OPTIMAL INVESTMENT

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1 Stochastic integration: a resumé.

This section aims to present the outline of stochastic integration. Proofs will be omitted, but can be found in any of a number of standard texts on the subject (add refs). Definitions and results will be properly stated.

Like any integration theory, stochastic integration starts from the definition of an integral for very simple situations, and extends, by way of linearity, and continuity. We are aiming to define what is meant by the stochastic integral

$$t \mapsto \int_0^t H_s \, dX_s \equiv (H \cdot X)_t \equiv I(H, X)_t$$

for some suitably large class of integrands H and integrators X. The stochastic integral will be thought of as a continuous-time process, and ultimately we shall find that

$$I : \mathrm{lb}\mathcal{P} \times \mathcal{S} \to \mathcal{S},$$

where $\mathrm{lb}\mathcal{P}$ is the space of *locally bounded previsible processes*, and $\mathcal{S}$ is the space of *semimartingales*, both of which will be defined in due course. But let's start small, and suppose firstly that H is a *basic integrand*

$$H = Z(S, T] \tag{1}$$

(the notation is a crisp way to write the more formal statement $H_t = Z I_{\{S<t\leq T\}}$), where $S \leq T$ are two stopping times, and Z is bounded and $\mathcal{F}_S$-measurable. In this case, the only definition we could possibly use is to set

$$(H \cdot X)_t = Z(X_{t\wedge T} - X_{t\wedge S}). \tag{2}$$

If $\mathcal{E}$ denotes the space of all finite linear combinations of basic integrands, then the required bilinearity of the integral I extends the definitions of $H \cdot X$ to all[1] $H \in \mathcal{E}$, but without further restrictions on X we can get no further.

We shall next suppose that X is in $\mathcal{M}^2 \equiv \{M : \; M \text{ is an } L^2\text{-bounded martingale }\}$. It is easy to see then that $H \cdot X$ defined by (34) is again an L^2-bounded martingale, and, as such, is closed on the right by its limit at infinity, $(H \cdot X)_\infty$. The key to the extension (due in essence to Paul-André Meyer) is the following.

Theorem 1 *For $X \in \mathcal{M}^2$, there exists a unique increasing adapted process $[M]$ vanishing at 0 such that*

(i) $M_t^2 - [M]_t$ is a uniformly-integrable martingale;

(ii) $\Delta[M]_t = (\Delta M)_t^2$ for all $t > 0$.

Using this, it is not hard to prove that

$$\|(H \cdot M)_\infty\|^2 \equiv E[(H \cdot M)_\infty^2] = \|H\|_M^2 \equiv E \int_0^\infty H_s^2 d[M]_s, \tag{3}$$

and so the map I from $\mathcal{E}$ (equipped with the norm $\|\cdot\|_M$) to L^2 is an isometry. We can therefore extend the definition of stochastic integral to all processes H in the closure of $\mathcal{E}$ in the norm $\|\cdot\|_M$, which coincides with the space of all *previsible*[2] processes H with finite $\|\cdot\|_M$ norm.

This is a good start to the construction of stochastic integrals, but the requirement to check that things are in the appropriate L^2 space is far too restrictive in practice. The way forward is to *localise*. We shall say that M is a *local martingale* if there exists a sequence $T_n \uparrow \infty$ of stopping times such that $M^{T_n} \equiv M(T_n \wedge \cdot)$ is a uniformly-integrable martingale for all n. We shall say that the previsible process H is *locally bounded* if there exists a sequence $T_n \uparrow \infty$ of stopping times such that $HI_{(0,\mathbb{T}_n]}$ is bounded; the space of all locally bounded previsible processes is denoted lb$\mathcal{P}$.

Because our only applications will be to continuous processes, and because the development of stochastic integration theory is substantially complicated by having jumps, we will from now on just discuss the case of *continuous* integrators X. Any continuous local martingale M is locally an L^2-bounded martingale[3], so this allows us to localise

[1] When $H \in \mathcal{E}$ can be represented in different ways as linear combinations of basic integrands, we have to ensure that all lead to the same definition of $H \cdot X$.

[2] A process $H : (0, \infty) \times \Omega \to \mathbb{R}$ is previsible if it is measureable with respect to the σ-field generated by all left-continuous adapted processes (equivalently, the σ-field generated by all basic integrands.)

[3] Not true without continuity.

M to L^2, and localise $H \in \mathrm{lb}\mathcal{P}$ to be bounded, and then use the stochastic integral so far constructed. The end point is the following.

Theorem 2 *If $H \in \mathrm{lb}\mathcal{P}$ and $M \in \mathcal{M}^c_{loc}$, then there is a unique continuous local martingale $H \cdot M$ such that for any stopping time T strongly reducing[4] M and H*

$$(H \cdot M)^T = (H(0,T]) \cdot M^T = (H(0,T]) \cdot M = H \cdot M^T.$$

We have that $(H \cdot M)^2_t - \int_0^t H^2_s d[M]_s$ is a local martingale.

Remark. Note that $M^2_t - [M]_t$ is a continuous local martingale; this follows from Theorem 3, but is really a more elementary consequence of the definition of a local martingale.

The final extension is to allow the integrator to be what is known as a *semimartingale:* the continuous process X is called a semimartingale[5] if it can be represented in the form

$$X_t = X_0 + M_t + A_t \tag{4}$$

for some local martingale M vanishing at 0, and finite-variation process A vanishing at zero. It turns out that the M and A can be taken to be *continuous*, and in that case the representation (36) of X is unique. We refer to M as the *(continuous) martingale part* of X, denoted X^c, and we use the notation $[X^c]_t = [X]_t = [M]_t$. It is now completely obvious how we shall define the stochastic integral $H \cdot X$ for a continuous semimartingale X; we set

$$(H \cdot X)_t = (H \cdot M)_t + (H \cdot A)_t, \tag{5}$$

where the first integral is in the sense of Theorem 3 and the second is just an ordinary Lebesgue-Stieltjes integral.

Important and obvious properties inherited from the integration of simpler integrands are

(i) $(H, X) \mapsto H \cdot X$ is bilinear;

(ii) $H \cdot (K \cdot X) = (HK) \cdot X$ for $H, K \in \mathrm{lb}\mathcal{P}$, $X \in \mathcal{S}^c$.

We have now presented the main points of the construction of stochastic integrals of locally-bounded previsible processes with respect to continuous semimartingales - so what? This is no sterile abstract theory, as we are about to see - there are endless applications, mainly of what is called *stochastic calculus.* The seed of stochastic calculus is the following result.

[4] A stopping time T reduces M strongly if M^T is bounded. We shall say that T reduces H strongly if $H(0,T]$ is bounded.

[5] The class of all such is denoted $\mathcal{S}^c$.

Theorem 3 *If X is a continuous semimartingale, then*

$$X_t^2 - X_0^2 - [X]_t = \int_0^t 2X_s\, dX_s. \tag{6}$$

The proof comes from the simple identity

$$X_t^2 - X_{t-h}^2 = 2X_{t-h}\Delta X_t + \Delta X_t^2.$$

Letting $h \downarrow 0$, the first term on the right tends to $2X_t dX_t$, the second to $d[X]_t$.

Polarisation of this quadratic equation (38) leads immediately to the following important extension.

Theorem 4 (Integration-by-parts formula.) *For X, Y continuous semimartingales,*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X,Y]_t. \tag{7}$$

Of course, $4[X,Y] = [X+Y] - [X-Y]$. Theorem 5 lets us express products of semimartingales in terms of stochastic integrals; extending this, we can get expressions for polynomial functions of semimartingales in terms of stochastic integrals; and then from polynomials we can to all the way to general C^2 functions.

Theorem 5 (Itô's formula.) *For any C^2 function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_0^t D_i f(X_s)\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^s D_{ij} f(X_s) d[X^i, X^j]_s. \tag{8}$$

It is for (40) that the entire theory of stochastic integration is worth the effort[6]. Here's a quick example of how it can be used.

Theorem 6 *Suppose that X is a continuous local martingale, and that $[X]_t = t$. Then X is Brownian motion.*

[6]Some people refer to Itô's formula as the change-of-variables formula, but perhaps that term should be reserved for Itô's personal use.

Proof. For $\theta \in \mathbb{R}$, consider the process

$$M_t^\theta = \exp\big[\, i\theta X_t + \frac{1}{2}\theta^2 t \,\big].$$

By Itô's formula[7], in the shorter differential notation

$$\begin{aligned} dM_t^\theta &= M_t^\theta\{i\theta dX_t + \frac{1}{2}\theta^2 dt + \frac{1}{2}(i\theta)^2 d[X]_t \,\} \\ &= M_t^\theta i\theta dX_t \end{aligned}$$

using the fact that $[X]_t = t$. Thus M^θ is a local martingale, and since it is obviously bounded on every interval $[0,T]$ for $T \in \mathbb{R}$, it is a true martingale. Thus for all $0 \le s \le t$ we have

$$E_s\big[\, \exp(i\theta(X_t - X_s)) \,\big] = \exp(-\frac{1}{2}\theta^2(t-s)).$$

Thus the distribution of $(X_t - X_s)$ conditional on $\mathcal{F}_s$ is $N(0, t-s)$, and hence X is Brownian motion. ∎

Corollary 1 *Let M be a continuous local martingale, $M_0 = 0$. Then M can be represented as*

$$M_t = B([M]_t)$$

where B is a Brownian motion on a suitably-defined probability space. (See Theorem IV.34.11 in RW).

Theorem 7 *If W is a standard Brownian motion, and $(\mathcal{F}_t)_{t \ge 0}$ is the usual augmentation[8] of the filtration generated by W, then every $Y \in L^2(\mathcal{F}_T)$ can be represented as*

$$Y = EY + \int_0^T H_s dW_s \tag{9}$$

for some previsible H such that $E\int_0^T H_s^2\, ds < \infty$. In particular, every L^2-bounded $(\mathcal{F}_t)$ martingale is a stochastic integral with respect to W.

Remark. In fact, any $Y \in L^1(\mathcal{F}_T)$ can be represented as a stochastic integral (41). Moreover, all local martingales in the Brownian filtration are represented as stochastic integrals with respect to W - see RW IV.36.5.

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[7] Of course, M^θ is complex-valued, but you can obviously work on the real and imaginary parts separately if you are bothered by this apparent extension of the validity of the formula.

[8] See RW II.67; this is needed for technical reasons.

Change of measure. Suppose that Q is a probability measure on $(\Omega, \mathcal{F}, (\mathcal{F})_{0 \le t \le T})$ which is equivalent to P. Then the process

$$Z_t \equiv \frac{dQ}{dP}\bigg|_{\mathcal{F}_t}$$

is a positive P-martingale. It is a straightforward exercise to prove that M is a Q-martingale if and only if MZ is a P-martingale; a little more effort establishes the result that M is a Q-local martingale if and only if MZ is a P-local martingale. See RW, IV.17.

The famous theorem of Cameron & Martin, Girsanov, Maruyama, ... concerning changes of measure in the Brownian filtration is useful and important; informally, it says that a change of measure is in effect a change of the *drift* of a Brownian motion. See RW Theorem IV.38.5 for the n-dimensional generalisation (and proof) of the following result.

Theorem 8 *Suppose that $(\Omega, \mathcal{F}, (\mathcal{F})_{0 \le t \le T}, P)$ is the (usual augmentation of) the filtered probablity space generated by Brownian motion X.*

(i) If $Q \sim P$, then there exists a previsible process c such that

$$Z_t \equiv \frac{dQ}{dP}\bigg|_{\mathcal{F}_t} = \exp\biggl(\int_0^T c_s dX_s - \frac{1}{2} \int_0^T c_s^2 \, ds \biggr), \tag{10}$$

and under Q,

$$\tilde{X}_t \equiv X_t - \int_0^t c_s \, ds \qquad \textit{is a martingale.} \tag{11}$$

(ii) Conversely, if γ is a previsible process such that

$$\zeta_t \equiv \exp\biggl(\int_0^t \gamma_s dX_s - \frac{1}{2} \int_0^t \gamma_s^2 \, ds \biggr) \tag{12}$$

is a martingale, and if we define a measure Q on $(\Omega, \mathcal{F}_T)$ by

$$\frac{dQ}{dP}\bigg|_{\mathcal{F}_T} = \zeta_T,$$

then under Q

$$\tilde{X}_t \equiv X_t - \int_0^t \gamma_s \, ds \qquad \textit{is a martingale.}$$