Diverse beliefs in a simple economy - Preprint

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Abstract

This paper introduces a new way of modelling the diverse beliefs of agents within an economy. We introduce a model in which all agents receive the same stream of information. However, their different beliefs about the system cause them to behave differently, yet still rationally. We assume that there is a single risky asset in our economy, but the agents differ in their beliefs about the behaviour of this asset. These differences in belief are expressed through a change of measure. The agents then seek to maximise their expected utility of consumption under their individual measure. Using this setup, we derive an expression for the state price density of the agents. This enables us to (theoretically) calculate the price of any contingent claim. To illustrate the model further, we then concentrate on the case of log investors. We then derive expressions for the stock price and interest rate process. We also fit the model to data, and it appears to behave very reasonably. Furthermore, we exhibit how our model can explain such features as rational overconfidence and the equity premium puzzle.

1 Introduction

This paper will look at diverse beliefs and their effects on asset pricing. The motivation for this is as follows; in any financial market there are

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many different agents and in general these agents will hold different views about the market. We seek to model these views. We assume that there is a single risky asset producing a dividend continuously in time. However, the agents are unsure about how this dividend process is evolving. More specifically, there is a stochastic differential equation which governs the behaviour of the dividend process, but the agents do not know what this SDE is. The agents will form beliefs about this SDE; this will often be equivalent to forming beliefs about some unknown parameter in the SDE. These beliefs will then feed through to how the agents act in the market and will thus affect the pricing of assets.

The evidence for these diverse beliefs has been frequently remarked upon in the literature. Kurz (2007) provides an excellent illustration of this; he compares the economic predictions of financial institutions and demonstrates the huge variation that occurs. However, there has been considerable debate about why these diverse beliefs exist. Some argue that it is the existence of private information which makes people's views differ. The alternative viewpoint is that all agents receive the same information, but they process it differently to derive their views. The model which we will introduce will fall into the second of these categories.

An excellent summary of the two different classes of models is given by Kurz (2007). He begins by examining the case in which there is private information. In such models each agent receives a different signal and therefore acts differently. For example, the model of Allen, Morris and Shin (2006) uses such a model to explain the "Beauty Contest" analogy of Keynes (1936). There have also been numerous other papers looking at these private information models, of which Kurz (2007) gives an excellent overview.

However, as Kurz remarks, there seem to be some fundamental problems with this concept of private information. Firstly, what is this private information? In reality, all agents have access to largely the same information. This information might include economic indicators or the past performance of the stock. However, the important point is that the information that the agents have access to will be largely the same. Kurz also raises another problem: if this private information does exist, what could we say about it? The private nature of the information would make it very difficult for us to verify any model that relied upon it.

For these reasons, we prefer to examine the second class of models in which all agents have the same information, but use it differently. This seems to be a very natural way to model the problem, as it models what we can actually observe happening. For example, some agents may be more responsive to data than other agents, as in the paper of Harris and Raviv (1993). Alternatively, as in the model of Kandel and Pearson (1995), the agents could believe that the data comes with some normally distributed noise added to it, yet they disagree about the parameters of this noise distribution. A further option is that some agents may attach more weight to certain economic indicators than other agents. Just as in the private information case, there is a large literature in this area, of which Kurz (2007) gives an excellent summary.

In our model there are multiple agents and they each receive the same information stream. We assume that there is a single risky asset which pays a dividend continuously in time. In addition there is a riskless asset (bond, bank account) which is in zero net supply. The dividend process of the risky asset obeys some stochastic differential equation, which is not initially known to the agents. They all have different beliefs about this SDE. For example, some agents may think that the dividend process has a larger drift than others. The objective of each agent is to maximise their expected integrated utilities of consumption. We can use these optimisation problems to derive a state price density. Using this state price density we can then price stocks as the expected return under the pricing measure.

The important point is that the beliefs of the agents are expressed via a change of measure. This change of measure will be equivalent to them assuming that the dividend process obeys a different SDE. This setup is very general. For example, agents may all agree on the form of the SDE that the dividend process obeys, yet disagree on one of the terms in the SDE. One setup would be for agents to start with a belief about the drift and always keep this belief, regardless of the data that they observe. A more convincing example would be that the agents are all Bayesian; thus they start with their beliefs about the parameter and as they observe more data, this causes them to update their beliefs. Our model can embrace such update of beliefs in a natural and coherent manner.

Having introduced the basic theory, we then proceed to give a concrete example. In this example all the agents have log-utility and we will see that this simplifies the analysis significantly; expressions for the state price density and the stock price are easily derived and we use this to fit our model to the Shiller data set.

The structure of this paper is as follows. We begin in Section 2 by introducing the model in the general case and explaining the optimisation problems of each of the agents. We work this model through and derive a state price density. In Section 3 we then move onto the specific example in which all the agents have log utilities.

This provides a simple example which we can work through to derive explicit expressions for various quantities of interest. In Section 4 we proceed to fit the data to the Shiller data set. We pick parameters so that our data agrees with various moments of interest from this data and the match is very close indeed. Finally in Section 5 we look at some simulations for the log agents and explain some of the interesting effects observed.

2 The Model

We begin by illustrating our model in a very general setup; we will later go on to give a much more concrete example. We take $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P}^0)$ as our filtered probability space[1]. Our model has a single risky asset (which we will call a stock), which produces dividends continuously in time. At time t, the dividend is given by δ_t. This dividend process is assumed to follow some SDE.

There are a total of J different agents in our economy. Agent j believes that the true measure is given by $\mathbb{P}^j$ (which we assume is equivalent to $\mathbb{P}^0$). Thus, we may define the density by:

$$\Lambda_t^j = \frac{d\mathbb{P}^j}{d\mathbb{P}^0}\bigg|_{\mathcal{F}_t} \tag{2.1}$$

When the agents look at their objective, they seek to maximise it under their own measure, $\mathbb{P}^j$, rather than the reference measure, $\mathbb{P}_0$. Turning now to those objectives, agent j seeks to maximise:

$$\sup \mathbb{E}^j \int_0^\infty U_j(t, c_t^j)dt \tag{2.2}$$

where the supremum is over all possible consumption policies which always keep the wealth of agent j positive. Here, U_j is some time-dependent utility, such that $U_j(t,\cdot)$ satisfies the Inada conditions. We may express the objective in terms of the reference measure as follows:

$$\sup \mathbb{E}^0 \int_0^\infty \Lambda_t^j U_j(t, c_t^j)dt \tag{2.3}$$

2.1 Deriving the state price density

We now seek to derive agent j's state price density. Once we have found this SPD, we can calculate how much the agent will pay for any given contingent claim. To this end, we let $\pi_s^j(Y_t)$ be the price that

[1]Here, $\mathbb{P}^0$ is our reference measure and is not necessarily the "true" measure

agent j is willing to pay at time s for a contingent claim which pays amount Y_t at time $t > s$. [2] By considering the change in agent j's objective from buying this (marginal) contingent claim, the first order conditions give us:

$$0 = \pi_s^j(Y_t)U_j'(s, c_s^j)\Lambda_s^j - \mathbb{E}^0\left[Y_t U_j'(t, c_t^j)\Lambda_t^j|\mathcal{F}_s\right] \tag{2.4}$$

Rearrangement gives:

$$\pi_s^j(Y_t) = \mathbb{E}^0\left[Y_t \frac{U_j'(t, c_t^j)\Lambda_t^j}{U_j'(s, c_s^j)\Lambda_s^j}\Big|\mathcal{F}_s\right] \tag{2.5}$$

So we see that agent j has state price density given by:

$$\zeta_t^j = U_j'(t, c_t^j)\Lambda_t^j \tag{2.6}$$

If we assume that the market is complete (or that we have a central planner equilibrium), then the agents must agree on the the price of all contingent claims. So looking at the expression for $\pi_s^j(Y_t)$ and recalling that Y_t is arbitrary, we see that we must have:

$$\zeta_{t,s}^j = \frac{U_j'(t, c_t^j)\Lambda_t^j}{U_j'(s, c_s^j)\Lambda_s^j} \tag{2.7}$$

is the same for all j. So we deduce that:

$$\boxed{\zeta_t \nu_j = U_j'(t, c_t^j)\Lambda_t^j} \tag{2.8}$$

where ν_j is some $\mathcal{F}_s$ random variable. In particular, if there exists some value t_0 such that $\mathcal{F}_{t_0}$ is trivial[3] then we deduce that ν_j is in fact just a constant.

2.2 Market clearing

The importance of (2.8) is that it gives us an equation for the state price density in terms of the optimal consumption of agent j and the change of measure martingale, Λ_t^j. In many examples it may be intractable to work out the optimal consumption policy explicitly, but we can use the market clearing condition as follows. Define I_j by:

$$I_j(t, U_j'(t, y)) = y \tag{2.9}$$

[2] Here, Y_t is some bounded $\mathcal{F}_t$ random variable

[3] This will be the case in the example looked at in this paper

for any $y \in \mathbb{R}$. This is well-defined by our assumptions on U_j. We then have that:

$$I_j(t, \frac{\zeta_t \nu_j}{\Lambda_t^j}) = c_t^j \tag{2.10}$$

In particular, summing on j and using market clearing gives:

$$\sum_j I_j(t, \frac{\zeta_t \nu_j}{\Lambda_t^j}) = \delta_t \tag{2.11}$$

Thus we have managed to obtain an expression for the state price density, ζ_t, in terms of the known dividend process δ_t. This should enable us to price all contingent claims in this model. In particular, we will be able to derive the stock price, using the expression:

$$S_t = \mathbb{E}^0 \left[\int_t^\infty \frac{\zeta_u \delta_u}{\zeta_t} du \middle| \mathcal{F}_t \right] \tag{2.12}$$

2.3 Remarks on the model

2.3.1 Rational overconfidence

Kurz remarks that 'a majority of people often expect to outperform the empirical frequency measured by the mean or median'. In other words, each of the agents believes that they will usually do better than the average. In our setup, this result comes for free. If $\tilde{c}_t$ is any consumption stream and c_t^j is agent j's optimal consumption stream, then we have:

$$\mathbb{E}^j \int_0^\infty U_j(t, c_t^j) dt \geq \mathbb{E}^j \int_0^\infty U_j(t, \tilde{c}_t) dt \tag{2.13}$$

This follows simply from the fact that c_t^j is agent j's optimal consumption stream. In general, different agents will choose a different consumption stream, *even if they have the same utility functions*; this is because their beliefs are different, hence the measure under which they perform the optimisation is different. So we see that each of the agents believes that he will do better (on average) than all the other agents.

3 Log Agents

In the previous section we explained a completely general method for modelling diverse beliefs. We have also explained how we may then analyse this model to work out the state price density and hence the stock price. The above analysis made no assumptions on U_j, δ_t or Λ_t^j. We will need to pick a specific example in order to examine the behaviour further.

We now turn to one of the simplest setups, namely the one in which all agents have log-utilities. This is the easiest model to look at and it will transpire that many of the calculations are much simpler in this case.

In this setup, all the agents are log-agents, but have different impatience factors. Formally, we have:

$$U_j(t,x) = e^{-\rho_j t} \log x \tag{3.1}$$

We can now use expression (2.8) to deduce that

$$\frac{e^{-\rho_j t}\Lambda_t^j}{c_t^j} = \nu_j \zeta_t \tag{3.2}$$

Market clearing then gives:

$$\delta_t = \zeta_t^{-1} \sum_j \frac{e^{-\rho_j t}\Lambda_t^j}{\nu_j} \tag{3.3}$$

3.1 Stock Price

Formula (3.3) gives us an expression for the state price density ζ_t in terms of δ_t, and the Λ^j, ν_j and ρ_j for each of the different agents. Since these are known, we have a simple expression for ζ_t. We may then use equation (2.12) to calculate the stock price. We have:

$$\begin{align}
S_t &= \mathbb{E}^0\left[\int_t^\infty \frac{\zeta_u \delta_u}{\zeta_t} du \middle| \mathcal{F}_t\right] \tag{3.4}\\
&= \mathbb{E}^0\left[\int_t^\infty \zeta_t^{-1} \sum_j \frac{e^{-\rho_j u}\Lambda_u^j}{\nu_j} du \middle| \mathcal{F}_t\right] \tag{3.5}\\
&= \int_t^\infty \zeta_t^{-1} \sum_j \frac{e^{-\rho_j u}\Lambda_t^j}{\nu_j} du \tag{3.6}\\
&= \zeta_t^{-1} \sum_j \frac{e^{-\rho_j t}\Lambda_t^j}{\rho_j \nu_j} \tag{3.7}
\end{align}$$

where we have used the fact that Λ^j is a $\mathcal{F}_t$-martingale under $\mathbb{P}_0$. We may rewrite this as:

$$\boxed{S_t = \delta_t \frac{\sum_j \frac{e^{-\rho_j t}\Lambda_t^j}{\rho_j \nu_j}}{\sum_j \frac{e^{-\rho_j t}\Lambda_t^j}{\nu_j}}} \tag{3.8}$$

Remark Note that we have not even specified the evolution of Λ_t^j or δ_t yet; the calculation of the stock price does not require us to do so. However, in the case in which agents have utilities that are not logarithmic, it will be much harder to work out the conditional expectation in the above calculation.

3.1.1 Price-Dividend Ratio

We remark that the price-dividend ratio is given by:

$$\frac{S_t}{\delta_t} = \frac{\sum_j \frac{e^{-\rho_j t}\Lambda_t^j}{\rho_j \nu_j}}{\sum_j \frac{e^{-\rho_j t}\Lambda_t^j}{\nu_j}} \tag{3.9}$$

Note that if all agents work under the same measure, then the price-dividend ratio is deterministic. Hence, if all the agents know what process the dividend process obeys then there is no volatility in the price-dividend ratio. However, when the agents disagree about the correct measure to work under, the price-dividend ratio becomes a genuinely random process.

Note further that the price-dividend ratio only depends upon the change of measure martingales, rather than the dividend process itself. Thus, in some sense, this model exhibits the 'beauty contest' metaphor outlined by Keynes. We see that it is the beliefs of the agents that determine the price-dividend ratio, rather than it just being determined from the behaviour of the dividend process.

3.2 The dividend process and change of measure martingale

We assume that the dividend process satisfies the following SDE:

$$d\delta_t = \delta_t \sigma_t dX_t \tag{3.10}$$

where $(\sigma_t)_{t\geq 0}$ is some adapted process and X is a standard Brownian motion under the reference measure $\mathbb{P}^0$. We also assume that:

$$d\Lambda_t^j = \Lambda_t^j \alpha_t^j dX_t \tag{3.11}$$

where again the α^j's are adapted processes. Hence, under the measure $\mathbb{P}^j$, X becomes a Brownian motion with drift α_t^j; this follows from the Cameron-Martin-Girsanov theorem[4]. Formally, we have that[5] $X_t =$

[4]See Rogers and Williams, IV.38 for an account

[5]In many cases α^j will be constant and so $X_t = \tilde{X}_t^j + \alpha^j t$

$\tilde{X}^j_t + \int_0^t \alpha^j_s ds$, where $\tilde{X}^j$ is a standard Brownian motion under $\mathbb{P}^j$. This gives:

$$d\delta_t = \delta_t \sigma_t (d\tilde{X}^j_t + \alpha^j_t dt) \tag{3.12}$$

so we see that under $\mathbb{P}^j$, the dividend process gains a drift. In terms of the diverse beliefs of the agents, we see that the diverse beliefs of the agents are equivalent to them believing that the dividend process has a different drift.

Recall that $\mathbb{P}^0$ is not necessarily the true measure, but is rather our reference measure. We denote the true measure by $\mathbb{P}^*$ and assume that:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}^0}\bigg|_{\mathcal{F}_t} = \exp\{\int_0^t \alpha^*_s dX_s - \frac{1}{2}\int_0^t (\alpha^*_s)^2 ds\} \tag{3.13}$$

We then have that:

$$d\delta_t = \delta_t \sigma_t (d\tilde{X}^*_t + \alpha^*_t dt) \tag{3.14}$$

where $\tilde{X}^*$ is a $\mathbb{P}^*$ Brownian motion.

3.3 Interest Rate Process

Recall that there is a riskless asset which is in zero net supply. We now move onto examining the interest rate process. We have from (3.3) that:

$$\zeta_t = \delta_t^{-1} \sum_j \frac{e^{-\rho_j t} \Lambda^j_t}{\nu_j} \tag{3.15}$$

so we may perform an Itô expansion and then express the state price density in the form:

$$d\zeta_t = \zeta_t(-r_t dt - \kappa_t dX^*_t) \tag{3.16}$$

where

$$r_t = \sigma_t \alpha^*_t - \sigma_t^2 + \frac{\sum_j e^{-\rho_j t} \Lambda^j_t \rho_j / \nu_j}{\sum_j e^{-\rho_j t} \Lambda^j_t / \nu_j} + (\sigma_t - \alpha^*_t) \frac{\sum_j e^{-\rho_j t} \Lambda^j_t \alpha^j_t / \nu_j}{\sum_j e^{-\rho_j t} \Lambda^j_t / \nu_j} \tag{3.17}$$

$$\kappa_t = \sigma_t - \frac{\sum_j e^{-\rho_j t} \Lambda^j_t \alpha^j_t / \nu_j}{\sum_j e^{-\rho_j t} \Lambda^j_t / \nu_j} \tag{3.18}$$

3.3.1 Remarks on interest rate

The expression for r_t gives us an interest rate process for the model. If we consider the simple case in which σ and α are positive constants, we see that increasing any α^j causes the interest rate to increase. This is because this agent will believe that the risky asset is going to perform better; thus he will not want to hold a riskless asset unless it can give a better rate of return.

We also see that in the case in which all the agents work under the same measure then the interest rate is deterministic. However, when the agents disagree, this adds volatility to the interest rate process.

3.4 Wealth Process

Let w_t^j be the wealth of agent j at time t. Then we must have that:

$$w_t^j = \mathbb{E}^0\left[\int_t^\infty \frac{\zeta_u c_u^j}{\zeta_t} du \big| \mathcal{F}_t\right] \tag{3.19}$$

$$= \mathbb{E}^0\left[\int_t^\infty \frac{e^{-\rho_j u}\Lambda_u^j/\nu_j}{\zeta_t} du \big| \mathcal{F}_t\right] \tag{3.20}$$

$$= \zeta_t^{-1} e^{-\rho_j t}\Lambda_t^j/\nu_j\rho_j \tag{3.21}$$

Note that:

$$c_t^j = \rho_j w_t^j \tag{3.22}$$

just as in the case in which all the case in which all agents have the same belief. However, note that in this model of diverse beliefs, the wealth and consumption process will depend on the diverse beliefs.

Note further that $\sum_j w_t^j = S_t$. Since there are only two assets (the stock and the bond) and the bond is in zero net supply, we could have deduced this directly. We may then perform an Itô expansion on w_t^j to deduce that:

$$dw_t^j = w_t^j\{-\rho_j dt + (\alpha_t^j + \kappa_t)dX_t + (r_t + \kappa_t^2 + \alpha_t^j\kappa_t)dt\} \tag{3.23}$$

But the wealth dynamics of agent j are:

$$dw_t^j = \pi_t^j(dS_t + \delta_t dt) - c_t^j dt + (w_t^j - \pi_t^j S_t)r_t dt \tag{3.24}$$

where π_t^j is the proportion of the risky asset held by agent j. Hence, we may deduce that:

$$\boxed{\pi_t^j = \frac{w_t^j(\alpha_t^j + \kappa_t)}{\sum_i w_t^i(\alpha_t^i + \kappa_t)}} \tag{3.25}$$

3.4.1 Volume of trade

Equation (3.25) gives us an expression for the amount of the risky asset held by agent j. We have that:

$$d\pi_t^j = \pi_t^j(\sigma_{\pi^j} dX_t + \mu_{\pi^j} dt) \tag{3.26}$$

where σ_{π^j} and μ_{π^j} are some processes. In the case where all the agents have the same belief, we have that:

$$\pi_t^j = \frac{e^{-\rho_j t}/\nu_j \rho_j}{\sum_i e^{-\rho_i t}/\nu_i \rho_i} \tag{3.27}$$

So in the case in which all the agents agree in their beliefs, there is no volatility in the evolution of π_t^j. However, when agents do disagree, then there will be a lot of volatility in π_t^j, which can explain the volume of trading. Indeed, we may show that, in the case where the α's are constant but different:

$$\sigma_{\pi^j} = \frac{(\kappa_t - \sigma_\kappa)\sum_i w_t^i(\alpha^j - \alpha^i) + \sum_i w_t^i \alpha^i(\alpha^j - \alpha^i)}{\sum_i w_t^i \alpha^i + \kappa_t S_t} \tag{3.28}$$

where

$$\sigma_\kappa = \left(\frac{\sum_i e^{-\rho_i t}\Lambda_t^i \alpha^i/\nu_i}{\sum_i e^{-\rho_i t}\Lambda_t^i/\nu_i}\right)^2 - \frac{\sum_i e^{-\rho_i t}\Lambda_t^i (\alpha^i)^2/\nu_i}{\sum_i e^{-\rho_i t}\Lambda_t^i/\nu_i} \tag{3.29}$$

3.5 Bayesian Agents

The case in which all the α^j are constant corresponds to that in which the agents all start with a belief about the behaviour of the dividend process and stick with this forever. Such a setup is in some senses unsatisfactory, because even if the agents were to observe that the behaviour of the dividend were very different to their initial beliefs about it, they would still keep with these initial beliefs.

Therefore, we now consider the case of Bayesian agents. All the agents believe that the dividend process satisfies the SDE:

$$d\delta_t = \sigma\delta_t(dX_t + b dt) \tag{3.30}$$

where σ is assumed known, but b is some constant unknown to the agents. Instead of making an initial guess at the value of b and sticking with it, the agents choose to give a prior distribution to the unknown parameter b and then update this prior distribution as time progresses. If the agents were sure about b, then they would have:

$$\Lambda_t^j = \exp\{bX_t - \frac{1}{2}b^2 t\} \tag{3.31}$$

However, these agents are unsure about b, so will instead give it a prior distribution. We assume that agent j has prior distribution which is normal with mean β^j and precision ϵ^j.[6] It follows that the change of measure that agent j works with is given by:

$$\Lambda_t^j = \int_{\infty}^{\infty} \sqrt{\frac{\epsilon^j}{2\pi}} \exp\{-\frac{\epsilon^j}{2}(b' - \beta^j)^2 + b'X_t - \frac{1}{2}(b')^2 t\} db' \tag{3.32}$$

$$= \sqrt{\frac{\epsilon^j}{\epsilon^j + t}} \exp\{\frac{X_t^2 + 2\beta^j \epsilon^j X_t - \epsilon^j (\beta^j)^2 t}{2(\epsilon^j + t)}\} \tag{3.33}$$

This gives:

$$\Lambda_t^j = \Lambda_t^j \alpha_t^j dX_t \tag{3.34}$$

where:

$$\alpha_t^j = \frac{X_t + \beta^j \epsilon^j}{\epsilon^j + t} \tag{3.35}$$

This is of the form described in the previous section, but the α_t^j are now adapted processes rather than constants. Thus, our model can deal with intelligent agents who update their beliefs as well as the simple agents who always hold the same beliefs.

4 Fitting the model to data

We now proceed to fit our model to data. In keeping with Kurz, we choose the parameters of our model so that various economic quantities from our model are matched with those from observed data. The data that we compare our model with is the Shiller data set. The quantities of interest are shown in the table below; we list both the empirical value (calculated by Kurz, based on the Shiller data set) and the values as produced by our model.

We assume that there are just two agents in our model. Furthermore, we assume that they never change their beliefs, so we assume that the α^j are constant. We also take σ_t to be constant.

The results shown were generated by choosing $\sigma = 0.044, \alpha^* = 1.614, \alpha^1 = -1.396, \alpha^2 = 1.460, \rho_1 = 0.031, \rho_2 = 0.035, \nu_1 = 1.819, \nu_2 = 5.523$

[6] This is equivalent to having variance $(\epsilon^j)^{-1}$

Table 1: Simulation Results

	Fitted from Model	Empirical
Mean price/dividend ratio	27.8	25
Standard deviation of price/dividend ratio	0.009	7.1
Mean return on equity	0.070	0.07
Mean riskless rate	0.0086	0.01
Standard deviation of riskless rate	0.0517	0.057
Equity Premium	0.0618	0.06
Sharpe Ratio	0.321	0.33

4.1 Comments on results and the equity premium puzzle

We see from the above results that the given parameters provide a very good fit to the Shiller data set. The only quantity that does not seem to match that well is the standard deviation of the price/dividend ratio. However, we see that the model provides an excellent match between the observed equity premium and the one computed from our model. So the model that we have illustrated seems to provide a possible resolution of the equity premium puzzle.

To see why this model can explain the equity premium puzzle, note that in the setup illustrated above we have two sets of agents with differing beliefs. The first group of agents is much more pessimistic about the dividend process, believing that in fact it has a negative drift. In contrast, the second group of agents is much more positive. This therefore explains why investors are happy to take such a low rate of return on the riskless asset; they believe that the dividend process will do badly and so they are happy just to hold the riskless asset. This is in keeping with the observations of Kurz in his model.

5 Simulations

We now proceed to look at some simulations of the various quantities that we have derived above. To do this, we generate a random simulation of the driving Brownian motion; we then use this to derive all the other quantities of interest. We then plot these quantities to illustrate their behaviour.

In order to generate our simulations, we begin by generating our driving process X_t, which is a Brownian motion under the reference measure $\mathbb{P}^0$. However, under the true measure, $\mathbb{P}^*$, X will be be a Brownian motion with drift α^*. We choose a constant drift of $\alpha^* =$

0.05. We also take the the volatility of the dividend process to be $\sigma = 0.1$. We produce simulations over 50 years. The evolution of the X_t and δ_t are shown in Figure 1.

We now come to specify the parameters of the agents. We will perform the simulations in two separate cases. In the first of these, we assume that the α^j's are constant. Thus, in this simulation the agents all have fixed beliefs and do not change them. We assume that there are two agents whose parameters are $\nu_1 = \nu_2 = 0.5, \rho_1 = \rho_2 = 0.4$, $\alpha^1 = 0$, $\alpha^2 = 0.2$. Thus, agent 1 is very pessimistic, as he believes that the dividend process has no upwards drift at all, whereas agent 2 is an optimist. The output is shown in Figures 2-6.

Having produced these simulations, we move onto the case in which agents are Bayesian. Thus they start with some prior belief and as they observe more and more of the dividend process they change their beliefs. In order to make our simulations comparable with the ones above, we use exactly the same realisation of the driving Brownian motion. We also take $\nu_1 = \nu_2 = 0.5, \rho_1 = \rho_2 = 0.4$, just as before. We must also specify the prior distribution that the agents have for the drift of the dividend process. Using the same notation as in section 3.5 , we choose $\beta^1 = 0, \beta^2 = 0.2$ and $\epsilon^1 = \epsilon^2 = 10$. Thus at the start of the simulation the agents' maximum likelihood estimate for the drift of the Brownian motion is the same as in the simulations above. However, as the agents observe more data, they will change their beliefs, thus affecting their behaviour.

5.1 Comments on Simulations

5.1.1 Simulation with fixed beliefs

We see from Figure 5 that initially agent 2 holds far more of the risky asset than agent 1. This is to be expected, since agent 2 is much more optimistic about the dividend process than agent 1, so will want to hold more of the risky asset. Hence, when the value of the stock falls, the wealth of agent 2 also falls heavily, as can be seen in Figure 4. This fall in wealth causes him to rebalance his portfolio so that he holds less of the risky asset. Furthermore, since the agents always consume a fixed proportion of their wealth, this explains why the consumption of agent 2 also falls when the stock price initially decreases.

Figure 6 shows the standard deviation of the proportion of the risky asset held by the agents. This can quantity can be interpreted as the volume of trade. The simulation indicates that if the stock price is falling, then there is a large volume of trade. In contrast, when the stock price rallies, the volume of trade decreases.

5.1.2 Simulation with Bayesian updating of beliefs

We now look at the results from the case in which the agents update their beliefs as time progresses. Figure 7 shows how the α^j's evolve for the different agents. We see that these α^j's start quite far apart, but get closer to each other as the agents observe more data. Hence, as time progresses the agents beliefs become closer to each other, as we would expect. Figure 8 shows that at first the consumption of agents is similar to the case in which they have fixed beliefs. However, as time progresses, the consumption of the agents seems to match each other more and more closely. This is again to be expected, since the agents have very similar beliefs and so they will consume in a similar manner. This is also reflected in Figure 9, where we see that the proportion of risky asset held by the different agents converges, as their beliefs converge.

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Figure 1: Evolution of X_t and the dividend process

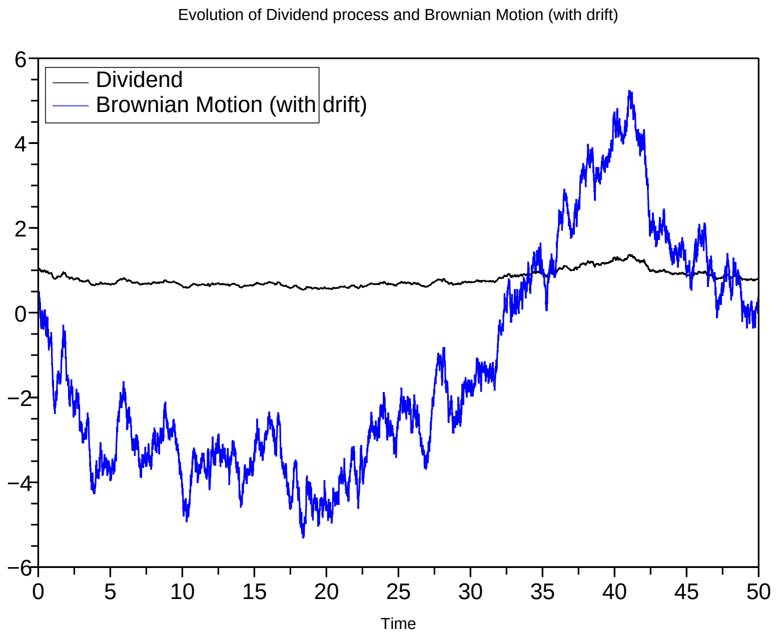

Figure 2: Evolution of S_t

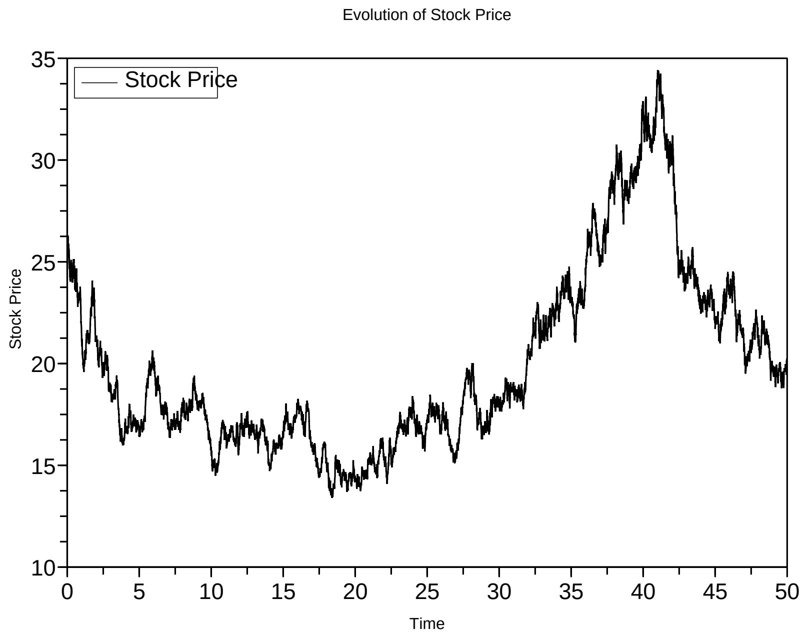

Figure 3: Consumption of agents (Fixed beliefs)

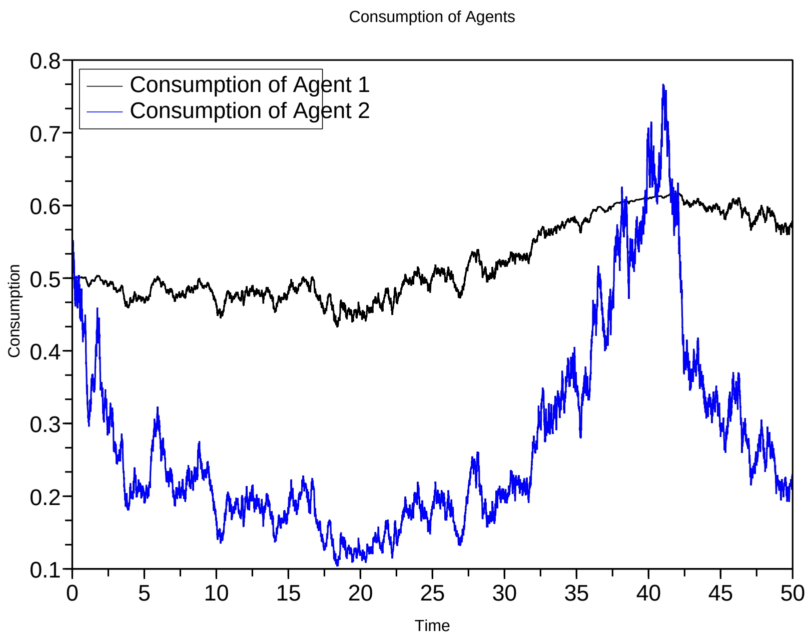

Figure 4: Wealth of agents (Fixed beliefs)

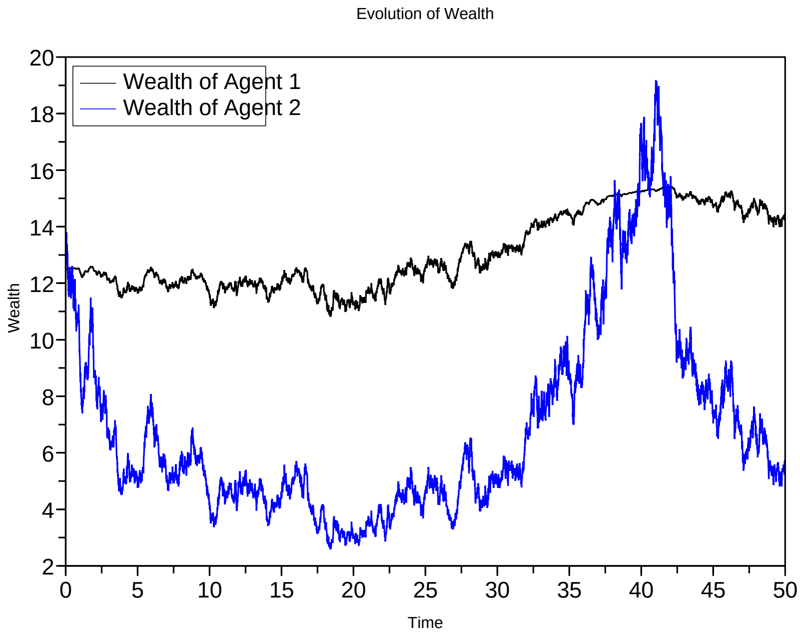

Figure 5: Proportion of risky asset held (Fixed beliefs)

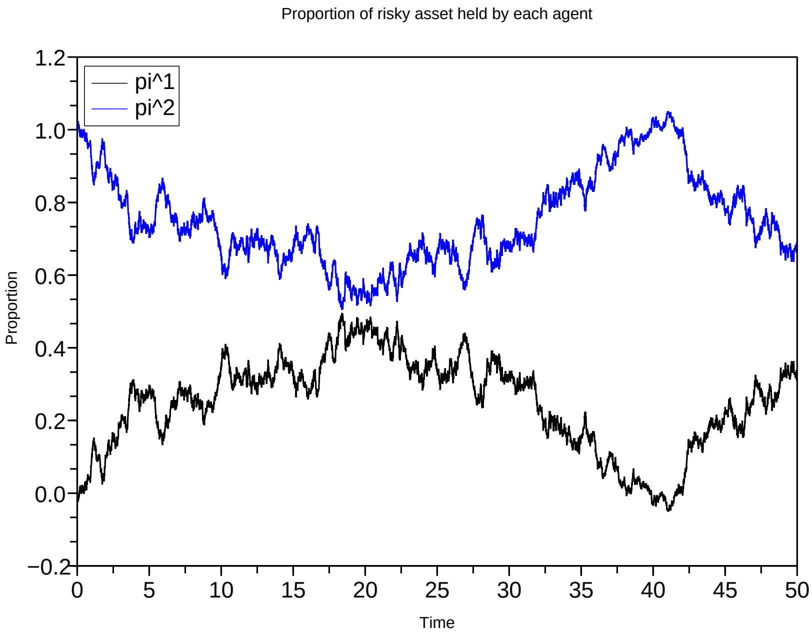

Figure 6: Standard deviation of proportion of risky asset held (Fixed beliefs)

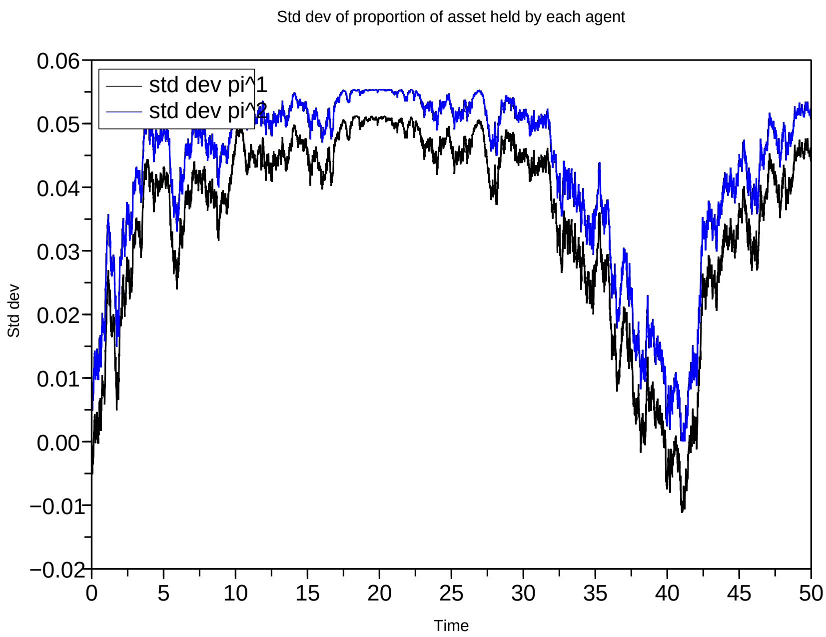

Figure 7: Evolution of α^j's of different agents (Bayesian updating)

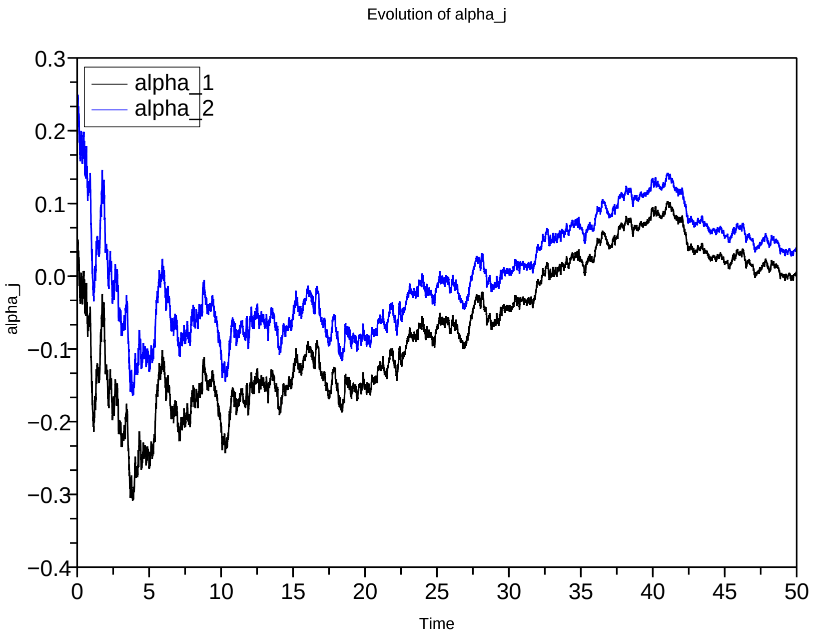

Figure 8: Consumption of agents (Bayesian updating)

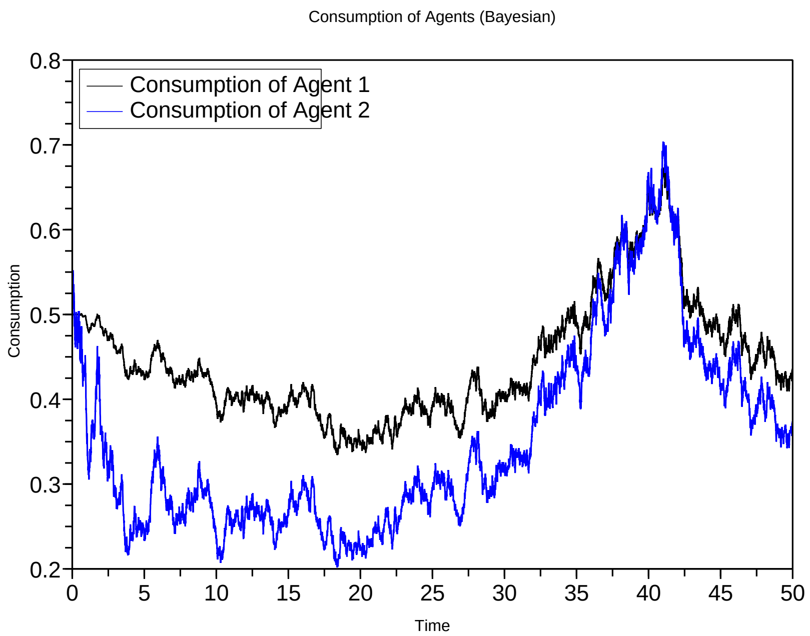

Figure 9: Proportion of risky asset held (Bayesian updating)