Price Comparison Results and Super-replication: An Application to Passport Options †

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Abstract

In this paper, we provide a new proof of the result that option prices are increasing in volatility. This has been shown to hold for convex payoff, path-independent options by El Karoui et al [6], Hobson [12] amongst others. Moreover, monotonicity results are established for path-dependent payoffs where the payoff depends on the maximum (or minimum) of the asset price process. The techniques used to prove each of these results are mean comparison theorems of Hajek [9] and coupling of stochastic processes.

Using these, we prove that the price of a passport option is increasing in volatility for general diffusion models for the asset price. It is shown that the seller of a passport option can super-replicate if the volatility is overestimated, regardless of the strategy followed by the holder. Prices of fixed-strike lookback options are also shown to be monotonic.

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1 Introduction

A natural question which has been answered by El Karoui [6] , Hobson [12] and Bergman et al [3] is whether option price monotonicity in volatility holds for more general models than Black-Scholes. Their result, which we prove in a simpler way, says that for diffusion models with convex payoff the (non path-dependent) option price is increasing in volatility. However, nothing has been said about *path-dependent* options.

We extend the monotonicity result for a payoff involving the maximum (or minimum) of the asset price process over a time interval. This covers the case of a lookback option, refer to Goldman et al [8]. The approach used is to adapt a comparison theorem of Hajek [9] to obtain the result directly. This represents an extension of the new proof for the simple non path-dependent option price monotonicity result. Alternatively, we could use coupling of stochastic processes to extend a result by Hobson [12]. For a description of coupling techniques see Lindvall [14].

This new result involving maximums will be applied to a new type of option called a passport option (see Henderson and Hobson [10]). A passport option is a call option on a trading account where the holder(buyer) of the option undertakes a trading strategy which is subject to a constraint. At expiry, the holder receives from the option seller either the positive gains from trading or nothing if a loss was made. It is possible to prove, using our result, that the price of a passport option is increasing in volatility, where the asset price follows a general diffusion process with a realistic assumption on the diffusion coefficient. This is of interest, since the optimal strategy for the holder is in fact the same for any diffusion with non-decreasing diffusion coefficient. A second theorem of Hajek [9] is used to give a price monotonicity result for fixed-strike lookback options.

We also consider the question of hedging with a model which differs from the true dynamics of the price process. What will happen if the seller of a passport option believes and uses a model in which the volatility is consistently higher or lower than the 'true' volatility? We examine the hedging strategy which is calculated under the sellers' assumed model using market prices which are from the 'true' model. Since the seller can never know the 'true' model, this is an important practical consideration.

For a diffusion price process of the type mentioned, we prove that if volatility is overestimated by the seller, they will always have at least enough to cover the option payout and may have more than the required amount. This is called super-replication, see El Karoui and Quenez [7]. We show this occurs *regardless of the strategy* followed by the holder.

A number of authors mentioned earlier have considered this question in the context of non path-dependent options (see El Karoui et al [6] and Hobson [12]). Recent work of Dudenhausen et al [5] examines Gaussian interest rate models and concludes that overestimating volatility can cause the seller to superreplicate if the hedges are implied by Black-Scholes-type pricing formulas (said to include Gaussian term structure models and lognormal interest rate "market" models) and if certain hedging instruments are available.

The paper is structured as follows. In §2 we provide a simple proof of price monotonicity for non path-dependent options using Hajek's [9] theorem. The important extension to path-dependent payoffs is made in §3 with Theorem 3.1. The conclusion of this theorem is applied to passport options in §4.1, with the main result given in Theorem 4.1. In §4.2, the fact that the seller can superreplicate if volatility is overestimated, despite the strategy followed by the holder is proved. The final section considers the fixed-strike lookback option.

2 A Simple Proof of Option Price Monotonicity

Consider the situation of an agent misspecifying volatility and the effect this has on option prices. The question we answer is: when does the option seller overcharge for the option? The idea is to compare two models, one being that used by the seller, the other is a true reflection of the market. The comparison is done for stochastic volatility models, which have been used by Bensoussan et al [2], Cox and Ross [4] and Rubinstein [16] and shown to be more realistic than the Black-Scholes model.

Previously a number of authors have dealt with this problem using different techniques. Each has shown that (when the price is a diffusion) if the misspecified volatility dominates the true volatility then the option prices are ordered the same way, provided the payoff is convex. Bergman et al [3] analyse the pricing partial differential equation whilst El Karoui et al [6] use stochastic flow theory. Hobson [12] constructs a coupling proof based on time changing the continuous local martingale into Brownian motion.

The approach taken here is to apply a comparison theorem of Hajek [9] to achieve the result. This provides a short alternative proof, relying on an existing theorem. We do not require the true model to be a diffusion, although we do need this for the pricing model. Hajek's result was previously used in finance by Shreve and Večeř [17].

Consider a continuous time model for the economy with a finite horizon T. There is a risky asset with price S_t and for simplicity we assume that interest rates are zero, until stated otherwise. Markets are frictionless with no transactions costs or taxes and assets are infinitely divisible.

We assume that the asset price process is a continuous martingale and that a martingale measure exists. This is the pricing measure under which the asset price is a martingale. As a corollary the price of any option can be written as the expectation of the payoff under $\mathbb{P}$.

The 'true' model is as follows: under $\mathbb{P}$, the measure used for pricing, the price solves

$$d\tilde{S}_t = \tilde{\eta}_t dW_t \tag{1}$$

where $\tilde{\eta}_t$ is non-negative and adapted.

Now suppose the option seller believes (and uses) another model where $\hat{S}$ solves:

$$d\hat{S}_t = \hat{\eta}(\hat{S}_t) dW_t \tag{2}$$

with $\tilde{S}_0 = \hat{S}_0$. We assume $\hat{\eta}$ has sufficient continuity properties to ensure the solution to (2) is unique in law (for example, a Lipschitz condition on $\hat{\eta}$, see Rogers and Williams [15], Remark V.16.4). This assumption will be used throughout the paper to ensure uniqueness of weak solutions to stochastic differential equations.

We need the following theorem (see Hajek [9, Theorem 3]).

Theorem 2.1 (Hajek)

Let x be a continuous martingale with representation

$$x_t = x_0 + \int_0^t \sigma_s dW_s \tag{3}$$

such that for some Lipschitz continuous function ρ on $\mathbb{R}$

$$(4) \qquad |\sigma_s| \leq \rho(x_s)$$

and let y be the unique solution to the SDE

$$(5) \qquad y_t = x_0 + \int_0^t \rho(y_s) dW_s.$$

Then for any convex function Φ and any $t \geq 0$

$$(6) \qquad \mathbb{E}\Phi(x_t) \leq \mathbb{E}\Phi(y_t).$$

□

Theorem 2.1 will be used to prove the following monotonicity result.

Theorem 2.2 *Given a convex payoff* Φ *and* $\tilde{\eta}_t \leq \hat{\eta}(\tilde{S}_t)$*, the option price is higher under the misspecified model than the true model.*

Proof:
Given the martingale $\tilde{S}$ follows (1) and $\hat{S}$ solves (2) with $|\tilde{\eta}_t| = \tilde{\eta}_t \leq \hat{\eta}(\tilde{S}_t)$, applying Theorem 2.1 gives

$$\mathbb{E}\Phi(\tilde{S}_t) \leq \mathbb{E}\Phi(\hat{S}_t).$$

□

Remark 2.3 In Hobson [12] the diffusion coefficient in (2) may also depend on the time parameter. However, this is at the cost of requiring both models to be diffusions.

3 Price monotonicity for path-dependent options

We consider two models for the asset price with $\hat{S}_0 = \tilde{S}_0 = S_0 > 0$ which are the same as those given in (1) and (2)

$$(7) \qquad d\tilde{S}_t = \tilde{\eta}_t dW_t$$

$$(8) \qquad d\hat{S}_t = \hat{\eta}(\hat{S}_t) dW_t$$

with $\hat{\eta}$ Lipschitz continuous and $\tilde{\eta}_t$ non-negative and adapted. Denote $S^*_T = \sup_{0\le t\le T} S_t$.

Theorem 3.1 *Given $\hat{\eta}(x) \ge \tilde{\eta}_t \,\forall\, t, x$ then*

$$\mathbb{E}h(\hat{S}^*_T) \ge \mathbb{E}h(\tilde{S}^*_T)$$

for any increasing function h.

Proof:
The method will be to adapt the proof of Theorem 2.1 by Hajek [9] to deal with maximums of processes. We refer the reader to Hajek [9] for details of the proof.

In Hajek's proof of Theorem 2.1, a process z_t is introduced as a time change of x_t (solving (3)):

$$z_{\delta_t} = x_t$$

where $\delta_t = \int_0^t \frac{\sigma_u^2}{\rho(x_u)^2} du$ and $\delta_t \le t$. Analogously, define $\tau(s) = \inf\{t : \delta_t > s\}$ and

$$z_s = x_{\tau(s)}.$$

Then

$$\begin{aligned} z_s^2 - \int_0^s \rho(z_t)^2 dt &= z_s^2 - \int_0^{\tau(s)} \rho(z_{\delta_t})^2 d\delta_t \\ &= x_{\tau(s)}^2 - \int_0^{\tau(s)} \sigma_t^2 dt \end{aligned}$$

and the optional sampling theorem implies that z_t and $z_t^2 - \int_0^t \rho(z_s)^2 ds$ are continuous martingales. So z is a weak solution to (5). By uniqueness (see comments in §2), $z_t = y_t$ in law.
At this point, a convex function Φ and the optional sampling theorem are used on the submartingale $\Phi(z_s)$ to give $\mathbb{E}\Phi(z_t) \ge \mathbb{E}\Phi(z_{\delta_t})$ and thus give the result (6).

To proof, we look at the maximum process and note that

$$\sup_{t\le T} x_t \equiv \sup_{r\le \delta_T} z_r \le \sup_{r\le T} z_r \equiv \sup_{t\le T} y_t, \tag{9}$$

since $\delta_T \le T$.
Now set $|\sigma_t| = \sigma_t = \tilde{\eta}_t$ and $\rho(x_t) = \hat{\eta}(\hat{S}_t)$ giving

$$\sup_{t\le T}\tilde{S}_t \le \sup_{t\le T}\hat{S}_t. \tag{10}$$

Then

$$h(\sup_{t\le T}\tilde{S}_t) \le h(\sup_{t\le T}\hat{S}_t)$$

for h increasing and taking expectations gives the result.

□

Remark 3.2 Defining $(S_*)_T = \inf_{0\le t\le T} S_t$ we can modify the proof to obtain the corresponding result for infimums:

$$\mathbb{E}h(\hat{S}_*)_T \le \mathbb{E}h(\tilde{S}_*)_T.$$

Remark 3.3 We could have proved Theorem 3.1 using coupling by extending the result in Hobson [12]. This is done by time changing the continuous martingale to Brownian motion. By using the same Brownian motion for both models, a comparison of the time changes may be made. In this case, both models need to be diffusions with Lipschitz continuous diffusion coefficients. However, both coefficients may be time dependent: $\hat{\eta}(\hat{S}_t, t)$ and $\tilde{\eta}(\tilde{S}_t, t)$.

4 An Application to Passport Options

A passport option is a call option on the balance of a trading account. The buyer of the option pays an upfront premium and trades according to a strategy of their choice, subject to the constraint that the number of units of risky asset held (long or short) is bounded by a fixed constant. At the expiry date T, either the gains from this strategy are paid to the holder, or if the account lost money, the loss is borne by the seller giving the buyer a zero net position. The passport option was first introduced by Bankers Trust and the initial paper by Hyer et al [13] appeared in 1997.

This type of structure could be used by active fund managers to offer products with principal protection. Whilst limiting the fund participants downside risk, the fund manager is able to engage in potentially high risk strategies with the knowledge that

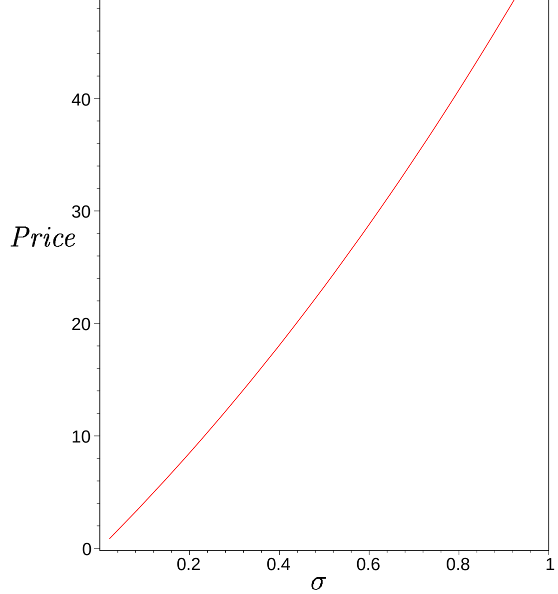

Figure 1: *Price of a Passport Option using the Black-Scholes model with* $S_0 = 100$, $G_0 = 0$ *and* $T = 1$.

they will be protected in the event of loss. However, as with other risk management tools, protection comes at a cost in the form of an initial premium.

A key problem in the pricing of passport options is determining the holders' optimal strategy. For the model in §4.1, Andersen et al [1] and Hyer et al [13] show that the holder should invest up to the allowed limit, buying when the value of the trading account is negative and selling otherwise, when the price follows exponential Brownian motion. This is shown to hold for more general diffusion models (with non-decreasing diffusion coefficient) by Henderson and Hobson [10].

It is clear that in a Black-Scholes world of exponential Brownian motion, the price of a passport option is increasing in the volatility. In Figure 1, the price of a passport option is plotted against volatility to show this result. This can also be seen directly from the pricing formula in Example 1, Henderson and Hobson [10] or Hyer et al [13]. However, if we only assume that the asset price follows a diffusion (with non-decreasing diffusion coefficient) does this property still hold? We show that the answer is yes.

We show in Theorem 4.2 that if the seller of a passport option overestimates volatility then they can superreplicate. This is independent of the strategy followed by the holder of the option. Coupling is applied to achieve the result in Lemma 4.3.

4.1 Passport Option Price Monotonicity

Assuming the (discounted) asset price S_t follows

$$dS_t = \eta(S_t, t)dW_t \tag{11}$$

whilst the undiscounted price is denoted by P_t, where $S_t = e^{-rt}P_t$. For notational simplicity, the interest rate r is constant and we further assume η is Lipschitz continuous.

Denote the holders' strategy by q_t, representing the number of units of asset held at time t. Then the gains from trade process ψ_t is given by

$$d\psi_t = r\psi_t dt + q_t\sigma_P(P_t, t)P_t dW_t$$

where $\eta(S_t, t) \equiv \sigma(S_t, t)S_t \equiv \sigma_P(e^{rt}S_t, t)S_t$.
The discounted gains from trade, $G_t = e^{-rt}\psi_t$ follows

$$dG_t = q_t dS_t.$$

A passport option with expiry T is a call option (with zero strike) on the trading account ψ_t and is defined by the payoff

$$\psi_T^+ \equiv \max(\psi_T, 0).$$

Assuming the holder follows the optimal strategy, the time 0 price of a passport option is given by

$$\sup_{|q|\leq 1} \mathbb{E}G_T^+(q). \tag{12}$$

In Henderson and Hobson [10] this price is shown to reduce to:

$$\tfrac{1}{2}\mathbb{E}(S_T^* - S_0 - |G_0(q)|)^+ + G_0(q)^+ \tag{13}$$

when $\eta(S_t, t)$ is non-decreasing in S.

Theorem 4.1 *Given $\hat{\eta}(x, t) \geq \tilde{\eta}(x, t)\ \forall\, x, t$ with $\hat{\eta}, \tilde{\eta}$ non-decreasing in x, the price of a passport option is higher under $\hat{\eta}$ than the diffusion coefficient $\tilde{\eta}$.*

Proof: Using Theorem 3.1 or particularly Remark 3.3, we know for h increasing

$$\mathbb{E}h(\hat{S}_T^*) \geq \mathbb{E}h(\tilde{S}_T^*).$$

Take $h(x) = \frac{1}{2}(x - S_0 - |G_0(q)|)^+ + G_0(q)^+$ and the result is true.

□

The transformation of the price of the passport option from (12) to a quantity involving the maximum of the price process has allowed us to use Theorem 3.1 in a straightforward fashion. The result is interesting as it is important to see the effect of a different volatility on the option price, and show that the monotonicity property remains true even for general diffusion models.

4.2 Hedging and Super-replication of Passport Options

We will now consider the situation of the option seller using an incorrect model. This question has been considered by El Karoui et al [6] and Hobson [12] in the context of non-path-dependent options with convex payoffs. They show the seller can super-replicate if volatility is overestimated and the asset price follows a diffusion under the model used. It is an important question as it means the sellers' hedging strategy is robust to model misspecification of the dynamics of the underlying asset.

Assume the seller believes and uses for pricing the model

$$d\hat{S}_t = \hat{\eta}(\hat{S}_t, t) dW_t \tag{14}$$

and the price of a passport option is given by

$$V^P(t, \hat{S}_t, \hat{G}_t(q)) = \sup_{|q| \le 1} \mathbb{E}_t \hat{G}_T^+(q). \tag{15}$$

We assume $r = 0$ for simplicity in this section.

Let q^* be the holder's optimal strategy which attains the supremum above. The price V^P is a martingale when $q = q^*$ and a supermartingale otherwise, as shown in Henderson and Hobson [10].

Suppose that the true volatility is $\tilde{\eta}(\tilde{S}_t, t)$ and that

$$\hat{\eta}(x, t) \ge \tilde{\eta}(x, t) \; \forall x, t. \tag{16}$$

Theorem 4.2 *If the seller of the passport option overestimates volatility then they can super-replicate the option, regardless of the holders' strategy.*

Proof:
Set $S_0 = \hat{S}_0 = \tilde{S}_0$ and $G_0 = \hat{G}_0 = \tilde{G}_0$. The buyer follows some strategy q, not necessarily q^*. Since we are in a complete market, we write the option payoff as the sum of the initial price plus gains from trade, under the writers' model:

$$\hat{G}_T^+ = V^P(0, S_0, G_0) + \int_0^T \theta_u d\hat{S}_u \tag{17}$$

where θ is the sellers' hedge.
Also since $V^P(t, \hat{S}_t, \hat{G}_t(q))$ is a super-martingale we have

$$\begin{aligned} \dot{V}^P(t,s,g) &+ \tfrac{1}{2}V_{SS}^P(t,s,g)\hat{\eta}^2(s,t) + \tfrac{1}{2}V_{GG}^P(t,s,g)\hat{\eta}^2(s,t)(q)^2 \\ &+ V_{SG}^P(t,s,g)\hat{\eta}^2(s,t)q \leq 0 \end{aligned} \tag{18}$$

with equality if $q = q^*$.
Using Ito also gives us the sellers' hedge as

$$\theta_t = (V_S^P(t,s,g) + qV_G^P(t,s,g)). \tag{19}$$

Now if the seller begins with an amount $V^P(0, S_0, G_0)$ and follows this hedge in the real world, he will have

$$V^P(0, S_0, G_0) + \int_0^T \theta_u d\tilde{S}_u$$

by time T. This is equivalent to

$$V^P(0, S_0, G_0) + \int_0^T (V_S^P(u, \tilde{S}_u, \tilde{G}_u) + qV_G^P(u, \tilde{S}_u, \tilde{G}_u))d\tilde{S}_u. \tag{20}$$

We may also represent the payoff in the real world by:

$$\tilde{G}_T^+ = V^P(0, S_0, G_0) + \int_0^T dV^P(u, \tilde{S}_u, \tilde{G}_u) \tag{21}$$

as $\tilde{G}_T^+ = V^P(0, S_T, \tilde{G}_T^+)$.
Using (20) and (21),

$$\begin{aligned} V^P(0, S_0, G_0) + \int_0^T \theta_u d\tilde{S}_u &= \tilde{G}_T^+ - \int_0^T dV^P(u, \tilde{S}_u, \tilde{G}_u) \\ &+ \int_0^T (V_S^P(u, \tilde{S}_u, \tilde{G}_u) + qV_G^P(u, \tilde{S}_u, \tilde{G}_u))d\tilde{S}_u. \end{aligned} \tag{22}$$

Then Ito on $V^P(u, \tilde{S}_u, \tilde{G}_u)$ gives,

$$
\begin{aligned}
dV^P(u,\tilde{S}_u,\tilde{G}_u) &= (V_S^P(u,\tilde{S}_u,\tilde{G}_u) + qV_G^P(u,\tilde{S}_u,\tilde{G}_u))d\tilde{S}_u \\
&\quad + (\dot{V}^P(t,\tilde{S}_t,\tilde{G}_t) + \tfrac{1}{2}V_{SS}^P(t,\tilde{S}_t,\tilde{G}_t)\tilde{\eta}^2(\tilde{S},t) \\
&\quad + \tfrac{1}{2}V_{GG}^P(t,\tilde{S}_t,\tilde{G}_t)\tilde{\eta}^2(\tilde{S},t)(q)^2 + V_{SG}^P(t,\tilde{S}_t,\tilde{G}_t)\tilde{\eta}^2(\tilde{S},t)q)du \\
&\leq (V_S^P(u,\tilde{S}_u,\tilde{G}_u) + qV_G^P(u,\tilde{S}_u,\tilde{G}_u))d\tilde{S}_u \qquad (23)
\end{aligned}
$$

if and only if

$$
\begin{aligned}
\dot{V}^P(t,s,g) &+ \tfrac{1}{2}V_{SS}^P(t,s,g)\tilde{\eta}^2(s,t) \\
&+ \tfrac{1}{2}V_{GG}^P(t,s,g)\tilde{\eta}^2(s,t)(q)^2 + V_{SG}^P(t,s,g)\tilde{\eta}^2(s,t)q) \leq 0.
\end{aligned}
$$

Using (18) with $q = q^*$ the above is equivalent to

$$
\begin{aligned}
\tfrac{1}{2}V_{SS}^P(t,s,g)\hat{\eta}^2(s,t) &+ \tfrac{1}{2}V_{GG}^P(t,s,g)\hat{\eta}^2(s,t)(q^*)^2 + V_{SG}^P(t,s,g)\hat{\eta}^2(s,t)q^* \\
&- (\tfrac{1}{2}V_{SS}^P(t,s,g)\tilde{\eta}^2(s,t) + \tfrac{1}{2}V_{GG}^P(t,s,g)\tilde{\eta}^2(s,t)(q)^2 \\
&+ V_{SG}^P(t,s,g)\tilde{\eta}^2(s,t)q) \geq 0
\end{aligned}
$$

There are now two cases to consider. If $q = q^*$, using (16) we need to show

$$
(\tfrac{1}{2}V_{SS}^P(t,s,g) + \tfrac{1}{2}V_{GG}^P(t,s,g)(q^*)^2 + V_{SG}^P(t,s,g)q^*) \geq 0. \qquad (24)
$$

We show this in Lemma 4.3 below.
Then for general q, we need

$$
\begin{aligned}
\tfrac{1}{2}V_{SS}^P(t,s,g) &+ \tfrac{1}{2}V_{GG}^P(t,s,g)q^{*2} + V_{SG}^P(t,s,g)q^* \\
&\geq \tfrac{1}{2}V_{SS}^P(t,s,g) + \tfrac{1}{2}V_{GG}^P(t,s,g)q^2 + V_{SG}^P(t,s,g)q
\end{aligned}
$$

which is immediate from (18).
This proves the result since now

$$
V^P(0,S_0,G_0) + \int_0^T \theta_u d\tilde{S}_u \;\geq\; \tilde{G}_T^+
$$

using (22) and (23).
It remains to prove the lemma.

Lemma 4.3 $(\frac{1}{2}V_{SS}^P(t,s,g) + \frac{1}{2}V_{GG}^P(t,s,g)(q^*)^2 + V_{SG}^P(t,s,g)q^*) \geq 0.$

Proof:
Using (18) with $q = q^*$ it is equivalent to show $\dot{V}^P(t,s,g) \le 0$. Rewrite the price as follows using (13), where $S_t = s$ and $G_t = g$:

$$\begin{aligned} V^P(t,s,g) &= \tfrac{1}{2}\mathbb{E}_t(\sup_{t<r\le T} S_r - (s+|g|))^+ + g^+ \\ &= \tfrac{1}{2}\mathbb{E}_t[\sup_{t<r\le T} S_r \vee (s+|g|)] + g^+ - \tfrac{1}{2}(s+|g|) \\ &= \tfrac{1}{2} f(s+|g|, s, T-t) + \tfrac{1}{2}(g-s) \end{aligned} \tag{25}$$

where f is given by $f(S_t^*, S_t, T-t) = \mathbb{E}_t[S_T^* | S_t^*, S_t] = \mathbb{E}_t[S_t^* \vee \sup_{t<r\le T} S_r | S_t^*, S_t]$.

We want

$$\dot{V}^P(t,s,g) \equiv \tfrac{1}{2}\dot{f}(s+|g|, s, T-t) \ \le 0. \tag{26}$$

Rewriting f as in Henderson and Hobson [10] as

$$f(S_t^*, S_t, T-t) = S_t^* + \mathbb{E}_t(\sup_{t<r\le T} S_r - S_t^*)^+$$

we will show

$$\dot{f}(S_t^*, S_t, T-t) = \frac{\partial}{\partial t}\left[S_t^* + \mathbb{E}_t(\sup_{t<r\le T} S_r - S_t^*)^+\right] \le 0. \tag{27}$$

which is equivalent to showing (26).

Intuition tells us that this result is surely true. If there is less time until maturity, the expected maximum of S should be lower. This is clearly true if the diffusion coefficient is time-independent. However in our setting, it turns out that it requires a subtle proof, involving starting S at two different times, but with the same value.

Consider two processes S^1 and S^2 which satisfy the same SDE and have the same initial conditions but start at different times. We write

$$dS_u^1 = \eta(S_u^1, u)dB_u \tag{28}$$

and

$$dS_u^2 = \eta(S_u^2, u)dB_u \tag{29}$$

with $S^1_t = S^2_{t+h} = x$ and $(S^1_t)^* = (S^2_{t+h})^* = y$. Then condition (27) will follow if

$$y + \mathbb{E}_t(\sup_{t<r\le T} S^1_r - y)^+ \ge y + \mathbb{E}_{t+h}(\sup_{t+h<r\le T} S^2_r - y)^+. \tag{30}$$

We prove this using a coupling argument. Use the Brownian motion W_t to define τ^1 as the solution to

$$\frac{d\tau^1_r}{dr} = \frac{1}{\eta(W_r + x, \tau^1_r)^2}. \tag{31}$$

Denote the inverse to τ^1 by A^1_r and define S^1 via

$$S^1_r \equiv W_{A^1_r} + x \tag{32}$$

Now

$$\frac{dA^1_r}{dr} = \eta(W_{A^1_r} + x, \tau^1_{A^1_r})^2 \equiv \eta(S^1_r, r)^2 \tag{33}$$

so that S^1 is a weak solution to (28).
Now we can use the same Brownian motion W to construct τ^2, A^2, S^2 such that $S^2_r = W_{A^2_r} + x$.
So

$$\sup_{t<r\le T} S^1_r(\omega) = \sup_{0\le m\le A^1_T} W_m(\omega) + x$$

and

$$\sup_{t+h<r\le T} S^2_r(\omega) = \sup_{0\le m\le A^2_T} W_m(\omega) + x$$

so to achieve the result it is sufficient to show that the time changes are ordered correctly, $A^1_T(\omega) \ge A^2_T(\omega)\ \forall\, \omega$.

We will prove the equivalent result for the inverse: $\tau^1_t \le \tau^2_t\ \forall\, t, \omega$.
Using the definition (31), we write

$$\frac{d\tau^1_r}{dr} = \frac{1}{\eta(W_r + x, \tau^1_r)^2}$$

and

$$\frac{d\tau^2_r}{dr} = \frac{1}{\eta(W_r + x, \tau^2_r)^2}$$

where $\tau^1_0 = t$ and $\tau^2_0 = t + h$. Hence τ^1_r and τ^2_r solve the same differential equation with different starting positions, $\tau^2_0 > \tau^1_0$.

Thus $\tau_t^1 \leq \tau_t^2 \,\forall\, t, \omega$ and so $A_t^1(\omega) \geq A_t^2(\omega) \;\forall\, t, \omega$.

□

□

Remark 4.4 If instead, volatility is underestimated, the seller can sub-replicate. This is only true when the holder follows the optimal strategy q^* and is not true in general.

5 Lookback Option Price Monotonicity

We now consider fixed-strike lookback options, see Goldman et al [8] and Heynen and Kat [11]. A fixed-strike lookback call option is a call on the highest price over the life of the option, with payoff $(P_T^* - K)^+$. The time zero price under the martingale measure $\mathbb{P}$ is:

$$e^{-rT}\mathbb{E}(P_T^* - K)^+.$$

In a zero-interest rate world, $P_T^* = S_T^*$ so Theorem 3.1 can be applied immediately. Thus the price of a fixed-strike lookback call is higher under the larger diffusion coefficient.

However, once we have non-zero interest rates, the same theorem cannot be used in a straightforward way. Assuming S_t follows (11) and $P_t = e^{rt}S_t$, then

$$dP_t = e^{rt}dS_t + re^{rt}S_t dt = e^{rt}\eta(e^{-rt}P_t, t)dW_t + rP_t dt$$

which on setting $e^{rt}\eta(e^{-rt}P_t, t) \equiv \xi(P_t, t)$ becomes

$$dP_t = \xi(P_t, t)dW_t + rP_t dt. \tag{34}$$

Theorem 5.1 *Given* $\hat{\xi}(x,t) \geq \tilde{\xi}(x,t) \,\forall\, x, t$ *and* $\hat{\xi}(x,t)$ *convex in* x, $\mathbb{E}(\hat{P}_T^* - K)^+ \geq \mathbb{E}(\tilde{P}_T^* - K)^+$.

Proof:
We have $\hat{P}_t$ solves

$$dP_t = \hat{\xi}(P_t, t)dW_t + r\hat{\mu}(P_t)dt$$

with $\hat{\mu}(P_t) = P_t$ and $\tilde{P}_t$ solves

$$dP_t = \tilde{\xi}(P_t, t)dW_t + r\tilde{\mu}(P_t)dt$$

with $\tilde{\mu}(P_t) = P_t$.
We now apply Hajek's Theorem 4.1 (see [9]) which applies to stochastic differential equations with drift. His result says that under the stated conditions, $\mathbb{E}\Phi(\hat{P}_t) \geq \mathbb{E}\Phi(\tilde{P}_t)$ for any non-decreasing convex function Φ. Again, this inequality may be strengthened to maximums, so covers the case $\mathbb{E}(\max_{t\leq T} \hat{P}_t - K)^+ \geq \mathbb{E}(\max_{t\leq T} \tilde{P}_t - K)^+$.

□

Similar arguments apply for a fixed strike lookback put with price $e^{-rT}\mathbb{E}(K - (P_*)_T)^+$, however the option price relationship is reversed.

6 Conclusion

Firstly, we have provided a short alternative proof of price monotonicity in a misspecified volatility model using a comparison theorem of Hajek [9]. This monotonicity result has been extended for path-dependent options whose payoff is a function of the maximum (or minimum) of the process in Theorem 3.1. A well known example is a lookback option, however a relatively new type of option called a passport option also falls into this category.

The extended monotonicity result has been used to obtain a comparison for passport option prices. For general diffusion models, the price of a passport option is increasing in the volatility of the asset price. This is similar to the questions asked by El Karoui et al [6] and Hobson [12] in a level dependent stochastic volatility model context. We also obtain a result for fixed-strike lookback options using a different theorem of Hajek [9].

Following from this, we have shown that regardless of the strategy followed by the option holder, the seller will always super-replicate when they overestimate volatility. This again holds for general diffusion models and is proved using a coupling argument.

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