OPTIMAL INVESTMENT: Example Sheet 1

1. In the context of the discrete-time Merton model, find as explicity as you can the value function, optimal investment rule and wealth evolution for an agent who aims to

$$\max E \log(w_T).$$

2. The price of a stock evolves as $S_t = S_{t-h}\exp(X_t)$, where the X_t are IID $N(\mu, \sigma^2)$. The riskless rate of return is r, constant, and the wealth evolves as

$$w_t - w_{t-h} = rhw_{t-h} + \alpha(e^{X_t} - rh) - C_t.$$

An agent wishes to maximise $E\sum_{t\geq 0}\beta^t U(C_t)$, while keeping his wealth non-negative. If the utility is of the form $U(x) = x^{1-R}/(1-R)$, show that the value function must be of the form $V(w) = AU(w)$ for some constant A. Can you decide when A is finite?

3. A discrete-time controlled Markov process evolves according to

$$X_{t+1} = \lambda X_t + u_t + \varepsilon_t, \quad t = 0, 1, \ldots,$$

where the ε are independent zero-mean random variables with common variance σ^2, and λ is a known constant.

Consider the problem of minimising

$$F_{t,T}(x) = E\big[\sum_{j=t}^{T-1}\beta^{j-t}C(X_j, u_j) + \beta^{T-t}R(X_T)\,\big],$$

where $C(x, u) = \frac{1}{2}(u^2 + ax^2)$, $\beta \in (0, 1)$ and $R(x) = \frac{1}{2}a_0x^2 + b_0$. Show that the optimal control at time j takes the form $u_j = k_{T-j}X_j$ for certain constants k_i. Show also that the minimised value for $F_{t,T}(x)$ is of the form

$$\frac{1}{2}a_{T-t}x^2 + b_{T-t}$$

for certain constants a_j, b_j. Find the forms of these constants as explicitly as you can. Do they tend to limits as $j \to \infty$? If they do, identify what the limits are.

4. If $Z_t = X_t + iY_t$ is a complex Brownian motion (so X and Y are independent standard Brownian motions), and f is an analytic function, prove that $f(Z)$ is a local martingale. What is its covariation process? Find the Itô expansion of $\log Z_t$.

5. If X is a Brownian motion in $\mathbb{R}^n$, $n > 1$, started away from 0, prove that

$$M_t \equiv |X_t| - \int_0^t \frac{n-1}{2|X_s|}\,ds$$

is a local martingale. What is its covariation process? In the case $n = 3$, show that

(i) $Y_t \equiv 1/|X_t|$ is a local martingale;

(ii) Y is bounded in L^2;

(iii) Y is not a martingale.

6. If X, Y are independent Brownian motions, find the (formal) Itô expansions of the following semimartingales:

(i) Y_t/X_t;

(ii) $\tan^{-1}(Y_t/X_t)$;

(iii) $(X_t, Y_t)/\sqrt{X_t^2 + Y_t^2}$;

(iv) $X_t/(X_t^2 + Y_t^2)$.

In each case, comment on any possible issues concerning the application of Itô's formula.

7. An economy has J agents, whose preferences over consumption streams are given by

$$E \sum_{t \geq 0} \beta_j^t U_j(C_t),$$

where $U_j(x) = -\gamma_j^{-1} \exp(-\gamma_j x)$ for positive constants $\gamma_j, j = 1, \ldots, J$. If there are n assets producing dividends $d_t = (d_t^1, \ldots, d_t^n)^T$ in period t, find the equilibrium pricing kernel as explicitly as you can (under the assumption that all agents agree on the prices of all contingent claims) in the two cases:

(i) the d_t are IID $N(\mu, V)$;

(ii) the $\xi_t \equiv d_t - d_{t-h}$ are IID $N(\mu, V)$.

In each case, find the time-t prices of the shares, and of a one-period bond which delivers 1 unit of consumption good at time $t + h$.

8. An economy has J agents, and N assets which generate dividends $d_t^i, i = 1, \ldots, N$ in period t. Suppose however that there are K different consumption goods produced, so that each d_t^i is a K-row vector, and write $D_t \equiv (d_t^{ik})_{i=1,\ldots,N,k=1,\ldots,K}$ for the $N \times K$ matrix of dividends in period t. If agent j consumes C_t^j (a K-row vector) in period t, then he values this stream of consumption according to

$$E \sum_{t \geq 0} \beta^j U_j(C_t^j),$$

where $U_j : \mathbb{R}^K \to \mathbb{R}$ is concave and increasing in each argument.

Suppose that in period t, the price of the different consumption goods in terms of the first is given by the K-column vector p_t (whose first component is of course 1), and the prices of the N assets in terms of the first consumption good are $S_t = (S_t^1, \ldots, S_t^N)^T$. Write down the portfolio dynamics for this problem, and then optimise agent j's objective by Lagrangian methods; show that the first-order conditions are

$$\beta_j^t \nabla U_j(C_t^j) = \lambda_t^j p_t$$
$$E_{t-h}\left[\, \lambda_t^j (D_t p_t + S_t) \,\right] = \lambda_{t-h}^j S_{t-h}$$