not fall into a leverage-forced sales trap; they simply borrow slightly less when the times are good.

Having described various relationships holding in the relatively steady economy seen so far, we proceed to consider economies where the paths of Z_t oscillate more. As we will see, this leads to other interesting effects.

Business cycles. Let us now consider paths of the shock process Z_t where the departures from the mean level are more persistent than in the case of an exponentiated OU process. We want to investigate how the economy responds to persistent series of shocks over a prolonged period of time. What we will see is not just short-term price fluctuations, but effects that impact the whole economy: business cycles.

We take the time horizon of 30 years and consider paths of Z varying around the mean productivity by 5% in the economy on Figure 4 and 10% on Figure 5. Let us analyze each figure in turn.

On Figure 4, it is very clear that fluctuations in Z_t filter through the whole economy, causing cycles. Take, for example, the cycle peaking between years $4-5$, and ending around year 9. During the recession period we see effects typically associated with contracting economy: fall in consumption, investment, real wages, prices and profitability, as unemployment rises. This of course is in line with our expectations. There are a couple of interesting points to note though.

The beginning of virtually every recession starts with the forced sales. This happens when the economy peaked, output begins to fall, prices begin to fall, which in turn makes the leverage constraints bite. Forced sales lead to losses of capital, which in turn magnifies the effects of the coming recession and plays a role of a financial accelerator.

Amplification of the capital losses is not the only place where our model behaves in a non-linear way. The consumption path goes on excursions of around 10% from its mean value, twice the amount that shocks path Z does. Labour is more stable, varying by about 1%, which is in line with our assumption of stable labour markets.

The timing of the fluctuations is also of interest. Looking at the first recession, labour, nominal wages, debt and price level all bottom out at year 5, only about a year after the peak of capital, but consumption, capital and profitability keep falling for another 2.5 years, investment and real wages turn around a little earlier.

Finally, it is worth mentioning the effects on interest rates. We do not expect huge variations in interest rates due to a fixed money supply and a cap on the debt. However, we see that the responses of the rates are sensible; they fall when debt is falling.

Similar trends can be observed in Figure 5. One interesting feature is here is that our coefficient of severity of forced sales $\gamma = 0.5$ is quite high. This is shows up in the economy when the leverage constraint gets hit: we see a huge drop in capital in years $4-5$. However, the economy subsequently grows back to its original level, and the recession around year 23 is much less severe than the first recession, largely because the price level at that time was a lot higher, so the pressure of the leverage constraint was less intense.

One can in theory pick up the Scilab code and tell an interesting story about any simulation run; therefore the list of features of the model is by no means exhaustive. We find it rather remarkable that a simple model we presented leads to so many different possible scenarios; and, even more so, that the behaviour exhibited is very much in line with what actually happens.

Conclusions.

This paper has presented a complete continuous-time model of a simplified but plausible single-nation economy. By setting the model in continuous time, much of the analysis becomes substantially easier than it would be in discrete time; and we escape from the stylized but implausible stories of a sequence of trading, production then consumption which has to happen in each time period. Closed-form solution of the economy is impossible, but from equilibrium considerations we can derive sufficient relations to close the model, and this allows us to simulate an evolution of the economy.

The model has a five-dimensional state-vector, consisting of the current level of productivity shocks, firm capital, debt, labour employed, and the price level. These state variables determine via the derived equilibrium relations: real wages, consumption, investment, gross profitability, interest rates for deposits and lending, default loss rates, and the dividend rate on firm equity. Numerical examples display credible features, and offer the prospect of solving for the entire economy without resorting to various linear approximations. We believe that this is important, because we cannot expect linear approximations to work reliably when the economy is in a state which is substantially different from its recent history, as it is now. Much remains to be done: numerous parameter choices need to be made to set up the model, and it will be challenging in practice to do this well; a rôle for government, government debt and taxation needs to be introduced; reaching further, it would be good to extend the story to cover several nations, and perhaps also to allow for different types of good, and different types of labour. However, it is evident that any such model will be far too complex to be solved by paper and pencil; we will *have* to proceed numerically in the end, so our view is that we may as well therefore go numerical from the beginning. We also believe that in order to understand systemic risk it will be necessary to model the *system*; modelling only some parts of it will only lead to partial understanding. This paper is a step in that direction.

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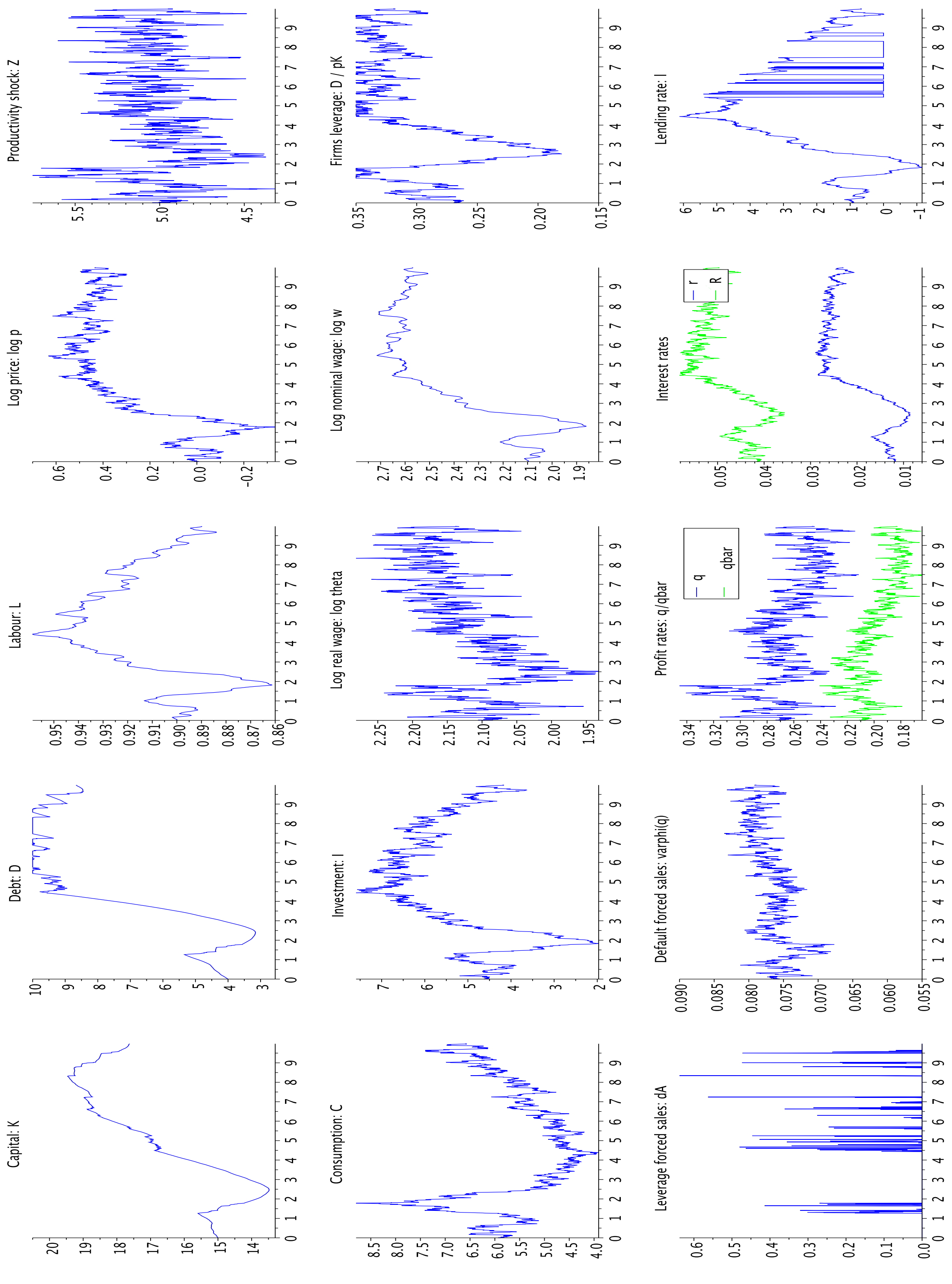

Figure 1: **Steady-state economy - base case**. Here we take $K_0 = 15$, $L_0 = 0.9$, $D_0 = 4$, $p_0 = 1$, $Z_0 = 5$,$\epsilon = 2$, $\nu = 10$, $A = 1.05$, $\mathbf{b = 0.35}$, $\kappa = 2$, $\alpha = 2$, $\mathbf{M = 10}$, $\rho = 0.05$, $\delta = 0.2$, $\gamma = 0.3$, $\sigma = 0.2$, $\lambda_\psi = 0.7$, $\psi_0 = 0.4$, $\beta = 0.3$.

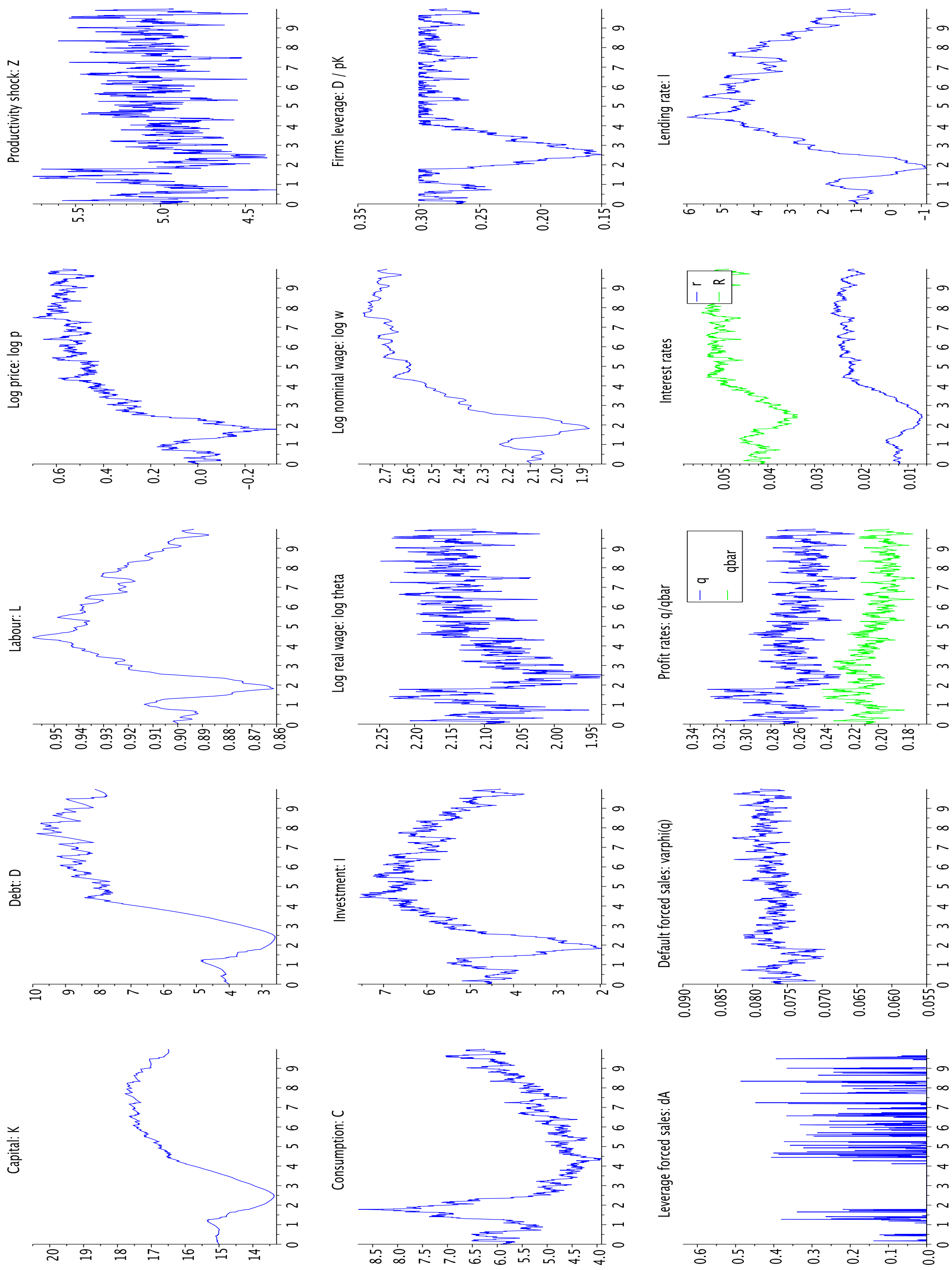

Figure 2: **Steady-state economy - firm's regulation**. Here we take $K_0 = 15$, $L_0 = 0.9$, $D_0 = 4$, $p_0 = 1$, $Z_0 = 5$,$\epsilon = 2$, $\nu = 10$, $A = 1.05$, $\mathbf{b = 0.30}$, $\kappa = 2$, $\alpha = 2$, $\mathbf{M = 10}$, $\rho = 0.05$, $\delta = 0.2$, $\gamma = 0.3$, $\sigma = 0.2$, $\lambda_\psi = 0.7$, $\psi_0 = 0.4$, $\beta = 0.3$.

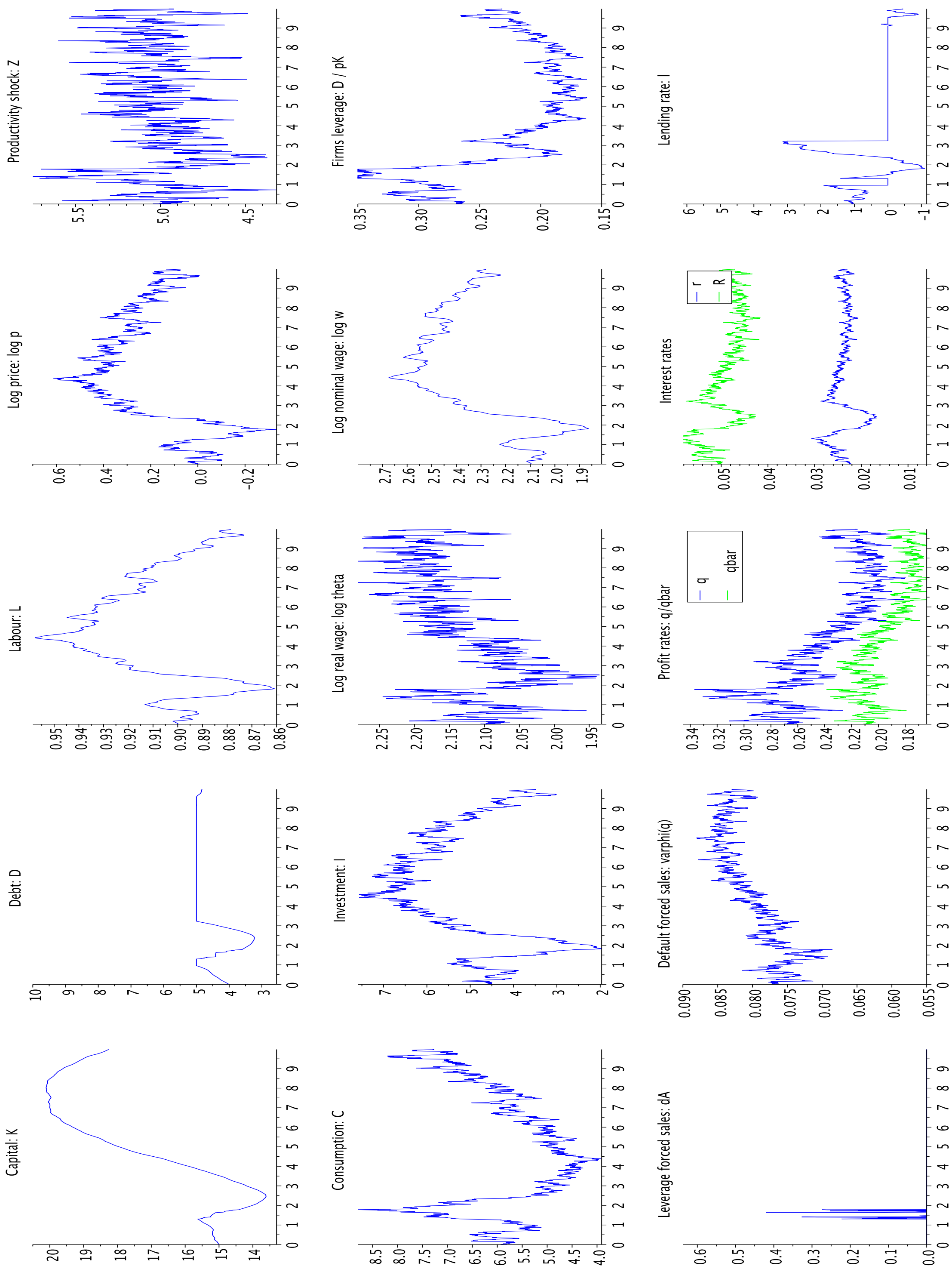

Figure 3: **Steady-state economy - lower monetary supply.** Here we take $K_0 = 15$, $L_0 = 0.9$, $D_0 = 4$, $p_0 = 1$, $Z_0 = 5$,$\epsilon = 2$, $\nu = 10$, $A = 1.05$, $\mathbf{b = 0.35}$, $\kappa = 2$, $\alpha = 2$, $\mathbf{M = 5}$, $\rho = 0.05$, $\delta = 0.2$, $\gamma = 0.3$, $\sigma = 0.2$, $\lambda_\psi = 0.7$, $\psi_0 = 0.4$, $\beta = 0.3$.

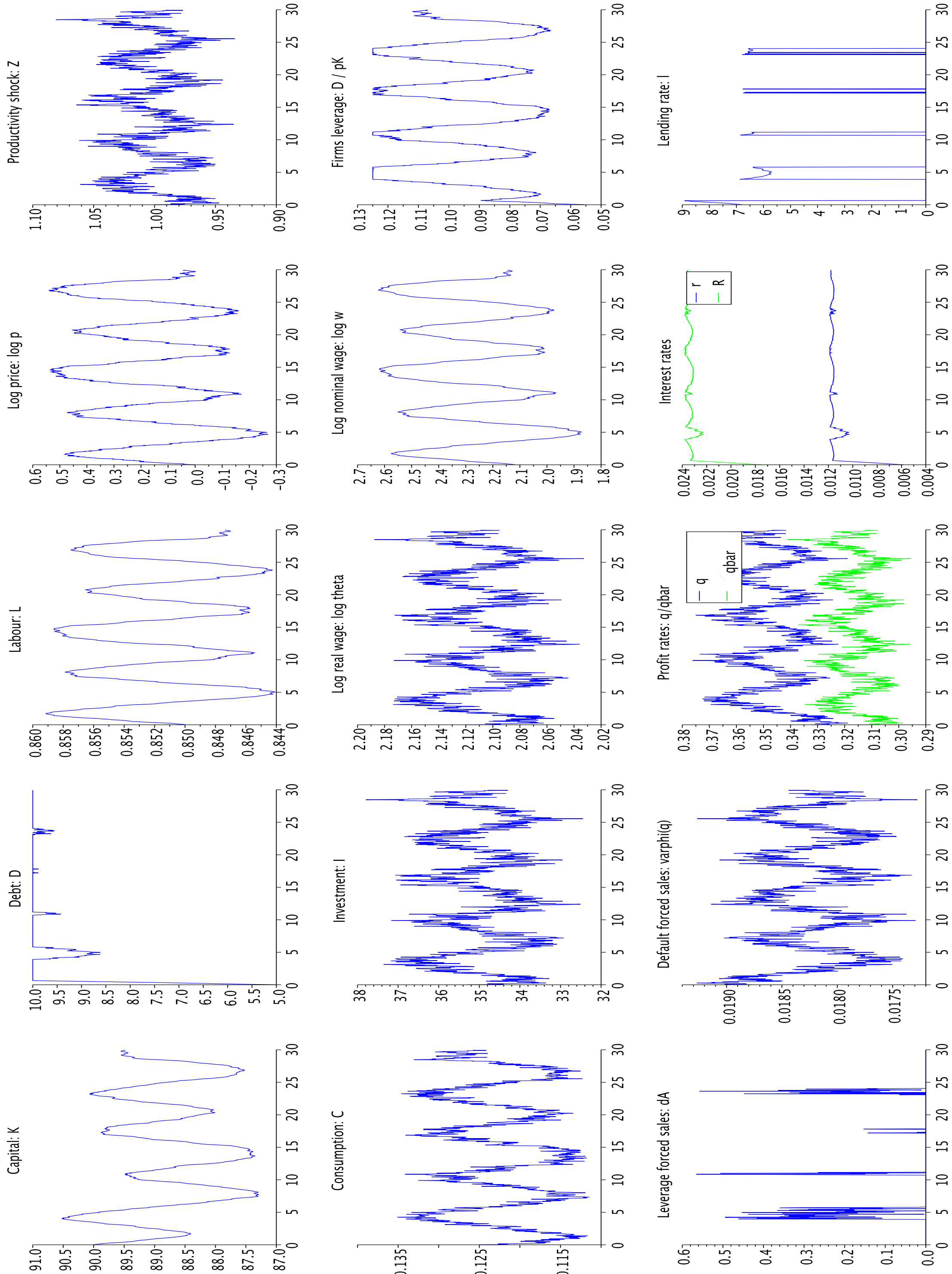

Figure 4: **Business cycle. Example 1.** Here we take utility (4.10), $K_0 = 90$, $L_0 = 0.85$, $D_0 = 5$, $p_0 = 1$, $Z_0 = 1$, $L^* = 1$, $\epsilon = 1.5$, $\nu = 15$, $A = 1$, $b = 0.125$, $\kappa = 2$, $\alpha = 2$, $M = 10$, $\rho = 0.06$, $\delta = 0.375$, $\gamma = 0.1$, $\sigma = 0.2$, $\lambda_\psi = 1$, $\psi_0 = 0.4$, $\beta = 0.8$.

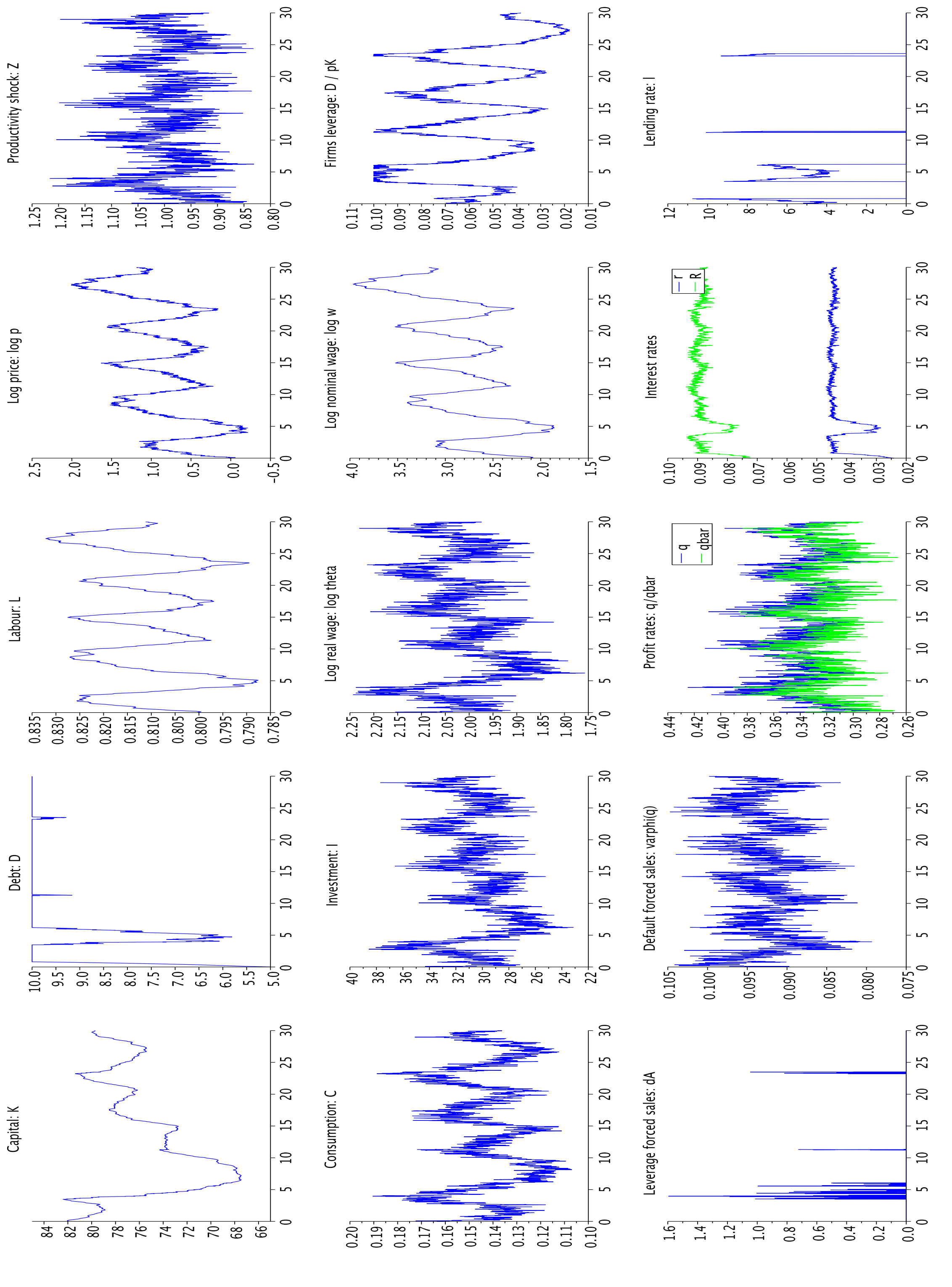

Figure 5: **Business cycle. Example 2.** Here we take utility (4.10), $K_0 = 85$, $L_0 = 0.8$, $D_0 = 5$, $p_0 = 1$, $Z_0 = 1$, $L^* = 1$, $\epsilon = 1.5$, $\nu = 15$, $A = 1$, $b = 0.1$, $\kappa = 2$, $\alpha = 2$, $M = 10$, $\rho = 0.06$, $\delta = 0.3$, $\gamma = 0.5$, $\sigma = 0.25$, $\lambda_\psi = 1$, $\psi_0 = 0.4$, $\beta = 0.8$.