The Bayesian agent and the co-movement of security prices

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Abstract

This paper introduces a Bayesian approach to explain co-movements in stock prices. The economy consists of a single agent and n production activities (stocks) paying a Brownian dividend flow δ with unknown drift μ. The drift of the dividend process is inferred in a Bayesian fashion from observed dividends. In the observation filtration of the agent, the dividend process is no longer drifting Brownian motion, but follows some more complicated (yet completely explicit) time-dependent dynamics. A simple equilibrium analysis allows us to specify the state price density of the economy uniquely. Stock prices are recovered by appropriately discounting future dividends via the state price density. We show that

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in this set-up a shock to the observed dividend of one stock will affect the prices of all the other stocks, primarily because of market clearing, but also because of the Bayesian learning.

1 Introduction

In this paper we introduce a simple multi-asset intertemporal financial model to explain co-movements and contagion effects between security prices.

The economy consists of a single agent and n activities, each one producing a stochastic output (dividend) at rate δ_t^i at time t. The dividend process δ^i is assumed to be observable. We further assume that the drift μ of the stochastic dividends are not known by the agent but must be inferred from the observation of the vector δ_t. The agent maximizes his utility from consumption subject to a budget constraint. Market clearing determines the equilibrium prices of the assets, allowing us to express the state price density of the economy ζ as a function of δ.

In this set-up, stock prices are calculated as the net present value of all future dividends. Thus changes in dividend rate cause changes in the stock prices. When observing a change in the dividend level of the production activities, the Bayesian agent updates his posterior on μ^i to incorporate this new information. But this will affect, through the state price density and market clearing, the distribution of the drift of the dividend processes, and thus the price of the remaining assets. The key point here is that the pricing kernel ζ is a function of the vector process δ, that is, of *all* dividends, so changes in the dividend rate of one asset will change the posterior distribution of ζ, and thus affect the prices of *all* assets in the market.

This (Bayesian) point of view offers a possible mechanism to explain why trading in the New York stock exchange may have an impact on asset prices in London or Tokyo.

In a related paper, Veronesi [4] proposes a model in which the dividend of a single risky asset evolves as a log-Brownian motion, whose drift is a finite-state Markov chain observed noisily. He then derives expressions for the dynamics of the equilibrium price of the asset in this model. Our modelling assumptions differ significantly from his: we take multiple risky assets, we assume that the growth rate of the dividend process is constant but unknown, and we assume that the observation process *is* the dividend process. Neither set of modelling assumptions contains the other. Our objective here is to understand the co-movement of many assets, and particularly to see the extent to which simple Bayesian updating can account for what is loosely described as contagion. Our modelling assumptions appear to be the most parsimonious for our objective.

2 Model set-up

The economy consists of a single representative agent and n production units (firms). The n firms generate a stochastic dividend flow, paid continuously, which we model by the vector process

$$d\delta_t \equiv \sigma(dW_t + \alpha dt), \tag{1}$$

where W_t is an n-dimensional Brownian motion, σ is the $n \times n$ volatility matrix of the dividend process and α is the *volatility adjusted* drift of δ. The volatility matrix σ is assumed to be known[1] and non-singular. In our set-up the parameter α is not known with certainty; rather, the agent takes a prior density

$$f_0(\alpha) \propto \varphi(\alpha) \exp\big[\, -\tfrac{1}{2}(\alpha - \hat{\alpha}_0) \cdot \tau_0(\alpha - \hat{\alpha}_0) \, \big] (2\pi)^{-n/2} \sqrt{\det \tau_0} \tag{2}$$

for α. Ignoring the prefactor φ, this is just the assumption that the parameter α has a multivariate Gaussian prior. For reasons that we shall explain in Section 3, we shall take for the prefactor

$$\varphi(\alpha) \equiv (\rho - \tfrac{1}{2}|v|^2 - \alpha \cdot v)^2 I_A, \tag{3}$$

where $A \equiv \{\alpha : \tfrac{1}{2}|v|^2 + \alpha \cdot v < \rho\}$, and

$$v \equiv -\gamma \sigma^T \mathbf{1}. \tag{4}$$

The agent will have to filter the value of α from observation of the dividend process δ_t. Equivalently, we can rewrite

$$\delta_t \equiv \sigma X_t \equiv \sigma(W_t + \alpha t)$$

and estimate α from X_t. More precisely, if we define $\mathcal{G}_t \equiv \sigma(\{X_u : 0 \leq u \leq t\})$, then we are looking to derive $\hat{\alpha}_t \equiv E[\alpha \mid \mathcal{G}_t]$.

As a piece of notation, we shall write $E^0[\cdot]$ for expectation computed under the assumption that the prior is just the standard Gaussian prior (that is, we take $\varphi \equiv 1$ in (2)), so that we have for any bounded Y, and any $t \geq 0$

$$E[\, Y \mid \mathcal{G}_t] = \frac{E^0[\, \varphi(\alpha) Y \mid \mathcal{G}_t]}{E^0[\, \varphi(\alpha) \mid \mathcal{G}_t]}. \tag{5}$$

[1] The assumption of known σ is quite substantive, but when estimating the dynamics of a stock it is well known that the rate of growth is far harder to estimate well than the volatility.

2.1 The filtering problem

The filtering problem facing the agent is of conventional type; see, for example, [2] for a more detailed account. Note that the appearance of the prefactor φ in (2) does not affect the prior-to-posterior Bayesian updating; it simply stands in front of the posterior as it stood in front of the prior. We shall briefly summarise here the results and their derivation, as we shall have need of their precise form. A simple approach to the problem is to update the law of α using the likelihood for a path $(X_s)_{0\leq s\leq t}$ with respect to the Wiener measure[2]:

$$\exp\left(\alpha \cdot X_t - \tfrac{1}{2}|\alpha|^2 t\right) \tag{6}$$

The posterior density for α, given $(X_s)_{0\leq s\leq t}$ will be

$$\begin{aligned} f_t(\alpha) &\propto \varphi(\alpha)\exp\left(\alpha \cdot X_t - \tfrac{1}{2}|\alpha|^2 t - \tfrac{1}{2}(\alpha - \hat{\alpha}_0)\cdot \tau_0(\alpha - \hat{\alpha}_0)\right) && (7)\\ &\propto \varphi(\alpha)\exp\left(-\tfrac{1}{2}(\alpha - \hat{\alpha}_t)\cdot \tau_t(\alpha - \hat{\alpha}_t)\right) && (8) \end{aligned}$$

where $\hat{\alpha}_t$ admits a closed form expression given by

$$\begin{aligned} \tau_t &\equiv \tau_0 + tI && (9)\\ \hat{\alpha}_t &\equiv \tau_t^{-1}(\tau_0\hat{\alpha}_0 + X_t). && (10) \end{aligned}$$

Under this updating rule, the posterior for α is again a multivariate gaussian, apart from the prefactor. Notice however that we may not interpret $\hat{\alpha}_t$ as the posterior mean of α given $\mathcal{G}_t$, because of the prefactor.

2.2 Equilibrium analysis

As in the classical equilibrium approach, we assume that there exists a single representative agent, maximizing his expected utility from consumption. More precisely the agent will face the following problem:

$$\sup_c E\left[\int_0^\infty e^{-\rho u} U(c_u)\, du\right] \tag{11}$$

subject to the budget constraint

$$E\left[\int_0^t \zeta_u(\mathbf{1}\cdot\delta_u)\, du\right] = E\left[\int_0^t \zeta_u c_u\, du\right], \tag{12}$$

[2] This is the celebrated Cameron-Martin-Girsanov Theorem; see, for example, [3], IV.38.

where the process ζ is the state price density, or pricing kernel, of the simple economy.

The above optimization problem yields the following standard equality

$$e^{-\rho t}U'(c_t^*) = \lambda\zeta_t, \tag{13}$$

for some Lagrange multiplier $\lambda > 0$. By requiring market clearing, $c_t = \mathbf{1}\cdot\delta_t$, we can write the state price density in terms of the dividend vector process,

$$\zeta_t \propto e^{-\rho t}U'(\mathbf{1}\cdot\delta_t). \tag{14}$$

In order to do computations, we need to put a bit more structure to the model, so we assume that the utility function of the agent is of CARA type, with coefficient γ. The state price density becomes

$$\zeta_t = \exp\left(-\rho t - \gamma\mathbf{1}^T\delta_t\right) \equiv \exp\left(-\rho t + v\cdot X_t\right), \tag{15}$$

where $v \equiv -\gamma\sigma^T\mathbf{1}$.

3 Asset prices

3.1 The stock price

The vector of stock prices is simply the net present value of all future dividends:

$$S_t \equiv \frac{1}{\zeta_t}E\left[\int_t^\infty \zeta_u\delta_u du \,\middle|\, \mathcal{G}_t\right]. \tag{16}$$

Our goal in this section is to derive (29) an explicit form for the stock price. Substituting the expression for the dividend vector δ and the state price density ζ into (16), we have

$$\begin{aligned} S_t &= E\left[\int_t^\infty \exp\left(-\rho(u-t) + v\cdot(X_u - X_t)\right)\sigma X_u du \,\middle|\, \mathcal{G}_t\right] \\ &= E\left[\int_t^\infty \exp\left(-\rho(u-t) + v\cdot(X_u - X_t)\right)\sigma(X_t + \{X_u - X_t\})du \,\middle|\, \mathcal{G}_t\right]. \end{aligned} \tag{17}$$

Observing that for $t < u$

$$E[\ \exp\{\ v\cdot(X_u - X_t)\ \} \mid \mathcal{G}_t \vee \sigma(\alpha)\ ] = \exp\{\ (\tfrac{1}{2}|v|^2 + \alpha\cdot v)(u-t)\ \},$$

we learn by differentiating with respect to v that

$$E[\ \exp\{\ v\cdot(X_u - X_t)\ \}\ (X_u - X_t)\ |\ \mathcal{G}_t \vee \sigma(\alpha)\ ] = \exp\{\ (\tfrac{1}{2}|v|^2 + \alpha\cdot v)(u-t)\ \}(\alpha + v)(u-t),$$

and so after conditioning first on the larger σ-field $\mathcal{G}_t \vee \sigma(\alpha)$ the expression for the stock price (17) becomes

$$S_t = \sigma E\left[\int_t^\infty \exp\{-(\rho - \tfrac{1}{2}|v|^2 - \alpha\cdot v)(u-t)\}(X_t + (\alpha+v)(u-t))\ du\ \middle|\ \mathcal{G}_t\right]. \quad (18)$$

To understand why the prefactor φ was needed in the prior density (2), let us suppose that in fact $\varphi \equiv 1$, so that the stock price is

$$S_t = \sigma E^0\left[\int_t^\infty \exp\{-(\rho - \tfrac{1}{2}|v|^2 - \alpha\cdot v)(u-t)\}(X_t + (\alpha+v)(u-t))\ du\ \middle|\ \mathcal{G}_t\right].$$

This expression is always *infinite*[3], because the set $A^c \equiv \{\alpha : \rho - \frac{1}{2}|v|^2 - \alpha\cdot v \le 0\}$ gets positive weight under the law P^0. However, converting P^0 into P by introducing the prefactor[4] φ as at (3) rescues the situation; the integral with respect to u in the expression (18) for the stock price can be evaluated, since the term $\rho - \frac{1}{2}|v|^2 - \alpha\cdot v$ in the exponent is almost always negative, and we end up (as at (5)) with

$$\begin{aligned} S_t &= \sigma E\left[\frac{X_t}{\rho - \frac{1}{2}|v|^2 - \alpha\cdot v} + \frac{\alpha+v}{(\rho - \frac{1}{2}|v|^2 - \alpha\cdot v)^2}\ \middle|\ \mathcal{G}_t\right] && (19)\\ &= \sigma\ \frac{E^0[\ \{\ X_t(\rho - \frac{1}{2}|v|^2 - \alpha\cdot v) + \alpha + v\ \}I_A\ |\ \mathcal{G}_t]}{E^0[\ \varphi(\alpha)\ |\ \mathcal{G}_t]}, && (20)\end{aligned}$$

and our task now is to compute the numerator and denominator in the expression (20). Everything we need is contained in the following little result.

Proposition 1 *For any $\lambda \in \mathbb{R}^n$ and any $t \ge 0$,*

$$\begin{aligned} E^0[e^{\lambda\cdot(\alpha - \hat{\alpha}_t)}\ I_A\ |\ \mathcal{G}_t] &= \exp\left\{\ \tfrac{1}{2}\lambda\cdot M_t\lambda + \tfrac{1}{2}(\theta_t\cdot\lambda)^2 k_t\ \right\}\Phi\left(\ \frac{b_t - k_t(\theta_t\cdot\lambda)}{\sqrt{k_t}}\ \right) && (21)\\ &\equiv H_t(\lambda), \end{aligned}$$

[3] This problem has also been observed by Geweke [1].

[4] This is not as contrived as it might appear. No-one would propose a prior distribution which gave positive weight to impossible parameters, and yet the Gaussian prior gives positive weight to parameters which imply an infinite stock price!

say, where

$$\begin{aligned}
\theta_t &= \frac{\tau_t^{-1}v}{v\cdot\tau_t^{-1}v}, &(22)\\
M_t &= \tau_t^{-1} - \frac{\tau_t^{-1}vv^T\tau_t^{-1}}{v\cdot\tau_t^{-1}v}, &(23)\\
b_t &= \rho - \tfrac{1}{2}|v|^2 - \hat{\alpha}_t\cdot v &(24)\\
k_t &= v\cdot\tau_t^{-1}v, &(25)
\end{aligned}$$

and $\hat{\alpha}_t$ and τ_t are as at (10), (9), and Φ is the standard normal distribution function.

We defer the proof of this to an appendix. Using this now, we have by differentiating a couple of times that[5]

$$P^0[A|\mathcal{G}_t] = H_t(0) = \Phi(b_t/\sqrt{k_t}), \tag{26}$$

$$E^0[(\alpha-\hat{\alpha}_t);A\mid\mathcal{G}_t] = -\tau_t^{-1}v\,q(k_t,b_t), \tag{27}$$

$$E^0[\,(v\cdot(\alpha-\hat{\alpha}_t))^2\,;A\mid\mathcal{G}_t] = k_t\Phi(b_t/\sqrt{k_t}) - b_tk_tq(k_t,b_t), \tag{28}$$

where

$$q(T,x)\equiv\exp(-x^2/2T)/\sqrt{2\pi T}$$

is the Gaussian density.

Now substituting these into the expression (20) for the stock price, we arrive after some calculations at the expression

$$S_t = \sigma\frac{(b_tX_t+\hat{\alpha}_t+v)\,\Phi\left(b_t/\sqrt{k_t}\right) + \left(X_tk_t-\tau_t^{-1}v\right)q(k_t,b_t)}{(b_t^2+k_t)\,\Phi\left(b_t/\sqrt{k_t}\right) + b_tk_tq(k_t,b_t)} \tag{29}$$

for the stock, expressed explicitly as a function of $X_t\equiv\sigma^{-1}\delta_t$.

From the previous equation it is clear that a shock to one component of the dividend vector δ is going to impact all the stocks in the economy. In order to have a more precise understanding of the contagion mechanism we can calculate the derivative of S_t

[5]The notation $E^0[Y;A]$ is equivalent to $E^0[YI_A]$

with respect to the observation process X_t; the derivative with respect to the dividend process δ is a corollary. To this end define,

$$SN(X_t) \equiv (b_t X_t + \hat{\alpha}_t + v)\,\Phi\left(b_t/\sqrt{k_t}\right) + \left(X_t k_t - \tau_t^{-1} v\right) q(k_t, b_t) \tag{30}$$

and

$$SD(X_t) \equiv \left(b_t^2 + k_t\right)\Phi\left(b_t/\sqrt{k_t}\right) + b_t k_t q(k_t, b_t). \tag{31}$$

The derivative of SN with respect to X_t is the matrix

$$(b_t I + \tau_t^{-1} - X_t v^T \tau_t^{-1})\Phi\left(b_t/\sqrt{k_t}\right) + q(k_t, b_t)\left[\, k_t I - (\hat{\alpha}_t + v + k_t^{-1} b_t \tau_t^{-1}) v^T \tau_t^{-1}\,\right], \tag{32}$$

where I denotes the $n \times n$ identity matrix, and the derivative of SD with respect to X_t is the row vector

$$2\left[\, b_t \Phi\left(b_t/\sqrt{k_t}\right) + k_t q(k_t, b_t)\,\right](-v^T \tau_t^{-1}). \tag{33}$$

3.2 The bond price

Our goal in this section is to derive an explicit expression for the bond price

$$\begin{aligned} B(t, t+s) &\equiv \zeta_t^{-1} E\left[\zeta_{t+s} \mid \mathcal{G}_t\right] \\ &= E\left[\exp(-\rho s + v \cdot (X_{t+s} - X_t)) \mid \mathcal{G}_t\right] \\ &= E\left[\exp(-\rho s + \tfrac{1}{2}|v|^2 s + s\alpha \cdot v) \mid \mathcal{G}_t\right] \\ &= e^{-b_t s} E\left[e^{s(\alpha - \hat{\alpha}_t)\cdot v} \mid \mathcal{G}_t\right] \\ &= e^{-b_t s}\, \frac{E^0\left[\varphi(\alpha) e^{s(\alpha - \hat{\alpha}_t)\cdot v} \mid \mathcal{G}_t\right]}{E^0\left[\varphi(\alpha) \mid \mathcal{G}_t\right]} \end{aligned} \tag{34}$$

where b_t is as at (24). To simplify this, we again use Proposition 1. Notice that the denominator in (34) is the same as in the expression (29) for the stock, so only the numerator of (34) needs to be calculated. Writing $Y \equiv (\alpha - \hat{\alpha}_t) \cdot v$ as before, the expression in the numerator which we need to evaluate is

$$\begin{aligned} E^0\left[\varphi(\alpha) e^{s(\alpha - \hat{\alpha}_t)\cdot v} \mid \mathcal{G}_t\right] &= E^0\left[(b_t - Y)^2 e^{sY}; A \mid \mathcal{G}_t\right]. \\ &= b_t^2 E^0\left[e^{sY}; A \mid \mathcal{G}_t\right] - 2b_t E^0\left[Y e^{sY}; A \mid \mathcal{G}_t\right] \\ &\qquad + E^0\left[Y^2 e^{sY}; A \mid \mathcal{G}_t\right]. \end{aligned} \tag{35}$$

Now from Proposition 1, we have that

$$\psi_0(s) \equiv E^0\left[e^{sY}; A \mid \mathcal{G}_t\right] = e^{s^2 k_t/2}\,\Phi\left(\frac{b_t - sk_t}{\sqrt{k_t}}\right), \tag{36}$$

which deals with the first term in (35), and the other two terms are obtained by differentiating (36) with respect to s. After some routine calculations, we arrive at the expression

$$E^0\left[\varphi(\alpha)e^{s(\alpha-\hat{\alpha}_t)\cdot v} \mid \mathcal{G}_t\right] = \{\,(b_t - sk_t)^2 + k_t\,\}\psi_0(s) + (b_t - sk_t)k_t q(k_t, b_t - sk_t)e^{sk_t^2/2} \tag{37}$$

for the numerator in the expression (34) for the bond price. From (29), the denominator in (34) is

$$\left(b_t^2 + k_t\right)\Phi\left(b_t/\sqrt{k_t}\right) + b_t k_t q(k_t, b_t),$$

which is the form of the numerator (37) when $s = 0$, as of course it must be.

4 Numerical simulations

In this section we present some simulated data in order to shed light on some of the main features of the model. In particular we fix the parameter set at some reasonable values, we simulate the vector process δ, and observe how stock prices evolve. In particular we set: $n = 3$, $\rho = 0.15$, $\gamma = 0.06$, $\tau_0 = 10\sigma^{-1}$, $\hat{\alpha}_0 = (0, 0, 0)^T$ and

$$\sigma = \begin{pmatrix} 3.6 & 0 & 0 \\ 0 & 3.6 & 0 \\ 0 & 0 & 3.6 \end{pmatrix} \tag{38}$$

In order to simulate the stock price, we fix the level of the unknown dividend drift α. We then simulate the Brownian motion X with drift, and apply formula (29) to recover the corresponding stock price vector process S.

Figure 4 shows the simulated paths of the n stock prices. To exhibit the contagion effect from one stock to the others, we impose an upward shock in X_t^3 at time $t = 0.5$. Such a jump in the dividend of an asset is of course impossible under the model we have presented, but this experiment makes evident the contagion effects in a way that a continuous perturbation of the dividend processes would not show so dramatically.

The level of stock price 3 jumps upwards. At the same time we observe a downward shock for the remaining assets in the economy, a price contagion phenomenon. Market clearing on its own will generate contagion, but the Bayesian transmission of information about the assets will affect the extent of contagion. The magnitude of the contagion effect in this example is affected by various parameters; for example, in the limit of infinite prior precision, there would be *no* contagion in this (independent asset) example.

5 Conclusions and discussion

We have presented a very simple representative agent model for the prices of many assets in an uncertain world, where the agent knows the volatility of the (multivariate) dividend process, but not the growth rates. The agent then estimates the growth rates as a Bayesian; in his observation filtration, the dynamics of the dividends are no longer drifting Brownian, but are something more complicated, yet completely explicit. The agent then optimises his investment and consumption when faced with this dividend dynamic. The equilibrium prices of the stocks and the bond are computed, and allow us to study the effect on the prices of shocks to individual dividend processes. We find that a shock to one dividend process impacts the prices of *all* assets, thus providing a mechanism for the transmission of price shocks across assets.

A key part of the analysis of the problem is to introduce a prefactor into the prior distribution of the growth rates. This prefactor is essential to guarantee finite stock prices, but the choice we made, though convenient, is obviously not unique. If we assumed that σ were diagonal, and that τ_0 were diagonal, then in the reference (Gaussian-prior) measure P^0, the assets would be independent, but the introduction of the prefactor makes them dependent. Is the co-movement of stock prices we observe just an artefact of the dependence imposed by the prefactor? We could study this further by changing our prefactor to

$$\tilde{\varphi}(\alpha) = \prod_{i=1}^{n} I_{\{\alpha_i < -v_i/2\}}$$

which would certainly keep stock prices finite, and would ensure that the dividend processes in the measure P would be *independent*. However, inspection of the expression (19) for the stock price (valid for general prefactors) shows that the stock prices will *not* be independent - the involvement of the terms in $\rho - \frac{1}{2}|v|^2 - \alpha \cdot v$ in the denominator is the cause. The dependence in the stock prices is not coming from the assumptions on the priors (because the prior makes the dividend processes independent), nor is it coming from the Bayesian updating (because under the posterior the dividend processes are still independent), it is in fact coming from market clearing. However, the

Bayesian learning process contributes to the effect, because if the drift were known with certainty then there would be *no* transmission of shocks.

References

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Appendix A Proof of Proposition 1.

Conditional on $\mathcal{G}_t$, the law of α under P^0 is $N(\hat{\alpha}_t, \tau_t^{-1})$, so the law of $\alpha - \hat{\alpha}_t \equiv \eta$ given $\mathcal{G}_t$ is $N(0, \tau_t^{-1})$. Thus if we condition on the value y of $Y \equiv v \cdot \eta$, then the law of η given this is

$$\big(\,\eta \;\big|\; v \cdot \eta = y\,\big) \sim N(\,\theta_t y, M_t\,),$$

where θ_t and M_t are given by (22), (23). The set A is the set

$$A = \{\alpha : v \cdot \eta < \rho - \tfrac{1}{2}|v|^2 - v \cdot \hat{\alpha}_t\},$$

so we evaluate the expectation $E^0[e^{\lambda \cdot (\alpha - \hat{\alpha}_t)}\,;A \mid \mathcal{G}_t]$ by firstly conditioning on Y. We find that (with b_t as at (24))

$$\begin{aligned}
E^0[e^{\lambda \cdot (\alpha - \hat{\alpha}_t)}\,;A \mid \mathcal{G}_t] &= E\big[\,E(e^{\lambda \cdot (\alpha - \hat{\alpha}_t)}|Y)\,;A \mid \mathcal{G}_t\big] \\
&= E\big[\,\exp(\tfrac{1}{2}\lambda \cdot M_t\lambda + \lambda \cdot \theta_t\,Y)\,;A \mid \mathcal{G}_t\big] \\
&= e^{\frac{1}{2}\lambda \cdot M_t \lambda} \int_{-\infty}^{b_t} \exp(-y^2/2k_t + \lambda \cdot \theta_t\,y)\,\frac{dy}{\sqrt{2\pi k_t}} \\
&= \exp\big\{\,\tfrac{1}{2}\lambda \cdot M_t\lambda + \tfrac{1}{2}(\theta_t \cdot \lambda)^2 k_t\,\big\}\Phi\bigg(\,\frac{b_t - k_t(\theta_t \cdot \lambda)}{\sqrt{k_t}}\,\bigg)
\end{aligned}$$

as claimed.

Figure 1: Stock prices simulation. X axis: t, Y axis: S_t

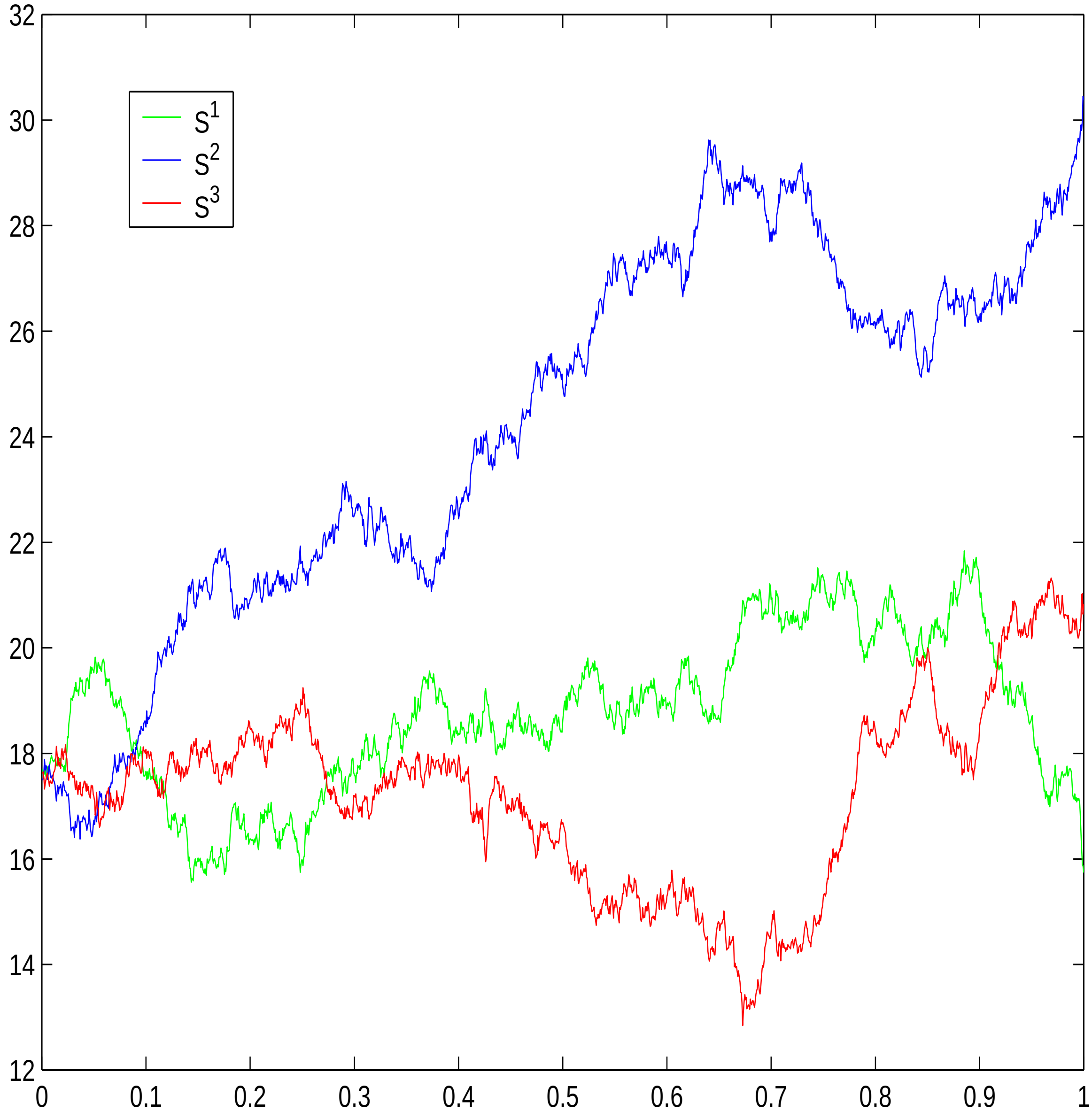

Figure 2: Stock prices simulation. X axis: t, Y axis: S_t