Optimal Investment: case studies

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1 Initial remarks.

The basic wealth dynamics are

$$dw_t = r_t w_t \, dt + \pi_t \cdot (dS_t + \delta_t dt - r_t S_t dt) - c_t \, dt, \tag{1.1}$$

where the n-vector semimartingale S denotes the price processes of the risky assets, r_t is the instantaneous riskless rate, δ is the n-vector dividend process, and π_t is n-vector of numbers of assets held at time t, a previsible process.

The objective which the agent wishes to optimize varies from example to example. Commonly, the objective is of the form

$$E\left[\int_0^T U(t, c_t) \, dt + U(T, w_T) \right], \tag{1.2}$$

or the infinite-horizon version of this:

$$E\left[\int_0^\infty U(t, c_t) \, dt \right]. \tag{1.3}$$

1.1 The basic Merton problem.

The basic Merton problem is one for which the solution to the optimal investment problem can be found in closed form. For the asset dynamics we take

$$dS_t = S_t(\sigma dW_t + \mu dt), \tag{1.4}$$

which is shorthand for the fuller but clumsier specification

$$dS_t^i = S_t^i \left(\sum_{j=1}^n \sigma_{ij} dW_t^j + \mu^i dt \right), \quad i = 1, \ldots, n. \tag{1.5}$$

Here, σ and μ are constants. We also assume that $\delta \equiv 0$, and that r is a constant. For the objective, we may take the finite-horizon objective (1.2) with

$$U(t,c) = ae^{-\rho t}u(c) \quad (0 \le t < T), \quad U(T,c) = bu(c) \tag{1.6}$$

where $a \ge 0$, $b \ge 0$, $\rho \ge 0$, and

$$u(c) = \frac{c^{1-R}}{1-R} \tag{1.7}$$

for some positive R different from 1. This problem can be solved in closed form, and we find the form of the optimal solution is to take

$$\begin{aligned} V(t,w) &= f(t)u(w) && (1.8)\\ \pi_t &= \pi_M\, w_t && (1.9)\\ c_t &= \gamma(t)\, w_t && (1.10) \end{aligned}$$

where

$$\begin{aligned} f(t) &= \left\{ b^{1/R}e^{-q(T-t)/R} + \frac{Ra^{1/R}}{\rho+q}e^{-\rho t/R}\big(1 - e^{-(\rho+q)(T-t)/R} \big) \right\}^R && (1.11)\\ \pi_M &= R^{-1}(\sigma\sigma^T)^{-1}(\mu - r\mathbf{1}) && (1.12)\\ \gamma(t) &= a^{1/R}e^{-\rho t/R}f(t)^{-1/R} && (1.13) \end{aligned}$$

and

$$q \equiv (R-1)(r + |\kappa|^2/2R), \quad \kappa \equiv \sigma^{-1}(\mu - r\mathbf{1}). \tag{1.14}$$

Two special cases are worthy of note.

Terminal wealth. If $a = 0$, $b = 1$, then there is no intermediate consumption, and we aim to mazimise $Eu(w_T)$. Here the value function simplifies to

$$V(t,w) = e^{-q(T-t)}u(w). \tag{1.15}$$

Infinite-horizon. Here we set $a = 1$, $b = 1$, but let $T \to \infty$. The asymptotic forms we find for the solution are

$$\begin{aligned} e^{\rho t}V(t,w) &= \gamma_M^{-1/R}u(w), && (1.16)\\ \pi_t &= \pi_M w_t, && (1.17)\\ c_t &= \gamma_M w_t, && (1.18) \end{aligned}$$

where

$$\gamma_M \equiv R^{-1}\big[\, \rho + (R-1)\big\{r + |\kappa|^2/2R\big\}\,\big]. \tag{1.19}$$

1.2 Efficiency.

The Merton problem has a particular feature that makes it ideal for comparing the effects of different constraints or objectives, and this is due to the scaling. Assuming a CRRA utility $u(c) = c^{1-R}/(1-R)$, we frequently find modifications of the Merton problem that the value takes the form $Au(w)$, where w is the initial wealth, and A is either some constant, or some function of other variables of the problem. In the case of the standard Merton problem we found that

$$A = A_M \equiv \gamma_M^{-1/R}.$$

If some variant $\mathcal{P}$ of the Merton problem has value $Au(w)$, we shall say that the *efficiency* of $\mathcal{P}$ is

$$\Theta \equiv (A/A_M)^{1/(1-R)}. \tag{1.20}$$

The interpretation is that *the standard Merton investor with initial wealth* Θ *would achieve the same objective as the investor in problem* $\mathcal{P}$ *would starting with wealth 1.*

2 Variants of the basic problem.

We list here a range of interesting optimal investment problems which are all variants in some form or another of the basic Merton problem. Most are infinite-horizon problems.

2.1 Optimising under portfolio constraints

The Merton rule of investing fixed proportions of wealth in the different assets need not keep non-negative wealth in each asset; it may well turn out that the conclusion of the Merton analysis is that some assets should be shorted (perhaps even cash should be shorted). Understandably, this is not a conclusion that everyone feels comfortable with, so we could attempt to optimise under the constraint that $\theta_t \in K$ for all t, where K is some convex constraint set, perhaps the positive cone. Likewise, we might be constrained not to allow the volatility of the wealth process to be too large, and this can also be handled.

2.2 Markov-modulated stock dynamics.

Here we suppose that there is some finite-state Markov chain ξ independent of the driving Brownian motion W such that the asset dynamics become

$$dS_t^i/S_t^i = \sigma_{ij}(\xi_t)dW_t^j + \mu_i(\xi_t)dt \tag{2.21}$$

for some functions σ, μ of the chain, and then the wealth dynamics change to

$$dw_t = \theta_t \cdot \sigma(\xi_t) dW_t + [r(\xi_t) w_t - c_t + \theta_t \cdot (\mu(\xi_t) - r(\xi_t)\mathbf{1})] dt. \tag{2.22}$$

Notice that we can without any real loss of generality allow the riskless rate to depend on the chain as well.

The agent's objective is the standard one for a consumption problem,

$$\max E\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right] \tag{2.23}$$

2.3 Transaction costs.

Consider the situation where

$$\begin{aligned} dX_t &= rX_t dt + (1-\epsilon) dM_t - (1+\epsilon) dL_t - c_t dt \\ dY_t &= Y_t(\sigma dW_t + \mu dt) - dM_t + dL_t, \end{aligned}$$

where X_t is value of holding of cash, Y_t is value of holding of stock at time t. $M_t(L_t)$ the cumulative sales (puchases) of stock by time t. The investor's goal is to achieve

$$V(x, y) = \sup E\left[\int_0^\infty e^{-\rho t} U(c_t) dt \,\middle|\, X_0 = x, Y_0 = y\right],$$

with $U(x) = x^{1-R}/(1-R)$ as in the Merton problem.

2.4 Effect of annual taxation.

What is the effect on the Merton problem of an annual tax on capital gains? Suppose that U is again CRRA, and at each time $t = nh$ we have to pay tax on wealth gain over the last time period of length h. Thus $w_{nh} = w_{nh-} - \tau(w_{nh-} - w_{nh-h}) = (1-\tau) w_{nh-} + \tau w_{nh-h}$. If we do this, then the problem becomes a finite-horizon problem,

$$V(w) = \sup E\left[\int_0^h e^{-\rho s} U(c_s) ds + e^{-\rho h} U(\tau w + (1-\tau) w_h)\right].$$

2.5 Infrequent portfolio revision.

Suppose we have a standard Merton investor, maximizing

$$E[\int_0^\infty e^{-\rho t} U(c_t) dt], \quad U'(x) = x^{-R}$$

and now instead of rebalancing continuously, we only allow the agent to rebalance the portfolio at times $t = 0, h, 2h, \dots$. Obviously the agent does less well, but does this actually matter? Let's also suppose the rate c is held constant in each interval, so the value solves

$$V(w) = \sup_{c,p} \left[U(c) \, \frac{1 - e^{-\rho h}}{\rho} + e^{-\rho h} EV\left((w - ch)(pS + (1 - p)e^{rh})\right) \right],$$

where $S = \exp\{\sigma\sqrt{h}Z + (\mu - \frac{1}{2}\sigma^2)h\}$ with $Z \sim N(0, 1)$.

2.6 Random rate of return for stock.

In this example, the basic wealth dynamics get modified by allowing the growth rate μ to vary with time. We shall take

$$dw_t = rw_t + \theta_t(\sigma dW_t + (\mu_t - r)dt) - c_t dt \tag{2.24}$$

$$d\mu_t = \sigma_m dW'_t + \beta(m - \mu_t)dt \tag{2.25}$$

where $dW_t dW'_t = \eta dt$, and $\beta > 0$ is known, as are σ_m, m, η.

2.7 A habit formation model.

Constantinides proposed a model where the agent's consumption is compared to an exponentially-weighted historical average of past consumption. One motivation for this was to try to explain the equity premium puzzle (EPP). The model proposed by Constantinides helps a bit in explaining the EPP, but it is in any case an interesting attempt to explore different objectives. The dynamics taken are a simple variant of the usual wealth equation:

$$dw_t = rw_t dt + \theta_t(\sigma dW_t + (\mu - r)dt) - c_t dt \tag{2.26}$$

$$d\bar{c}_t = \lambda(c_t - \bar{c}_t)dt. \tag{2.27}$$

The agent's objective in Constantinides' account is

$$\sup E \int_0^\infty e^{-\rho t} U(c_t - \bar{c}_t) \, dt$$

so that present consumption is in some sense evaluated relative to the exponentially-weighted (EW) average $\bar{c}_t$ of past consumption.

2.8 A better habit formation model.

What we propose to do here is to keep the dynamics (2.26) and (2.27), but to take as the objective

$$V(w, \bar{c}) \equiv \sup E\left[\int_0^\infty e^{-\rho t} U(c_t/\bar{c}_t)\, dt \,\middle|\, w_0 = w, \bar{c}_0 = \bar{c} \right] \tag{2.28}$$

which (more realistically) rewards the *ratio* of current consumption to the EW average. This objective permits current consumption to fall below the EW average of past consumption at various times, again a more realistic feature.

2.9 Recursive utility.

The wealth dynamics this time are the usual thing,

$$dw_t = rw_t dt + \theta_t(\sigma dW_t + (\mu - r)dt) - c_t dt,$$

with now the objective to maximize U_0, where the recursive utility process $(U_t)_{0 \le t \le T}$ satisfies the dynamics

$$U_t = E_t\big[\int_t^T F(s, c_s, U_s)\, ds + G(w_T) \,\big] \tag{2.29}$$

where $F(s, \cdot, \cdot)$ is concave increasing, and G is concave increasing.

2.10 Liquidity effects.

We take a model for an illiquid asset, where the number of units H_t of the asset held at time t cannot be changed very rapidly. We assume that H is differentiable with derivative h_t, and that there is a cost for rapid change of portfolio. Thus the dynamics are

$$dw_t = rw_t dt + H_t(dS_t - rS_t dt) - c_t dt - S_t l(h_t) dt \tag{2.30}$$

$$dH_t = h_t dt. \tag{2.31}$$

Here l is a non-negative convex function, vanishing at 0. The objective is the usual one of

$$\max E\left[\int_0^\infty e^{-\rho t} U(c_t) dt \right]$$

2.11 Parameter uncertainty.

The dynamics of wealth are as usual

$$dw_t = rw_t dt + \theta_t \cdot \sigma\{dW_t + (\alpha_t - r\sigma^{-1}\mathbf{1})dt\} - c_t dt \tag{2.32}$$

which we have written in a slightly unusual way, because we intend now to suppose that the parameter α is *not* known with certainty, rather that we shall have a prior $N(\hat{\alpha}_0, \tau_0^{-1})$ distribution for it. The volatility matrix σ is $n \times n$, and assumed known and non-singular. We write V for $\sigma\sigma^T$.

2.12 Stochastic interest rates.

Suppose that the riskless rate is not constant, but diffuses as in a Vasicek model:

$$dr = \sigma_r dW' + \beta(\bar{r} - r)dt,$$

where $dW'dW = \eta dt$. What happens?

2.13 Drawdown constraints.

In this problem, we assume the (by now) standard dynamics

$$dw_t = r(w_t - \theta_t)dt + \theta_t(\sigma dW_t + \mu dt) - c_t dt$$

for the wealth and objective

$$\sup E[\int_0^\infty e^{-\rho t} U(c_t)dt], \quad U'(x) = x^{-R},$$

but now we shall impose the constraint

$$w_t \geq b\bar{w}_t = b \sup_{s \leq t} w_s, \quad \forall t, \tag{2.33}$$

where $b \in (0,1)$ is fixed. This is called a *drawdown constraint*, in a natural terminology. Drawdown constraints are of practical importance for fund managers, because if their portfolio loses too much of its value, the investors are likely to take their money out and that is the end of the story, however clever (or even optimal!) the rule being used by the fund manager.

2.14 Business cycle.

Suppose that the growth rate μ_t of the risky asset is not constant, but varies with time in a known deterministic way:

$$dw_t = rw_t + \theta_t(\sigma dW_t + (\mu_t - r)dt) - c_t dt, \tag{2.34}$$

the objective being once again the standard objective (2.23). This story for the dynamic reflects the commonly-held belief that there is a so-called 'business cycle effect', where returns are depressed for a while, during a period where business is investing in new products and technology, and then once the new products become available the profitability of businesses is increased.

2.15 Ratcheting of consumption.

The standard wealth dynamics

$$dw_t = rw_t dt + \theta(\sigma dW_t + (\mu - r)dt - c_t dt$$

get combined with the constraint

$$c_t \textit{ is non-decreasing.} \tag{2.35}$$

The rationale for this problem is that in some situations (such as the use of a university endowment) it is hard to reverse commitments to consumption.

2.16 Keeping up with the Jones'.

Suppose there are two agents, $i = 0, 1$, who derive utility from consumption, but this utility is modified by the consumption of the other agent. Thus we assume the standard dynamics

$$dw_t^i = rw_t^i dt + \theta^i(\sigma dW_t + (\mu - r)dt - c_t^i dt$$

for their wealths, and then take the objective of agent i to be

$$\max E\Big[\int_0^\infty e^{-\rho_i t} U_i(c_t^i, c_t^{1-i})\, dt \Big] \tag{2.36}$$

A natural and simple form to take for U_i would be

$$U_i(c, y) = \frac{c^{1-R_i}}{1-R_i} \left(\frac{c}{y}\right)^{\alpha_i}, \tag{2.37}$$

where $R_i > 0$ and $(1 - R_i)\alpha_i > 0$.

2.17 Variable liquidity.

This is another story where there is a finite-state Markov chain ξ_t influencing the dynamics. This time, the single risky asset is assumed to be illiquid, in that you are only able to change your position in the asset at the times of a Poisson process, whose intensity is $\lambda(\xi_t)$. Suppose that when one of these times occurs, you are able to make an arbitrary change in your position. If x_t denotes the value of your bank account at time t, and y_t the value of your holding of the risky asset, then between the times of rebalancing we have the dynamics

$$\begin{aligned} dx_t &= (rx_t - c_t)dt, && (2.38) \\ dy_t &= y_t(\sigma dW_t + \mu dt). && (2.39) \end{aligned}$$

What do you do? How big an effect is this illiquidity if the aim is to

$$\max E \int_0^\infty e^{-\rho t} U(c_t)\, dt$$

for CRRA U?

2.18 Stochastic volatility

In the Hobson-Rogers model of asset dynamics, the single risky asset evolves as

$$dS_t = S_t(\sigma_t dW_t + \mu dt) \tag{2.40}$$

where $\sigma_t = f(Z_t)$ is some function of the offset

$$Z_t = \int_{-\infty}^t e^{\lambda(s-t)}(X_s - X_t)\, ds \tag{2.41}$$

where $X_t \equiv \log S_t$. Thus we could suppose that $-f$ is unimodal, with its maximum at 0; this would give an increased volatility when the asset was farther away from its (exponentially-weighted) historical level. If the objective is the standard one (2.23), how does the solution look?

2.19 Heston's model of stochastic volatility

In the Heston model of asset dynamics, we have

$$\begin{aligned} dS_t &= S_t(\sqrt{v_t} dW_t + \mu dt) \\ dv_t &= a\sqrt{v_t} dW'_t + \beta(\bar{v} - v_t)dt \end{aligned} \tag{2.42}$$

where W and W' are correlated Brownian motions, $dW dW' = \rho dt$ and a, β, $\bar{v}$ are positive constants. If the objective is again (2.23), what can we say about the solution?

2.20 Non-constant relative risk aversion

While the assumption of constant relative risk aversion makes for nice scaling, it is arguably not how people would behave; if you are very wealthy, then you may be less risk-averse than if you are not wealthy. Or perhaps if you are poor, desperation may drive you to take greater risks than someone who is not! If we take $R_1 > 1 > R_2$, then defining the inverse marginal utility by

$$I(y) = y^{-1/R_1} + ky^{-1/R_2} \tag{2.43}$$

for some positive constant k gives an agent whose coefficient of relative risk aversion is roughly R_2 for large levels of consumption, and roughly R_1 for small levels of consumption - the adventurous rich. Alternatively, we could define

$$I(y) = (y^{1/R_1} + ky^{1/R_2})^{-1}, \tag{2.44}$$

which gives an agent whose coefficient of relative risk aversion is roughly R_1 for large levels of consumption, and roughly R_2 for small levels of consumption - the desperate poor.

How do the optimal investment/consumption decisions look now?

2.21 Optimizing under risk-management constraints

Suppose there is a single risky asset, the usual dynamics

$$dS_t = S_t(\sigma dW_t + \mu dt)$$

and that the objective is to max $EU(w_T)$ subject to a constraint on the value-at-risk (V@R); we insist that

$$P(w_T < 0.9w_0) \leq 0.05 \tag{2.45}$$

How does the optimal policy look now?

2.22 Log-Lévy asset dynamics

Suppose now that the asset dynamics are such that $\log S_t$ is a Lévy process, and the agent wishes to invest so as to maximise $E[U(w_T)]$, for some CRRA utility U.

2.23 Production economy

Suppose that the rate of output Y_t from the economy is a function $F(K, L)$ of the capital deployed, and the labour employed in production. We suppose that F is increasing in both variables, concave, and homogeneous of degree 1:

$$F(\lambda K, \lambda L) = \lambda F(K, L) \quad \forall \lambda > 0.$$

The controller of the economy chooses the rate I_t at which to invest in capital, which then evolves as

$$\dot{K}_t = I_t - \delta K_t,$$

where $\delta > 0$ is constant. The consumption rate is $C_t = F(K_t, L_t) - I_t$. Suppose that the objective is to attain

$$\sup E \int_0^\infty e^{-\rho t} U(C_t/L_t)\, dt$$

where $\rho > 0$ is constant, and the labour force (= population) evolves as

$$dL_t = L_t(\sigma dW_t + \mu dt).$$

Assuming CRRA utility U, how should the controller of the economy invest?

2.24 Equilibrium with return for investment

Suppose that a productive asset delivers a dividend process δ where

$$d\delta_t = \delta_t(\sigma dW + \mu_t dt).$$

Here $\mu_t = g(i_t)$, where $i_t = \delta_t - c_t$ is the rate of investment into the productive asset. We assume that g is increasing and concave. The agent aims to

$$\sup E \int_0^\infty e^{-\rho t} U(c_t)\, dt.$$

What is the optimal investment policy, and what is the equilibrium price of the productive asset?

2.25 Choosing a time to retire

An agent works until a time τ of his choosing, at which time he retires. While working, he receives a constant income stream of ε, which incurs a disutility $\lambda > 0$. He invests his wealth in a riskless bank account bearing interest rate r, and in a risky stock with constant volatility σ and rate of growth μ. His wealth therefore evolves as

$$dw_t = rw_t dt + \theta(\sigma dW_t + (\mu - r)dt) - c_t dt + \varepsilon I_{\{t \le \tau\}} dt$$

(where W is a standard Brownian motion, and θ_t is the time-t value of his holding of the stock) and he seeks to maximise

$$E \int_0^\infty e^{-\rho t}(U(c_t) - \lambda I_{\{t \le \tau\}})\, dt.$$

Assume that $U'(x) = x^{-R}$ for some positive constant $R \neq 1$. Show that the critical level at which he retires is $\gamma_M^{-1}(\varepsilon/\lambda)^{1/R}$. By introducing the dual variable $z = V'(w)$, solve his problem as completely as you can.

2.26 Utility from the slice of the cake

A continuous-time model of an economy contains a single productive asset, whose dividend process $(\delta_t)_{t\geq 0}$ evolves as

$$d\delta_t = \delta_t(\sigma dW_t + \mu dt),$$

where W is a standard Brownian motion. Agent $i \in \{1, \dots, J\}$ has preferences over consumption streams $(c_t^i)_{t\geq 0}$ given by

$$E\int_0^\infty e^{-\rho_i t} U_i(p_t^i)\, dt,$$

where

$$p_t^i = \frac{c_t^i}{\sum_j c_t^j}$$

and $U_i : (0,\infty) \to \mathbb{R}$ is C^2, strictly increasing and strictly concave, $U'(0) = \infty,\ U'(\infty) = 0$. Agent i initially holds a fraction π_0^i of the productive asset.

By considering marginal pricing of future consumption relative to present consumption, or otherwise, derive the equilibrium for this economy as explicitly as you can. Show that in equilibrium the process $(p_t)_{t\geq 0}$ is non-random. Show that if all the agents have same ρ_i and U_i then in equilibrium the proportion of the output of the productive asset which agent i consumes is constant and equal to π_0^i.

2.27 History-dependent preferences

Suppose we have the usual dynamics (1.1), but now the objective of the agent is given as

$$\sup E\int_0^\infty e^{-\rho t} U(\xi_t)\, dt, \tag{2.46}$$

where the process ξ solves

$$d\xi_t = \lambda(c_t^\alpha - \xi_t)dt \tag{2.47}$$

for positive constants α and λ.

2.28 Performance relative to a benchmark.

The geometric market index J is defined to be

$$J_t \equiv \left\{ \prod_{i=1}^n S_t^i \right\}^{1/n}.$$

Prove that

$$\log(J_t/J_0) = n^{-1}\big[\,\mathbf{1}\cdot\sigma W_t + \{\mathbf{1}\cdot\mu - \frac{1}{2}\mathrm{tr}(V)\}\,t\,\big]$$

where $\mathbf{1} = (1,\ldots,1)^T$, and $V \equiv \sigma\sigma^T$.

An agent with initial wealth w_0 invests in this market, choosing a portfolio process θ. His aim is to invest in such a way as to do well relative to the index; his objective is $E[\,U(w_T/J_T)\,]$, where U is the CRRA utility $U(x) = x^{1-R}/(1-R)$, for some positive $R \neq 1$. Show that the agent should optimally split his wealth among the stocks in fixed proportions, given by the vector

$$R^{-1}V^{-1}(\mu - r\mathbf{1}) + (1-R^{-1})n^{-1}\mathbf{1}.$$