OPTIMAL INVESTMENT: Example Sheet 2

1. The (infinite-horizon) Merton consumption problem has wealth dynamics

$$dw_t = rw_t dt + \theta_t(\sigma dW_t + (\mu - r)dt) - c_t dt \tag{1}$$

and objective ($u(x) = x^{1-R}/(1-R)$, $0 < R \neq 1$)

$$\sup_{c,\theta} E\Big[\int_0^\infty e^{-\rho t} u(c_t)\, dt \Big]. \tag{2}$$

Let $\pi = (\mu - r)/\sigma^2 R$. Assuming that $\gamma \equiv R^{-1}\big\{\rho + (R-1)(r + \frac{1}{2}\sigma^2\pi^2 R)\big\} > 0$, *prove* that the policy

$$c_t = \gamma w_t$$
$$\theta_t = \pi w_t$$

is optimal.

2. Find the value function and optimal policy for the Merton consumption problem with $u(x) = \log(x)$.

3. (i) Consider the problem of maximising $E[u(w_T)]$ for CRRA u and with wealth dynamics

$$dw_t = r(w_t - \theta)dt + \theta_t\sigma(dW_t + \alpha dt) - c_t dt, \tag{3}$$

but with uncertain α; assume a $N(\hat\alpha_0, \tau_0^{-1})$ prior for α. Show that the value of this problem is ($b \equiv 1 - R^{-1}$)

$$u(w)\tau_0^{1/2} \frac{(\tau_0 + T)^{(R-1)/2}}{(\tau_0 + bT)^{R/2}} \exp\Big[r(1-R)T - \frac{(\hat\alpha_0 - r/\sigma)^2 b\tau_0 T}{2(\tau_0 + bT)} \Big].$$

Confirm that as $\tau_0 \to \infty$ this expression converges to the value that would be obtained by the investor who knew for certain that the true value of α was α_0.

(ii) Suppose now that the wealth dynamics are given by (3), where α has a $N(\hat\alpha_0, \tau_0^{-1})$ distribution. Investor A is told the true value of α before he starts to invest, but investor B has to filter it from the observation of the stock price. By averaging over the distribution of α, calculate the expected value to investor A, and compare with the expected value to investor B. Deduce that investor B has efficiency

$$\left(1 - \frac{T}{R(\tau_0 + T)}\right)^{1/2}$$

relative to investor A.

4. As a variant on the Constantinides habit formation example, let us suppose that the historical level of consumption is measured (as before) by

$$x_t = e^{-at} x_0 + b \int_0^t e^{a(s-t)} c_s \, ds,$$

but that the objective is to obtain

$$V(w,x) \equiv \sup E\left[\int_0^\infty e^{-\rho t} u(c_t/x_t) dt \;\middle|\; w_0 = w, x_0 = x \right].$$

Find the HJB equation for V. Can you solve it?

5. (*Dybvig's ratcheting of consumption example - quite challenging!.*) Suppose an investor wishes to optimise (2) with wealth dynamics (1), but subject to the constraint that the process c should be *non-decreasing.* If V is the value function

$$V(w,c) = \sup_{c,\theta} E\Big[\int_0^\infty e^{-\rho t} u(c_t) \, dt \;\Big|\; w_0 = w, c_0 = c \Big],$$

show that $V(w,c) = c^{1-R} V(w/c, 1) \equiv c^{1-R} v(w/c)$. Derive the HJB equations

$$\begin{aligned} 0 &= \sup_\theta \big[(1-R)^{-1} - \rho v(z) + \tfrac{1}{2}\sigma^2 \theta^2 v''(z) + (rz + \theta(\mu - r) - 1) v'(z) \big], \\ 0 &\geq (1-R) v(z) - z v'(z), \quad \text{with equality for } z \geq z_1, \end{aligned}$$

where z_1 is a value that is to be found. By passing to the dual variables $y = v'(z)$, $J(y) = v(z) - zy$, re-express the HJB equation as

$$\begin{aligned} 0 &= \tfrac{1}{2}\kappa^2 y^2 J''(y) + (\rho - r) y J'(y) - \rho J(y) - y + \frac{1}{1-R}, \\ 0 &\geq (1-R) J(y) + R y J'(y) \end{aligned}$$

where $\kappa \equiv (\mu - r)/\sigma$.

Argue that if $z = 1/r$ then $v(z) = 1/\rho(1-R)$, and for $z < 1/r$, $v(z) = -\infty$. Show that (under well-posedness conditions), there exists some $y_0 > 0$ and positive A and A_0 such that

$$\begin{aligned} J(y) &= -A_0 \frac{(y/y_0)^{1-1/R}}{1 - 1/R} \quad (y \leq y_0) \\ &= -\frac{y}{r} + \frac{1}{\rho(1-R)} + A\left(\frac{y}{y_0}\right)^{-\alpha} \quad (y \geq y_0). \end{aligned}$$

Here $-\alpha$ is the negative root of the quadratic $Q(x) \equiv \frac{1}{2}\kappa^2 x(x-1) + (\rho - r)x - \rho$. Find y_0, A, and A_0 as explicitly as you can.